

A Conjecture of Guadagnini and Pilo on Quantum Invariants and the Fundamental Group: Counterexamples and a Classification by Gauge Group

Alper Ferudun*

October 4, 2026

Abstract

Guadagnini and Pilo conjectured that for a closed oriented 3-manifold M and a compact gauge group G the absolute value of the quantum invariant $\tau_r^G(M)$, when it does not vanish, depends only on the fundamental group of M ; this is Conjecture 7.5 of Ohtsuki's problem list. They proved it for lens spaces and $G = \mathrm{SU}(2)$. We show that the conjecture is false in general. With $\tau(S^3) = 1$: for $\mathrm{SU}(4)$ and $r = 12$ the lens spaces $L(8, 1)$ and $L(8, 3)$ have $|\tau| = 16 + 8\sqrt{3}$ and $56 + 32\sqrt{3}$; for $\mathrm{SU}(5)$ and $r = 10$ one has $|\tau(L(5, 1))|^2 = 3475 + 1550\sqrt{5}$ and $|\tau(L(5, 2))| = 45 + 20\sqrt{5}$; for $\mathrm{Spin}(7)$ and $r = 15$ the moduli for $L(10, 1)$ and $L(10, 3)$ are nonzero with ratio $2 - \sqrt{3}$. These values are obtained by exact computation from the Reshetikhin–Turaev surgery formula. On the basis of the lens space formula of Hansen and Takata we then determine the simply connected compact simple groups for which the conjecture holds. We prove a structure result for the lattice Gauss sums in that formula: only a coset of a stabilizer in the Weyl group contributes. From it we deduce that the lens-space statement fails for $\mathrm{SU}(N)$ for every $N \geq 4$, for $\mathrm{Spin}(m)$ with $m = 7$ and every $m \geq 9$, and for E_6, E_7, E_8 , and that it holds for $\mathrm{Sp}(2n)$ for every $n \geq 1$ and for $\mathrm{SU}(3), G_2, \mathrm{Spin}(8)$ and F_4 (for the last two with a computer-assisted lemma). By geometrization the conjecture for G is equivalent to its lens-space case, so, given that formula, it holds exactly for $\mathrm{SU}(2), \mathrm{SU}(3), \mathrm{Sp}(2n), G_2, \mathrm{Spin}(8)$ and F_4 . We also record that two closed formulas of Guadagnini and Pilo for $\mathrm{SU}(2)$ are inaccurate; their conclusion is not affected. This is an unrefereed note.

1 Introduction

Let M be a closed oriented 3-manifold (all manifolds in this note are connected) and let G be a simply connected compact simple Lie group with dual Coxeter number $h^\vee$. For every integer $k \geq 0$, the level, there is a quantum invariant $\tau_r^G(M) \in \mathbb{C}$, where $r = k + h^\vee$. It was proposed by Witten [Wit89] and constructed by Reshetikhin and Turaev [RT91] for $G = \mathrm{SU}(2)$, and later for all G ; see [Tur10, BK01]. We normalize it by $\tau_r^G(S^3) = 1$; the precise definition is recalled in Section 2. Guadagnini and Pilo [GP98] proposed the following conjecture. It is Conjecture 7.5 of Ohtsuki's problem list [Oht02, Section 7.1] and record AMR-103-0120 of the UnsolvedMath dataset [ula26].

For non-vanishing $\tau_r^G(M)$, the absolute value $|\tau_r^G(M)|$ depends only on the fundamental group $\pi_1(M)$.

That is: if M and N are closed oriented 3-manifolds with isomorphic fundamental groups and $\tau_r^G(M), \tau_r^G(N)$ are both nonzero, then $|\tau_r^G(M)| = |\tau_r^G(N)|$. In [GP98] the invariant is defined through Chern–Simons theory with a simple compact gauge group, and neither [GP98] nor [Oht02] restricts the group.

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: <https://github.com/AlperTheKing>.
Unrefereed preprint. CC BY 4.0.

