

Knots with Signature 4 and Determinant $4k + 1$: An Answer to a Question of Shinohara

Alper Ferudun*

October 2, 2026

Abstract

By a congruence of Murasugi, the determinant and the signature of a knot K satisfy $\det K \equiv (-1)^{\sigma(K)/2} \pmod{4}$; moreover, a knot of determinant 1 has signature divisible by 8, because its symmetrized Seifert form is even and unimodular. In 1971 Shinohara realized all pairs $(\det K, |\sigma(K)|)$ allowed by these conditions, except those with $\det K \equiv 1 \pmod{8}$ and $|\sigma(K)| \equiv 4 \pmod{8}$; he stated that the case $\det K \not\equiv 1 \pmod{24}$ can be treated as well and left the case $\det K \equiv 1 \pmod{24}$ open, expecting an affirmative answer. The question reappeared as Problem 12.21 of Ohtsuki's list of problems on invariants of knots and 3-manifolds: is there, for every $n = 4k + 1$ with $k > 0$, a knot with determinant n and signature 4? Stoimenow showed that every prime factor of a counterexample is $\equiv 1 \pmod{24}$ and at least 33049, and asked whether $\{\det K : \sigma(K) = 4\} = 5 + 4\mathbb{N}$. We answer these questions affirmatively. For every $n \equiv 1 \pmod{4}$ with $n \geq 5$ there is a knot of determinant n and signature 4; it can be taken of genus 2 unless n is a square all of whose prime factors are $\equiv 1 \pmod{4}$, and of genus at most 3 in that case. Consequently, an odd positive integer d and an even integer s are the determinant and the signature of some knot if and only if $d \equiv (-1)^{s/2} \pmod{4}$, where $d = 1$ requires $s \equiv 0 \pmod{8}$. The proof combines Seifert's realization theorem with an explicit family of genus-two Seifert matrices and a lemma based on Dirichlet's theorem on primes in arithmetic progressions and quadratic reciprocity. We also give explicit knots, among them a two-bridge knot of determinant 33049, and report exact computer checks. This is an unrefereed note.

1 Introduction

All knots are tame knots in S^3 . If V is a Seifert matrix of a knot K , of size $2g \times 2g$, then $V - V^T$ is the intersection form of the Seifert surface, so $\det(V - V^T) = 1$, and

$$\det K = |\det(V + V^T)| = |\Delta_K(-1)|, \quad \sigma(K) = \text{sign}(V + V^T)$$

are the *determinant* and the *signature* of K . They do not depend on the choice of V [Tro62, Mur65]. Under connected sum the determinant is multiplicative and the signature is additive; mirroring changes the sign of σ and keeps $\det$, so the sign convention for σ plays no role below. Murasugi proved [Mur65, Thm. 5.6] that

$$\det K \equiv (-1)^{\sigma(K)/2} \pmod{4}, \tag{1}$$

and a knot of determinant 1 has $\sigma(K) \equiv 0 \pmod{8}$, because $V + V^T$ is then even and unimodular ([Shi71, (3.7)], [Sto07, Thm. 1.1]; see Lemma 2.1). In particular, a knot of signature 4 has determinant in $\{5, 9, 13, \dots\} = 5 + 4\mathbb{N}$, where $\mathbb{N} = \{0, 1, 2, \dots\}$.

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: <https://github.com/AlperTheKing>.
Unrefereed preprint. CC BY 4.0.

Shinohara considered whether these are the only restrictions. In [Shi71, Thm. 6] he constructed, for all $m, j \geq 0$, knots with $(|\Delta_K(-1)|, |\sigma(K)|)$ equal to $(4m+1, 8j)$, $(8m+5, 8j+4)$ and $(4m+3, 4j+2)$; in the second case he used $T(2, 8m+5)$ and connected sums with copies of the torus knot $T(3, 5)$ or of its mirror image, which have determinant 1 and $|\sigma| = 8$. In the remark after the theorem he stated that the remaining pairs $(8m+1, 8j+4)$, $m > 0$, can also be realized when $m \not\equiv 0 \pmod{3}$, and that the case $m \equiv 0 \pmod{3}$, that is $|\Delta_K(-1)| \equiv 1 \pmod{24}$, remained open, with an affirmative answer expected. The question reappeared, attributed to Shinohara, as Problem 12.21 of the problem list edited by Ohtsuki [Oht02, p. 540]:

If $n = 4k + 1$ with $k > 0$, is there a knot with determinant n and signature 4?