The hypothesis of non-vanishing is needed: for $SU(2)$ and $r = 7$ the invariant vanishes on the lens space $L(7, 2)$ but not on $L(7, 1)$.

Guadagnini and Pilo proved the conjecture for lens spaces and $G = SU(2)$: for fixed p and r the nonzero numbers among $|\tau_r^{SU(2)}(L(p, q))|$ coincide [GP98, Section 4]. Their proof evaluates the invariant in closed form by an argument of Jeffrey [Jef92]. The absolute values of the $SU(2)$ invariants of lens spaces are also the subject of a paper of Yamada [Yam95], which gives an explicit formula for them, and of a paper of Sokolov [Sok97], which determines the lens spaces that are distinguished by the Turaev–Viro invariants; we know these two papers only from their summaries in zbMATH. For $SU(3)$, Guadagnini and Pilo report numerical agreement for some lens spaces with $p \leq 20$ and $3 \leq r \leq 50$.

For a group G we call *lens-space statement*, $LS(G)$, the assertion that for all $p \geq 1$ and all $r \geq h^\vee$ the nonzero numbers among $|\tau_r^G(L(p, q))|$, q prime to p , coincide. The lens spaces $L(p, q)$ and $L(p, q')$ have fundamental group $\mathbb{Z}/p$. So $LS(G)$ is a special case of the conjecture for G , and a failure of $LS(G)$ is a counterexample to it.

Theorem 1.1. *Let G be simply connected and let $\tau = \tau_r^G$ be the invariant of Section 2.2: the Reshetikhin–Turaev invariant of the modular category whose simple objects correspond to all integrable highest weights of level $k = r - h^\vee$, normalized by $\tau(S^3) = 1$.*

- (a) $G = SU(4)$, $r = 12$ (level 8): $|\tau(L(8, 1))| = 16 + 8\sqrt{3} = 29.856\dots$ and $|\tau(L(8, 3))| = 56 + 32\sqrt{3} = 111.425\dots$
- (b) $G = SU(5)$, $r = 10$ (level 5): $|\tau(L(5, 1))|^2 = 3475 + 1550\sqrt{5}$ and $|\tau(L(5, 2))| = 45 + 20\sqrt{5}$, so that $|\tau(L(5, 1))| = 83.312\dots$ and $|\tau(L(5, 2))| = 89.721\dots$
- (c) $G = Spin(7)$, $r = 15$ (level 10): $|\tau(L(10, 1))| = 10\sqrt{10} \cos(\pi/12)/\Pi = 1182.524\dots$ and $|\tau(L(10, 3))| = 10\sqrt{10} \sin(\pi/12)/\Pi = 316.856\dots$, where $\Pi = 0.0258305\dots$ is the number $\Pi(15)$ of (1) for the root system B_3 . The ratio of the two moduli is $2 - \sqrt{3}$.

In each case the two lens spaces have the same fundamental group and nonzero invariants with different absolute values. Hence the conjecture is false for $G = SU(4)$, $SU(5)$ and $Spin(7)$.

The proof of Theorem 1.1 is an exact computation in a cyclotomic ring. It starts from the Reshetikhin–Turaev surgery formula with the Kac–Peterson matrices S and T and uses no closed formula for lens spaces (Section 3). Floating-point computations with the same formula give further examples (Table 1). Among them are E_8 at $r = 35$ (level 5, where the modular category has only 15 simple objects), with moduli $24.5324\dots$ and $35.4504\dots$ for $L(5, 1)$ and $L(5, 2)$, and $SU(7)$ at $r = 14$, where the homotopy equivalent spaces $L(7, 1)$ and $L(7, 2)$ have the nonzero moduli $7152.09\dots$ and $8696.41\dots$

Our second result decides the lens-space statement for every group.

Theorem 1.2. *The following statements are consequences of the lens space formula of Hansen and Takata [HT04, Theorem 5.1] (Theorem 2.3 below), which we use without reproving it. Let G be a simply connected compact simple Lie group.*

- (a) $LS(G)$ holds for $G = Sp(2n)$, $n \geq 1$ (this includes $SU(2) = Sp(2)$ and $Spin(5) = Sp(4)$), and for $G = SU(3)$, G_2 , $Spin(8)$ and F_4 . For $Spin(8)$ and F_4 the proof uses a lemma that was verified by computer (finite exact tables over the Weyl group).
- (b) $LS(G)$ fails for every other G , that is, for $SU(N)$ with $N \geq 4$, for $Spin(m)$ with $m = 7$ or $m \geq 9$, and for E_6 , E_7 , E_8 . More precisely:

- (i) $\text{SU}(5)$: $0 < |\tau_r(L(5, 1))| < |\tau_r(L(5, 2))|$ for every $r \geq 10$ divisible by 5;
- (ii) $\text{SU}(4)$: $|\tau_r(L(8, 3))| = \cot(\pi/r) |\tau_r(L(8, 1))| \neq 0$ for every $r \equiv 4 \pmod{8}$, $r \geq 12$;
- (iii) $\text{SU}(N)$ with $N = ap$, $p \notin \{1, 2, 3, 4, 6\}$ and $(p-1)a$ even: $0 < |\tau_r(L(p, 1))| < |\tau_r(L(p, q))|$ for every $q \not\equiv \pm 1 \pmod{p}$ and every multiple $r \geq Np^5$ of p . Such a p exists for every $N \geq 4$ except $N = 4, 6, 8, 12$; for these, the pairs $(p, r) = (8, 12), (12, 18), (8, 16), (12, 24)$ give failures;
- (iv) $\text{Spin}(7)$: $|\tau_r(L(10, 3))| = \tan(5\pi/(4r)) |\tau_r(L(10, 1))| \neq 0$ for every $r \equiv 5 \pmod{10}$, $r \geq 15$;
- (v) $\text{Spin}(2n+1)$, $n \geq 4$: $p = 2n$ and every multiple $r > n^2/2$ of $2n$; $\text{Spin}(2n)$, $n \geq 5$: $p = 2n-2$ and every multiple $r > \max\{2n-2, (n-1)(n-2)/2\}$ of $2n-2$;
- (vi) E_6, E_7, E_8 : $(p, r) = (9, 18), (14, 28), (5, 35)$ (the proof evaluates finite sums over cosets in the Weyl group exactly, by computer).

For $\text{SU}(4)$, $\text{SU}(5)$ and $\text{Spin}(7)$, part (b) also follows from Theorem 1.1 and does not depend on [HT04].

Corollary 1.3. *Assume the formula of Hansen and Takata. For a simply connected compact simple Lie group G , the conjecture of Guadagnini and Pilo holds for G (for all closed oriented 3-manifolds and all r) if and only if G is one of $\text{SU}(2)$, $\text{SU}(3)$, $\text{Sp}(2n)$ with $n \geq 2$, G_2 , $\text{Spin}(8)$, F_4 . For $G = \text{SU}(4)$, $\text{SU}(5)$ and $\text{Spin}(7)$ the conjecture fails without this assumption.*

Corollary 1.3 follows from Theorems 1.1 and 1.2 and from the fact that for every G the conjecture is equivalent to $\text{LS}(G)$ (Proposition 7.1). This reduction uses geometrization, in the form of a theorem on 3-manifold groups stated in [AFW15], the multiplicativity of τ under connected sum and its behaviour under reversal of orientation. For $G = \text{SU}(2)$ it shows that the theorem of Guadagnini and Pilo on lens spaces implies their conjecture for all closed oriented 3-manifolds. We did not find this consequence stated in the literature.