In a remark there, Stoimenow notes that every prime factor of a counterexample $n > 1$ would be $\equiv 1 \pmod{24}$ and at least 2857. In [Sto07, §5.1] he asked again whether

$$S := \{\det K : \sigma(K) = 4\} \subseteq 1 + 4\mathbb{N}$$

equals $5 + 4\mathbb{N}$ [Sto07, Question 5.1]. He proved that S contains every $n \equiv 1 \pmod{4}$ that has a prime factor $\equiv 3 \pmod{4}$, and that S is closed under multiplication by positive integers $\equiv 1 \pmod{4}$ [Sto07, Prop. 5.1]. He raised the bound 2857 to 33049 [Sto07, Cor. 5.1] and reported a computer search showing that S contains every $n \equiv 1 \pmod{4}$ with $5 \leq n \leq 4 \cdot 10^9$. He also asked whether ± 1 plays a special role in his Theorem 1.1, which says that no knot has determinant 1 and signature $\equiv 4 \pmod{8}$. In [Sto04] he proved that a knot of signature 4 whose determinant is a square without prime factors $\equiv 3 \pmod{4}$ has unknotting number at least 3, and he observed by computer that every non-square $4k + 1 < 400$ is the determinant of a knot with unknotting number 2 and signature 4. The problem is record AMR-103-0226 of the UnsolvedMath dataset [ula26], with status “open”; the statement of the record agrees with [Oht02], but its title field is truncated.

We answer these questions.

Theorem 1.1. *Let $n \geq 5$ be an integer with $n \equiv 1 \pmod{4}$. Then there is a knot K with $\det K = n$ and $\sigma(K) = 4$. If n is not a square, or if n has a prime factor $\equiv 3 \pmod{4}$, then K can be chosen of genus 2; in all cases K can be chosen of genus at most 3.*

Genus 2 is the least possible, since $|\sigma(K)| \leq 2g(K)$. Together with (1) and the case of determinant 1, Theorem 1.1 gives

$$\{\det K : \sigma(K) = 4\} = \{\det K : \sigma(K) = -4\} = \{5, 9, 13, 17, \dots\} = 5 + 4\mathbb{N}.$$

Theorem 1.2. *Let d be an odd positive integer and s an even integer. There is a knot K with $\det K = d$ and $\sigma(K) = s$ if and only if $d \equiv (-1)^{s/2} \pmod{4}$ and, in case $d = 1$, $s \equiv 0 \pmod{8}$.*

Corollary 1.3 (Shinohara’s open case). *For all integers $m \geq 1$ and $j \geq 0$ there is a knot K with $|\Delta_K(-1)| = 8m + 1$ and $|\sigma(K)| = 8j + 4$.*

Theorem 1.2 also answers Stoimenow’s side question: apart from the values excluded by (1), 1 is the only determinant for which no signature $\equiv 4 \pmod{8}$ occurs.

Most ingredients are classical or due to Shinohara and Stoimenow: the necessary conditions, the cases $n \equiv 5 \pmod{8}$ and $n \not\equiv 1 \pmod{24}$, the case of a prime factor $\equiv 3 \pmod{4}$, the multiplicativity used for squares, and the shift of the signature by connected sums with $T(3, 5)$. By Stoimenow’s results, every prime factor of a counterexample would be a prime $p \equiv 1 \pmod{24}$ with $p \geq 33049$ (by his computer search even $p > 4 \cdot 10^9$), and by multiplicativity it would suffice to show that all such primes lie in S ; this case remained open. The new ingredient is Lemma 3.2, used with the genus-two Seifert matrices of Lemma 3.1: it shows that every non-square $n \equiv 1 \pmod{4}$, in particular every

prime $p \equiv 1 \pmod{24}$, lies in S . Its only non-elementary input is Dirichlet's theorem on primes in arithmetic progressions. Remark 5.3 sketches a second, less elementary route through the existence theory of even lattices, which would also give genus 2 in all cases. Section 5 discusses the squares, explicit knots and prime examples.

2 Seifert matrices

All matrices have integer entries. A symmetric matrix is *even* if its diagonal entries are even. We write $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and $V \oplus W$ for the block sum.

Lemma 2.1 (Murasugi). *Let V be a $2g \times 2g$ matrix with $\det(V - V^T) = 1$ and put $B = V + V^T$. Then $\det B \equiv (-1)^g \pmod{4}$. Consequently, for every knot K the determinant is odd, the signature is even, and (1) holds. If $\det K = 1$, then $\sigma(K) \equiv 0 \pmod{8}$.*

Proof. Put $A = V - V^T$ and $W = A^{-1}$; both are skew-symmetric integer matrices. Since $B = A + 2V^T = A(I + 2M)$ with $M = WV^T$, and $\det(I + 2M) \equiv 1 + 2 \operatorname{tr} M \pmod{4}$ (expand the determinant), it suffices to show that $\operatorname{tr} M = -g$. Using $W_{ii} = 0$ and $W_{ji} = -W_{ij}$,

$$\operatorname{tr} M = \sum_{i,j} W_{ij} V_{ij} = \sum_{i < j} W_{ij} (V_{ij} - V_{ji}) = \sum_{i < j} W_{ij} A_{ij} = \frac{1}{2} \operatorname{tr}(W A^T) = -\frac{1}{2} \operatorname{tr}(W A) = -g.$$

Hence $\det B \equiv 1 - 2g \equiv (-1)^g \pmod{4}$; in particular $\det B$ is odd and B is nondegenerate. If B has q negative eigenvalues, then $\sigma = 2g - 2q$ is even, $\det B = (-1)^q |\det B|$ and $q = g - \sigma/2$, so $|\det B| \equiv (-1)^{q+g} = (-1)^{\sigma/2} \pmod{4}$. If $|\det B| = 1$, then B is even and unimodular, and its signature is divisible by 8 [Ser73, Ch. V], [MH73]. $\square$

Theorem 2.2 (Seifert [Sei35]). *Every $2g \times 2g$ matrix V with $\det(V - V^T) = 1$ is a Seifert matrix of a knot, which then has a Seifert surface of genus g .*

If V_1 and V_2 are Seifert matrices of K_1 and K_2 , then $V_1 \oplus V_2$ is a Seifert matrix of $K_1 \# K_2$, and $-V_1^T$ is one of the mirror image of K_1 .

Lemma 2.3. *Let B be an even $2g \times 2g$ matrix with $\det B$ odd. Then there is a matrix V with $V + V^T = B$ and $\det(V - V^T) = 1$.*

Proof. Modulo 2, B is a nondegenerate alternating form, so $B \equiv \bar{P}(J \oplus \cdots \oplus J)\bar{P}^T \pmod{2}$ for some $\bar{P}$ in $GL_{2g}(\mathbb{F}_2) = SL_{2g}(\mathbb{F}_2)$. Lift $\bar{P}$ to $P \in SL_{2g}(\mathbb{Z})$ (reduction modulo 2 is onto) and put $A = P(J \oplus \cdots \oplus J)P^T$. Then A is skew-symmetric, $\det A = 1$ and $A \equiv B \pmod{2}$, so $V = (B + A)/2$ has integer entries, $V + V^T = B$ and $V - V^T = A$. $\square$

We use two families of 2×2 blocks. For $u \equiv 3 \pmod{4}$ with $u \geq 3$, and for $m \equiv 1 \pmod{4}$ with $m \geq 1$, put

$$E(u) = \begin{pmatrix} 1 & 1 \\ 0 & \frac{u+1}{4} \end{pmatrix}, \quad T(m) = \begin{pmatrix} -1 & 1 \\ 0 & \frac{m-1}{4} \end{pmatrix}.$$

Both satisfy $V - V^T = J$. The matrix $E(u) + E(u)^T = \begin{pmatrix} 2 & 1 \\ 1 & (u+1)/2 \end{pmatrix}$ has determinant $u > 0$, so it is positive definite and has signature 2; the matrix $T(m) + T(m)^T$ has determinant $-m < 0$, so its signature is 0.

3 Proof of Theorem 1.1

Lemma 3.1. *For integers b, c and an even integer y let*

$$G(b, c, y) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & b & y/2 & 0 \\ 0 & y/2 & c & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad B = G + G^T = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2b & y & 0 \\ 0 & y & 2c & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}.$$

Then $G - G^T = J \oplus J$ and $\det B = (4b - 1)(4c - 1) - 4y^2$. If $b \geq 1$ and $\det B > 0$, then B is positive definite, so a knot with Seifert matrix $G(b, c, y)$ has genus 2, determinant $\det B$ and signature 4.