The mechanism. By the formula of Hansen and Takata, $|\tau_r(L(p, q))|$ is, up to a factor that does not depend on q , the modulus of a sum over the Weyl group W of Gauss sums on the coroot lattice L modulo pL . We show (Proposition 4.2) that each of these Gauss sums is zero or has a fixed modulus, and that the nonzero ones belong to a coset W_q of the subgroup $H \subseteq W$ which fixes ρ modulo the lattice $rL + pX$, where X is the weight lattice. If H is trivial there is at most one term, and the lens-space statement holds for the pair (p, r) . If not, $|\tau_r(L(p, q))|$ is the modulus of a sum of $|H|$ roots of unity. For $\text{SU}(5)$, $p = 5$ and $5 \mid r$ the cosets W_1 and W_2 have five elements each, and their five-term sums have different moduli (Proposition 5.2). For $\text{SU}(4)$ and $\text{Spin}(7)$ the two terms of one coset carry a relative sign that depends on q . For the groups of Theorem 1.2(a) the sums over different cosets are related by $x \mapsto x^{\pm 1}$ and a monomial factor, or factor into Vandermonde determinants.

What was known. Formulas for quantum invariants of lens spaces were given by Jeffrey [Jef92] and by Guadagnini and Pilo [GP98] for $\text{SU}(2)$ (see also [Yam95, LL96, Sok97]), by Takata [Tak96] for the projective invariants of type $\text{PSU}(n)$, by Zhang and Carey [ZC96] and Zhang [Zha97] at odd roots of unity, and by Hansen and Takata [HT04] for every simple Lie algebra. When p is prime to mr (m the lacing number), [HT04, Proposition 5.2] expresses $\tau_r(L(p, q))$ as a product in which q enters only through a phase and a Gauss sum of the lattice; by Lemma 4.1(iii) its modulus does not depend on q . So the question is about the pairs (p, r) that are not coprime. We found no counterexample to the conjecture in the literature, and no proof of the lens-space statement for a group other than $\text{SU}(2)$. A preprint of Kuriya, which we could not obtain, is reported to treat a version of the conjecture for the perturbative invariants of lens spaces; see the paragraph “Scope and priority” at the end.

Organization. Section 2 fixes the conventions and recalls the surgery formula and the formula of Hansen and Takata. Section 3 contains the exact counterexamples. Section 4 analyses the Gauss sums. Section 5 treats the groups for which the statement fails, Section 6 those for which it holds, and Section 7 the passage from lens spaces to all closed oriented 3-manifolds. Section 8 contains remarks and Section 9 describes the computations.

2 Setting

2.1 Lie algebra data

Let $\mathfrak{g}$ be a simple complex Lie algebra of rank l and let G be the simply connected compact Lie group with complexified Lie algebra $\mathfrak{g}$. We use the conventions of [HT04, Sections 2 and 4]. On the real span of the roots, $\langle \cdot, \cdot \rangle$ is the invariant inner product for which long roots have square length 2, and $|x|^2 = \langle x, x \rangle$. The lacing number $m \in \{1, 2, 3\}$ is the ratio of the square lengths of long and short roots. We write Δ_+ for the set of positive roots, W for the Weyl group, $\det(w) = \pm 1$ for the determinant of $w \in W$, w_0 for the longest element, ρ for the half sum of the positive roots and θ for the highest root; $h^\vee = \langle \rho, \theta \rangle + 1$. Let X be the weight lattice and L the coroot lattice, spanned by the coroots $\alpha^\vee = 2\alpha/\langle \alpha, \alpha \rangle$ (it is denoted $Y^\vee$ in [HT04]). Then $L \subseteq X$, X is the dual lattice of L , and $|\nu|^2 \in 2\mathbb{Z}$ for $\nu \in L$; we put $Q(\nu) = |\nu|^2/2$. We write $\mathbf{e}(x) = \exp(2\pi i x)$ and, for an integer $\kappa \geq h^\vee$,

$$\Pi(\kappa) = \prod_{\alpha \in \Delta_+} 2 \sin \frac{\pi \langle \rho, \alpha \rangle}{\kappa} > 0. \quad (1)$$

It is positive because $0 < \langle \rho, \alpha \rangle \leq h^\vee - 1 < \kappa$.