Proof. The first claim is immediate. The leading principal minors of the tridiagonal matrix B are $D_1 = 2$, $D_2 = 4b - 1$, $D_3 = 2cD_2 - y^2D_1$ and $D_4 = 2D_3 - D_2$, so $D_4 = (4b - 1)(4c - 1) - 4y^2$ and $2D_3 = D_4 + D_2$. If $b \geq 1$ and $D_4 > 0$, then $D_1, D_2, D_3, D_4 > 0$ and B is positive definite by Sylvester's criterion. The genus is at most 2 by Theorem 2.2 and at least $|\sigma|/2 = 2$. $\square$

Lemma 3.2. *Let $n \equiv 1 \pmod{4}$ be a positive integer that is not a square. Then there are a prime $l \equiv 3 \pmod{4}$ not dividing n and an even integer $y \geq 0$ such that l divides $n + 4y^2$.*

Proof. Write $n = s^2m$ with m squarefree. Then $m > 1$, and $m \equiv n \equiv 1 \pmod{4}$ because s is odd. Choose a prime $r \mid m$ and a quadratic non-residue u modulo r . By the Chinese remainder theorem there is an integer t with

$$t \equiv u \pmod{r}, \quad t \equiv 1 \pmod{m/r}, \quad t \equiv 3 \pmod{4}.$$

Then $\gcd(t, 4m) = 1$, and the Jacobi symbol satisfies $\left(\frac{t}{m}\right) = \left(\frac{t}{r}\right)\left(\frac{t}{m/r}\right) = \left(\frac{u}{r}\right) \cdot 1 = -1$. By Dirichlet's theorem there are infinitely many primes $l \equiv t \pmod{4m}$; choose one with $l \nmid s$. Then $l \equiv 3 \pmod{4}$ and $l \nmid n$. Since $m \equiv 1 \pmod{4}$, quadratic reciprocity for the Jacobi symbol gives

$$\left(\frac{n}{l}\right) = \left(\frac{m}{l}\right) = \left(\frac{l}{m}\right) = \left(\frac{t}{m}\right) = -1.$$

As $\left(\frac{-1}{l}\right) = -1$, we get $\left(\frac{-n}{l}\right) = 1$: there is an integer x with $x^2 \equiv -n \pmod{l}$. Let $0 < y_0 < l$ with $2y_0 \equiv x \pmod{l}$ (here $y_0 \neq 0$ because $l \nmid n$), and put $y = y_0$ if y_0 is even and $y = y_0 + l$ otherwise. Then y is even and $4y^2 \equiv x^2 \equiv -n \pmod{l}$. $\square$

Proof of Theorem 1.1. Let $n \geq 5$ with $n \equiv 1 \pmod{4}$. We construct a matrix V with $\det(V - V^T) = 1$, $|\det(V + V^T)| = n$ and $\text{sign}(V + V^T) = 4$; then Theorem 2.2 gives the knot.

(a) n has a prime factor $l \equiv 3 \pmod{4}$. Then $n/l \equiv 3 \pmod{4}$, so $n/l \geq 3$. Take $V = G\left(\frac{l+1}{4}, \frac{n/l+1}{4}, 0\right)$; by Lemma 3.1, $\det(V + V^T) = l \cdot n/l = n$. This V is the block sum of $E(l)$ and $\begin{pmatrix} (n/l+1)/4 & 1 \\ 0 & 1 \end{pmatrix}$, a connected sum of two knots of genus one and signature 2, as in [Sto07, Prop. 5.1(1)].

(b) n is not a square. Take l and y from Lemma 3.2 and write $n + 4y^2 = lm'$. Since $n + 4y^2 \equiv 1$ and $l \equiv 3 \pmod{4}$, we have $m' \equiv 3 \pmod{4}$, so $m' \geq 3$. Take $V = G\left(\frac{l+1}{4}, \frac{m'+1}{4}, y\right)$; by Lemma 3.1, $\det(V + V^T) = lm' - 4y^2 = n$.

(c) Otherwise $n = s^2$, where $s > 1$ and every prime factor of s is $\equiv 1 \pmod{4}$. Let p be a prime factor of s . By (b) applied to p there is a 4×4 matrix V_p with $V_p - V_p^T = J \oplus J$, $\det(V_p + V_p^T) = p$ and $\text{sign}(V_p + V_p^T) = 4$. The integer $m = n/p$ satisfies $m \equiv 1 \pmod{4}$ and $m \geq 5$. Take $V = V_p \oplus T(m)$. Then $\det(V - V^T) = 1$, $|\det(V + V^T)| = pm = n$ and $\text{sign}(V + V^T) = 4 + 0 = 4$. This is Stoimenow's closure property [Sto07, Prop. 5.1(2)]: a connected sum with a knot of signature 0 and determinant m .

In cases (a) and (b), V is a 4×4 matrix and K has genus 2 by Lemma 3.1; in case (c), V is 6×6 and K has genus at most 3. $\square$

Remark 3.3. (a) The prime $l = 3$ handles two residue classes, both treated before. If $3 \mid n$, case (a) applies with $l = 3$. If $n \equiv 5 \pmod{12}$, then $3 \mid n + 16$, and $G(1, \frac{n+19}{12}, 2)$ has determinant n . These are Stoimenow's progressions $9 + 12k$ and $5 + 12k$ [Sto07, Prop. 5.1(6)]; together with Shinohara's case $n \equiv 5 \pmod{8}$ they cover every $n \not\equiv 1 \pmod{24}$. If $n \equiv 1 \pmod{12}$, then $n + 4y^2 \equiv 1 + y^2 \not\equiv 0 \pmod{3}$ for all y , and a larger prime is needed.

(b) For a prime $p \equiv 1 \pmod{4}$ one may take for l the least prime $l \equiv 3 \pmod{4}$ with $(\frac{l}{p}) = -1$, since then $(\frac{-p}{l}) = -(\frac{p}{l}) = -(\frac{l}{p}) = 1$. By computation, this l is at most 139 for every prime $p \equiv 1 \pmod{4}$ below 10^7 . With the least even y this gives, for example, $G(3, 71, 8)$ for $p = 2857$ (here $11 \cdot 283 - 4 \cdot 8^2 = 2857$) and $G(5, 437, 6)$ for $p = 33049$ (here $19 \cdot 1747 - 4 \cdot 6^2 = 33049$); these are the two lower bounds of Stoimenow quoted above, and both are primes $\equiv 1 \pmod{24}$.

4 Proof of Theorem 1.2 and Corollary 1.3

Proof of Theorem 1.2. Necessity is Lemma 2.1. For sufficiency, let E_8 be the Gram matrix of the E_8 root lattice with respect to the simple roots, numbered so that $1, \dots, 7$ form a path in the Dynkin diagram and 8 is joined to 3. It is even, positive definite and unimodular [Ser73, Ch. V]. Let V_8 be the matrix with $(V_8)_{ii} = 1$, $(V_8)_{ij} = (E_8)_{ij}$ for $i < j$ and $(V_8)_{ij} = 0$ for $i > j$. Then $V_8 + V_8^T = E_8$, and $V_8 - V_8^T$ is a signed adjacency matrix of the Dynkin diagram, a tree with exactly one perfect matching, so its Pfaffian is ± 1 and $\det(V_8 - V_8^T) = 1$. By Theorem 2.2 there is a knot K_8 with $\det K_8 = 1$ and $\sigma(K_8) = 8$ (Shinohara used the torus knot $T(3, 5)$ instead, and so did Stoimenow [Sto07, Remark 5.1]). Connected sums with copies of K_8 or of its mirror image change the signature by any multiple of 8 and keep the determinant. It therefore suffices to realize one signature in each admissible class modulo 8:

- $s \equiv 0 \pmod{8}$, $d \equiv 1 \pmod{4}$: the matrix $T(d)$, with determinant d and signature 0;
- $s \equiv \pm 2 \pmod{8}$, $d \equiv 3 \pmod{4}$: the matrix $E(d)$, with signature 2, and $-E(d)^T$, with signature -2 ;
- $s \equiv 4 \pmod{8}$, $d \equiv 1 \pmod{4}$, $d \geq 5$: Theorem 1.1.

The first two items are Shinohara's cases (1) and (3) [Shi71, Thm. 6]. □

Proof of Corollary 1.3. Apply Theorem 1.2 with $d = 8m + 1 \geq 9$ and $s = 8j + 4$. □

5 Further remarks

Proposition 5.1. *Let $n = s^2$, where $s > 1$ and every prime factor of s is $\equiv 1 \pmod{4}$. Then there are no integers $b, c \geq 1$ and y with $(4b - 1)(4c - 1) - 4y^2 = n$. In particular, no matrix $G(b, c, y)$ of Lemma 3.1 with $b \geq 1$ has determinant n , and case (c) of the proof of Theorem 1.1 cannot be avoided within this family.*

Proof. Since $4b - 1 \geq 3$ and $4b - 1 \equiv 3 \pmod{4}$, it has a prime factor $l \equiv 3 \pmod{4}$, which divides $s^2 + (2y)^2$. If $l \nmid s$, then $(2y/s)^2 \equiv -1 \pmod{l}$, which is impossible for $l \equiv 3 \pmod{4}$. So $l \mid s$, contrary to the assumption on s . □

Remark 5.2 (Squares). Proposition 5.1 rests on the same congruence as Stoimenow's theorem in [Sto04] quoted above: both need a prime $l \equiv 3 \pmod{4}$ modulo which $-n$ is a square. By Lemma 3.2 such a prime exists when n is not a square, and by the proof of Proposition 5.1 it does not exist when n is a square whose prime factors are all $\equiv 1 \pmod{4}$.