2.2 The invariant

Fix an integer $\kappa \geq h^\vee$ and put $k = \kappa - h^\vee$. Let $\mathcal{C}(\mathfrak{g}, \kappa)$ be the modular category of [BK01, Theorem 3.3.20], built from the quantum group of $\mathfrak{g}$ at $q = \exp(\pi i/(m\kappa))$. (For $\mathfrak{g} = \mathfrak{sl}_N$ the quantum integers are $[n] = \sin(\pi n/\kappa)/\sin(\pi/\kappa)$; in the notation of [KM91, Oht02] the parameter is $\exp(2\pi i/\kappa)$.) Its simple objects are labelled by the set I_κ of all $\lambda \in X$ with $\langle \lambda, \alpha^\vee \rangle > 0$ for the simple roots α and $\langle \lambda, \theta \rangle < \kappa$. (The weights $\lambda - \rho$, $\lambda \in I_\kappa$, are the integrable highest weights of level k ; the unit object has label ρ .) For a closed oriented 3-manifold M we denote by $\tau_\kappa(M) = \tau_\kappa^G(M)$ the Reshetikhin–Turaev invariant of this category [RT91, Tur10], normalized by $\tau_\kappa(S^3) = 1$. In the notation of [Tur10, Section II.2], where $\tau_{(\mathcal{V}, \mathcal{D})}(S^3) = \mathcal{D}^{-1}$, it is $\mathcal{D} \tau_{(\mathcal{V}, \mathcal{D})}(M)$ with $\mathcal{D} > 0$ the rank of [BK01, (3.3.9)].

This is the quantum G invariant $\tau_r^G(M)$ of [Oht02, Section 7.1] with $r = \kappa$; for $G = \mathrm{SU}(2)$ it is the invariant τ_r of Kirby and Melvin [KM91]. We stress three points. First, all labels of I_κ are used; the projective versions (such as the $\mathrm{SO}(3)$ invariant) are different invariants and are not treated in this note (Remark 8.4). Second, in [HT04] the invariant is written $\tau_r^\mathfrak{g}$ with $r = m\kappa$, so for the algebras that are not simply laced the r of [HT04] is m times the $r = k + h^\vee$ used here and in [Oht02]. Third, [HT04] use the normalization of [Tur10], in which the value on $S^1 \times S^2$ is 1. From now on we write κ instead of r .

The modular matrices of $\mathcal{C}(\mathfrak{g}, \kappa)$ are, for $\lambda, \mu \in I_\kappa$,

$$S_{\lambda\mu} = \frac{i^{|\Delta_+|}}{c} S'_{\lambda\mu}, \quad S'_{\lambda\mu} = \sum_{w \in W} \det(w) \mathbf{e}\left(-\frac{\langle w\lambda, \mu \rangle}{\kappa}\right), \quad T'_\lambda = \mathbf{e}\left(\frac{|\lambda|^2}{2\kappa}\right), \quad (2)$$

with $c = \kappa^{l/2} |X/L|^{1/2}$; these are the formulas of Kac and Peterson [KP84], see [BK01, Theorem 3.3.20] and [HT04, Section 2]. The matrix S is symmetric and unitary, and $S^2 = C$ is the permutation matrix

of the involution $\lambda \mapsto \lambda^* = -w_0\lambda$ of I_κ , which fixes ρ . The diagonal matrix T' is the T -matrix up to a scalar factor of modulus 1. Replacing w by w_0w in (2) and using $w_0\mu = -\mu^*$ gives $\overline{S'_{\lambda\mu}} = \det(w_0)S'_{\lambda\mu^*}$, hence

$$\overline{S} = SC = CS. \quad (3)$$

By Weyl's denominator formula, $|S'_{\rho\rho}| = \Pi(\kappa)$ and $S_{\rho\rho} = \Pi(\kappa)/c$.

2.3 Lens spaces and the surgery formula

For coprime integers $p \geq 1$ and q we denote by $L(p, q)$ the oriented 3-manifold obtained from S^3 by surgery on the unknot with coefficient $-p/q$. This is the convention of [HT04] and [GS99]. Its fundamental group is $\mathbb{Z}/p$. By the classification of lens spaces [Rei35, Bro60], $L(p, q)$ and $L(p, q')$ are homeomorphic by an orientation preserving homeomorphism if and only if $q' \equiv q^{\pm 1} \pmod{p}$; moreover $-L(p, q) = L(p, -q)$. They are homotopy equivalent if and only if $qq' \equiv \pm n^2 \pmod{p}$ for some n [Whi41]. The absolute values studied here are not changed by reversal of orientation (Proposition 2.1), so only the class $\{\pm q, \pm q^{-1}\}$ of q matters. For example, the lens spaces with group $\mathbb{Z}/5$ are $L(5, \pm 1)$ and $L(5, \pm 2)$, those with group $\mathbb{Z}/8$ are $L(8, \pm 1)$ and $L(8, \pm 3)$, and those with group $\mathbb{Z}/10$ are $L(10, \pm 1)$ and $L(10, \pm 3)$.

For integers $a_1, \dots, a_n$ we write

$$[a_1, \dots, a_n] = a_1 - \frac{1}{a_2 - \frac{1}{\ddots - \frac{1}{a_n}}} \in \mathbb{Q} \cup \{\infty\},$$

with the conventions $1/0 = \infty$ and $1/\infty = 0$.

Proposition 2.1 (Surgery formula for lens spaces). *Let $p \geq 1$ and q be coprime and let $a_1, \dots, a_n$ be integers with $[a_1, \dots, a_n] = p/q$ or $[a_1, \dots, a_n] = -p/q$. Put $R'(a_1, \dots, a_n) = (S'T'^{a_1}S'T'^{a_2} \dots T'^{a_n}S')_{\rho\rho}$. Then*

$$|\tau_\kappa(L(p, q))| = \frac{|(ST'^{a_1}S'T'^{a_2} \dots T'^{a_n}S)_{\rho\rho}|}{S_{\rho\rho}} = \frac{|R'(a_1, \dots, a_n)|}{c^n \Pi(\kappa)}. \quad (4)$$

In particular $|\tau_\kappa(L(p, -q))| = |\tau_\kappa(L(p, q))|$.

Proof. For $[a_1, \dots, a_n] = -p/q$ this is [HT04, Corollary 4.2], a formula of Hansen [Han01] that holds for every modular category. There the S -matrix is $\mathcal{D}S$, the T -matrix is T' up to a scalar of modulus 1, the invariant of S^3 is $\mathcal{D}^{-1} = S_{\rho\rho}$, and the formula carries a further factor of modulus 1. The tuple may be read in either order: S and T' are symmetric, and $[a_n, \dots, a_1] = \pm p/\tilde{q}$ with $q\tilde{q} \equiv 1 \pmod{p}$, which gives the same lens space. Now let $[a_1, \dots, a_n] = p/q$. Then $[-a_1, \dots, -a_n] = -p/q$. By (3) the complex conjugate of $(ST'^{a_1}S \dots T'^{a_n}S)_{\rho\rho}$ is $(ST'^{-a_1}S \dots T'^{-a_n}S C^{n+1})_{\rho\rho} = (ST'^{-a_1}S \dots T'^{-a_n}S)_{\rho\rho}$, because C commutes with S and T' and fixes the basis vector of ρ . So both cases give the same modulus. The last equality in (4) follows from (2) and $S_{\rho\rho} = \Pi(\kappa)/c$. $\square$