For squares there is a second obstruction. If $\text{coker}(V + V^T)$, the first homology of the double branched cover of K [Rol03], is cyclic of order s^2 , then its subgroup of order s is a metabolizer for the linking form, and Jabuka's theorem [Jab12] (which also follows from Milgram's formula [MH73]) gives $\sigma(K) \equiv 0 \pmod{8}$. So a knot of signature 4 and square determinant never has cyclic $\text{coker}(V + V^T)$; in particular it is not a two-bridge knot, whose double branched cover is a lens space [Sch56, Rol03]. This also confirms the first part of a conjecture stated at the end of the arXiv version of [Sto04]: no knot of signature 4 has cyclic homology of prime-square order. In case (c) of the proof the group is $\mathbb{Z}/p \oplus \mathbb{Z}/(n/p)$. By contrast, a computer search found for every non-square $n \equiv 1 \pmod{4}$ with $n \leq 20001$ a two-bridge knot of determinant n and $|\sigma| = 4$.

For the squares s^2 with $s \in \{5, 13, 17, 25, 29, 37, 41, 53, 61, 65, 73, 85, 89, 97, 101\}$, that is, for every $s \leq 101$ all of whose prime factors are $\equiv 1 \pmod{4}$, computer searches over even positive definite 4×4 matrices, combined with Lemma 2.3, found knots of genus 2. For example, the matrix V with rows $(1, 0, 0, -1)$, $(-1, 1, 0, -1)$, $(-1, 0, 2, -1)$ and $(-2, -2, -1, 14)$ has $\det(V - V^T) = 1$, $\det(V + V^T) = 25$ and $\text{sign}(V + V^T) = 4$.

Remark 5.3 (A lattice-theoretic route; sketch). By Lemma 2.3 and Theorem 2.2, Theorem 1.1 with genus 2 follows from the existence of an even positive definite 4×4 matrix of determinant n . Such a matrix is an even lattice of signature $(4, 0)$ with a discriminant quadratic form q on a finite abelian group A of order n . By Nikulin's existence theorem [Nik80, Thm. 1.10.1] (see also Wall [Wal63]), such a lattice exists as soon as A is generated by at most 3 elements and $\text{sign } q \equiv 4 \pmod{8}$, where $\text{sign } q$ is defined by Milgram's formula $\sum_{x \in A} e^{\pi i q(x)} = \sqrt{|A|} e^{\pi i \text{sign}(q)/4}$ [MH73]. If n is not a square, take $A = \mathbb{Z}/n$ and $q(x) = 2ux^2/n \pmod{2}$ with $(\frac{u}{n}) = -1$; the classical evaluation of quadratic Gauss sums gives $\sum_x e^{2\pi i ux^2/n} = (\frac{u}{n})\sqrt{n} = -\sqrt{n}$, so $\text{sign } q \equiv 4 \pmod{8}$. If n is a square and p is a prime factor of n , take $A = \mathbb{Z}/p \oplus \mathbb{Z}/(n/p)$, $q = q_1 \oplus q_2$ with $q_1(x) = 2x^2/p$ and $q_2(x) = 2ux^2/(n/p)$, where $(\frac{u}{n/p}) = -1$ if $p \equiv 1 \pmod{4}$ and $u = 1$ if $p \equiv 3 \pmod{4}$; again $\text{sign } q \equiv 4 \pmod{8}$. We have not written out this argument in detail and do not use it; it indicates that the result may be regarded as a consequence of the classical theory of quadratic forms, and it would give genus 2 in all cases, in accordance with the examples of Remark 5.2.

Remark 5.4 (Explicit knots). Table 1 lists knots of signature ± 4 for small determinants and for the two primes of Remark 3.3(b). The names refer to Rolfsen's table [Rol03]; the prime knots in the table are two-bridge knots, given by their Conway notation, and connected sums have summands of the same handedness. Their determinants and signatures were computed exactly from Seifert matrices obtained from even continued fractions, and the signatures also by a second formula; the second verification run recomputed both from the diagrams.

n	knot	Conway notation	n	knot	Conway notation
5	5_1	5	29	9_7	3 4 2
9	$3_1 \# 3_1$	3 # 3	33	$9_{10}, 9_{11}$	3 3 3; 4 1 2 2
13	7_3	4 3	37	9_{13}	3 2 1 3
17	$7_5, 8_2$	3 2 2; 5 1 2	41	$9_{18}, 9_{20}$	3 2 2 2; 3 1 2 1 2
21	9_4	5 4	45	9_{23}	2 2 1 2 2
25	$5_1 \# 4_1$	5 # 2 2	49	$5_2 \# 5_2$	3 2 # 3 2
2857: $S(2857, 652)$, Conway notation 4 2 1 1 1 1 1 1 3 3 (19 crossings)					
33049: $S(33049, 9229)$, Conway notation 3 1 1 2 1 1 2 2 1 1 2 3 1 1 2 (24 crossings)					

Table 1: Knots with determinant n and signature ± 4 . The last two rows are two-bridge knots of least crossing number among those with determinant n and $|\sigma| = 4$ (computation).

Remark 5.5 (Prime and hyperbolic examples). The knots of Theorem 1.1 can be chosen prime: Stoimenow notes that prime examples follow from composite ones by a method of Bleiler [Ble82], [Sto07, Remark 5.1]. Moreover, by a theorem of Stoimenow [Sto08], every knot has hyperbolic knots with a common Seifert matrix, so the knots of Theorem 1.1 can even be chosen hyperbolic.

Verification. The claims were checked as follows.

1. A self-contained program by the author, `verify_note.py` (standard library only, exact integer and rational arithmetic, about fifteen seconds), does the following.
 - It proves the identities of Lemma 3.1 and those for $E(u)$ and $T(m)$ by symbolic expansion.
 - It follows the proof of Lemma 3.2 literally for every non-square $n \equiv 1 \pmod{4}$ with $n \leq 10^6 + 1$, the Dirichlet step being a search for the least admissible prime; the largest prime needed is 287,024,779.
 - It builds the matrices of the proof of Theorem 1.1 for every $n \equiv 1 \pmod{4}$ with $5 \leq n \leq 10^6 + 1$ and checks $\det(V - V^T) = 1$, $|\det(V + V^T)| = n$ and signature 4 exactly.
 - It realizes all 530 admissible pairs (d, s) of Theorem 1.2 with $d \leq 101$ and $|s| \leq 20$ by the block sums of Section 4, and it checks the examples of Remarks 3.3, 5.2 and 5.4.
2. Earlier programs, written separately and with their own exact linear algebra, checked the construction for every $n \leq 2 \cdot 10^6$, tested Lemma 2.1 on 4000 random Seifert matrices, enumerated the two-bridge knots with at most 12 crossings, and scanned the two-bridge knots of determinant at most 20001.
3. Two independent verification runs, both AI-assisted, re-derived all proofs line by line and wrote their own code.
 - The first run followed the proof for every $n \leq 10^6 + 1$, tested Lemma 2.1 on 3000 random Seifert matrices with a general unimodular skew part, and computed determinants and signatures of two-bridge knots by three methods (from a diagram, by a floor-sum formula, and from a Seifert matrix), which agree on all 8,282 pairs (p, q) with $p \leq 201$.
 - The second run followed the proof again for every $n \leq 10^6 + 1$, with an exact inertia computation for every matrix, and obtained the same case counts and the same largest prime. It recomputed the determinants and signatures of all knots of Table 1 from their diagrams, by the Fox and Goeritz matrices and the formula of Gordon and Litherland [GL78] for both checkerboard colourings, and confirmed the crossing numbers stated there. It also implemented the proof of Lemma 2.3, realized the 530 pairs of Theorem 1.2 again, and found the genus-two matrices for $s = 97$ and $s = 101$ in Remark 5.2.

Neither run found a mathematical error. The runs asked for corrections to the history, to the attribution of known cases and to the statement of what is new, which have been made.

All programs and outputs are in the source files.

Scope and priority. Theorem 1.1 answers Problem 12.21 of [Oht02] as stated and Question 5.1 of [Sto07]. With Theorem 1.2 it settles the case that Shinohara left open [Shi71]. Much of the problem had been settled by Shinohara and Stoimenow, who reduced it to primes $p \equiv 1 \pmod{24}$ (see the introduction). The new part is Lemma 3.2 together with the matrices of Lemma 3.1, which settles every non-square $n \equiv 1 \pmod{4}$ and hence this remaining case, and the bound on the genus. We read [Mur65], [Shi71] and [Sto07] in the publishers' PDF files, [Sto04] and [Jab12] in their arXiv versions,

and the problem list in its published version. We searched arXiv, Crossref, OpenAlex, zbMATH and Semantic Scholar, including the works that cite [Shi71], [Sto04], [Sto07] and [Oht02], and ran three general web searches. We found no published answer and no later work addressing the question. MathSciNet and Google Scholar were not searched. Because the result also follows from the classical existence theory of even lattices (Remark 5.3), it may be known to experts. This negative search is not a proof of priority.