Remark 2.2. The right-hand side of (4) is the modulus of the Reshetikhin–Turaev surgery formula [Tur10, (II.2.2.a)] for the linear chain of n unknots with framings $a_1, \dots, a_n$, in which consecutive components form Hopf links: the invariant of the coloured chain is a product of entries of S and T' , and the sum over the colours is the matrix product. The last component of the chain is a meridian of the previous one, so $n-1$ slam-dunk moves [GS99, Section 5.3] show that surgery on the chain gives the same oriented manifold as surgery on the unknot with coefficient $[a_1, \dots, a_n]$; this is $L(p, -q)$ if $[a_1, \dots, a_n] = p/q$. The proof of Proposition 2.1 also shows that the modulus (4) does not depend on sign conventions for S and T' : it is unchanged if (S, T') is replaced by $(\overline{S}, \overline{T'})$ or if S is replaced by $\overline{S}$.

2.4 The formula of Hansen and Takata

For coprime integers $p \geq 1$ and q , and for $y \in X$, put

$$G_q(y) = \sum_{\nu \in L/pL} \mathbf{e}\left(\frac{q\kappa Q(\nu) + \langle \nu, y \rangle}{p}\right), \quad A_q = \sum_{w \in W} \det(w) \mathbf{e}\left(-\frac{\langle \rho, w\rho \rangle}{p\kappa}\right) G_q(q\rho - w\rho). \quad (5)$$

The summand of $G_q(y)$ depends only on the class of ν modulo pL , $G_q(y)$ depends only on y modulo pX and on q modulo p , and so does A_q .

Theorem 2.3 (Hansen–Takata [HT04, Theorem 5.1, first formula]). *For every simple $\mathfrak{g}$, every $\kappa \geq h^\vee$ and all coprime $p \geq 1$ and q ,*

$$|\tau_\kappa(L(p, q))| = \frac{|A_q|}{p^{l/2} \Pi(\kappa)}. \quad (6)$$

In [HT04] the theorem reads $\tau_r^{\mathfrak{g}}(L(p, q)) = \varepsilon(\kappa p)^{-l/2} \text{vol}(L)^{-1} A_q$ with an explicit ε of modulus 1, in the normalization with value 1 on $S^1 \times S^2$. Multiplying by the rank $\mathcal{D} = \kappa^{l/2} \text{vol}(L)/\Pi(\kappa)$ [HT04, Section 4] gives (6). The theorem is proved there for every complex simple Lie algebra by a reciprocity formula for Gauss sums; according to [HT04], it generalizes [Jef92, Theorem 3.4], which is the case $\mathfrak{g} = \mathfrak{sl}_2$.

Everything in Sections 4 to 6 is derived from (6). Theorem 1.1 is not. As a test of the transcription we compared (6) with (4): exactly in eight cases (the six lens spaces of Theorem 1.1 and two cases for $\text{SU}(2)$), and in floating point for many triples (p, q, κ) , including algebras that are not simply laced (Section 9).

3 Exact counterexamples

3.1 The computation

Let D be the least positive integer with $D\langle \lambda, \mu \rangle \in \mathbb{Z}$ for all $\lambda, \mu \in X$, put $M = 2D\kappa$ and $\zeta = \mathbf{e}(1/M)$. Then $T'_\lambda = \zeta^{D|\lambda|^2}$ and $S'_{\lambda\mu} = \sum_w \det(w) \zeta^{-2D\langle w\lambda, \mu \rangle}$, so the number $R'(a)$ of (4) lies in $\mathbb{Z}[\zeta]$. An element of $\mathbb{Z}[\zeta]$ is represented by an integer vector, that is, by a polynomial in x modulo $x^M - 1$; it is zero if and only if this polynomial is divisible by the cyclotomic polynomial Φ_M ; and complex conjugation is $x \mapsto x^{-1}$. So identities between such numbers are decided by integer arithmetic. We use $\sqrt{3} = \zeta^{M/12} + \zeta^{-M/12}$ when $12 \mid M$ and $\sqrt{5} = \zeta^{M/5} + \zeta^{-M/5} - \zeta^{2M/5} - \zeta^{-2M/5}$ when $5 \mid M$.