AI-assisted tools were used to search the problem corpus and the literature, to find the constructions and the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

References

- [Ble82] Steven A. Bleiler. Realizing concordant polynomials with prime knots. *Pacific Journal of Mathematics*, 100(2):249–257, 1982. doi:10.2140/pjm.1982.100.249.
- [GL78] C. McA. Gordon and R. A. Litherland. On the signature of a link. *Inventiones Mathematicae*, 47(1):53–69, 1978. doi:10.1007/BF01609479.
- [Jab12] Stanislav Jabuka. The signature of an even symmetric form with vanishing associated linking form, 2012. arXiv:1204.4965.
- [MH73] John Milnor and Dale Husemoller. *Symmetric Bilinear Forms*, volume 73 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer, Berlin, Heidelberg, 1973. doi:10.1007/978-3-642-88330-9.
- [Mur65] Kunio Murasugi. On a certain numerical invariant of link types. *Transactions of the American Mathematical Society*, 117:387–422, 1965. doi:10.1090/S0002-9947-1965-0171275-5.
- [Nik80] V. V. Nikulin. Integral symmetric bilinear forms and some of their applications. *Mathematics of the USSR-Izvestiya*, 14(1):103–167, 1980. Translated from Izv. Akad. Nauk SSSR Ser. Mat. 43 (1979), 111–177. doi:10.1070/IM1980v014n01ABEH001060.
- [Oht02] Tomotada Ohtsuki. Problems on invariants of knots and 3-manifolds. In *Invariants of Knots and 3-Manifolds (Kyoto 2001)*, volume 4 of *Geometry & Topology Monographs*, pages 377–572. Geometry & Topology Publications, Coventry, 2002. Problem list edited by T. Ohtsuki; Problem 12.21 (Y. Shinohara) is on p. 540. arXiv:math/0406190, doi:10.2140/gtm.2002.4.377.
- [Rol03] Dale Rolfsen. *Knots and Links*. AMS Chelsea Publishing, Providence, RI, 2003. Corrected reprint of the 1976 original (Publish or Perish). doi:10.1090/chel/346.
- [Sch56] Horst Schubert. Knoten mit zwei Brüchen. *Mathematische Zeitschrift*, 65(1):133–170, 1956. doi:10.1007/BF01473875.
- [Sei35] H. Seifert. Über das Geschlecht von Knoten. *Mathematische Annalen*, 110(1):571–592, 1935. doi:10.1007/BF01448044.
- [Ser73] Jean-Pierre Serre. *A Course in Arithmetic*, volume 7 of *Graduate Texts in Mathematics*. Springer, New York, 1973. doi:10.1007/978-1-4684-9884-4.
- [Shi71] Yaichi Shinohara. On the signature of knots and links. *Transactions of the American Mathematical Society*, 156:273–285, 1971. doi:10.1090/S0002-9947-1971-0275415-1.

- [Sto04] A. Stoimenow. Polynomial values, the linking form and unknotting numbers. *Mathematical Research Letters*, 11(6):755–769, 2004. `arXiv:math/0405076`, `doi:10.4310/MRL.2004.v11.n6.a4`.
- [Sto07] A. Stoimenow. Determinants of knots and Diophantine equations. *Acta Arithmetica*, 129(4):363–387, 2007. `doi:10.4064/aa129-4-6`.
- [Sto08] A. Stoimenow. Vassiliev invariants, Seifert matrix, and hyperbolic volume of knots. *International Mathematics Research Notices*, 2008. Art. ID rnn119, 26 pp. `doi:10.1093/imrn/rnn119`.
- [Tro62] H. F. Trotter. Homology of group systems with applications to knot theory. *Annals of Mathematics*, 76(3):464–498, 1962. `doi:10.2307/1970369`.
- [ula26] ulamai. UnsolvedMath. Hugging Face dataset, <https://huggingface.co/datasets/ulamai/UnsolvedMath>, 2026. Version 1.6.0; record AMR-103-0226.
- [Wal63] C. T. C. Wall. Quadratic forms on finite groups, and related topics. *Topology*, 2(4):281–298, 1963. `doi:10.1016/0040-9383(63)90012-0`.