Theorem 3.1. *The following identities hold in $\mathbb{Z}[\zeta]$, where $\Pi^2 = |S'_{\rho\rho}|^2 = \Pi(\kappa)^2$ and $c^2 = \kappa^l |X/L|$.*

- (a) *Type A_3 , $\kappa = 12$ (165 labels, $M = 96$, $c^2 = 6912$): $|R'(8)|^2 = (448 + 256\sqrt{3}) c^2 \Pi^2$ and $|R'(3, 3)|^2 = (6208 + 3584\sqrt{3}) c^4 \Pi^2$.*
- (b) *Type A_4 , $\kappa = 10$ (126 labels, $M = 100$, $c^2 = 50000$): $|R'(5)|^2 = (3475 + 1550\sqrt{5}) c^2 \Pi^2$ and $|R'(3, 2)|^2 = (4025 + 1800\sqrt{5}) c^4 \Pi^2$.*
- (c) *Type B_3 , $\kappa = 15$ (161 labels, $M = 120$, $c^2 = 13500$): $|R'(10)|^2 = 250(2 + \sqrt{3}) c^2$ and $|R'(3, -3)|^2 = 250(2 - \sqrt{3}) c^4$.*

Proof. This is a finite computation with integers. It was carried out by the author's programs and, independently, by programs of the first and of the third verification run, which were written separately and describe the root systems in other ways (Section 9). $\square$

Proof of Theorem 1.1. We have $[8] = 8$, $[3, 3] = 8/3$, $[5] = 5$, $[3, 2] = 5/2$, $[10] = 10$ and $[3, -3] = 10/3$. By Proposition 2.1, $|\tau_\kappa(L(p, q))|^2 = |R'(a)|^2/(c^{2n}\Pi^2)$ for the corresponding chain a of length n . So Theorem 3.1 gives $|\tau(L(8, 1))|^2 = 448 + 256\sqrt{3} = (16 + 8\sqrt{3})^2$, $|\tau(L(8, 3))|^2 = 6208 + 3584\sqrt{3} = (56 + 32\sqrt{3})^2$, $|\tau(L(5, 1))|^2 = 3475 + 1550\sqrt{5}$, $|\tau(L(5, 2))|^2 = 4025 + 1800\sqrt{5} = (45 + 20\sqrt{5})^2$, and, for $\text{Spin}(7)$, $|\tau(L(10, 1))|^2 = 250(2 + \sqrt{3})/\Pi^2$ and $|\tau(L(10, 3))|^2 = 250(2 - \sqrt{3})/\Pi^2$. Since $\cos^2(\pi/12) = (2 + \sqrt{3})/4$ and $\sin^2(\pi/12) = (2 - \sqrt{3})/4$, these are the values in (c), and their ratio is $\tan(\pi/12) = 2 - \sqrt{3}$. All six numbers are nonzero and the two numbers of each pair are different. The lens spaces of each pair have the same p , hence isomorphic fundamental groups. $\square$

The programs of the first and of the third verification run (Section 9) also confirm, by exact computation in the three cases, that $S'^2 = (-1)^{|\Delta|}c^2C$, that $(S'T')^3$ is a scalar multiple of C , that the chains (1), (-1) and $(2, 1)$ for S^3 give $|\tau| = 1$, and that other chains for the same lens spaces give exactly the same values of $|\tau|^2$: for instance (-8) , $(9, 1)$, $(7, -1)$ and $(2, 2, 2, 2, 2, 2, 2)$ for $L(8, 1)$, and $(-3, -3)$, $(2, 3, 2)$, $(3, 4, 1)$, $(3, 2, -1)$ and $(4, 1, 4)$ for $L(8, 3)$. The author's programs and those of these two runs confirm the identity $p^l|R'(a)|^2 = c^{2n}|A_q|^2$ in $\mathbb{Z}[\mathbf{e}(1/(pM))]$ in all six cases, that is, the equality of (4) and (6).

Remark 3.2. The two spaces of each pair are not even homotopy equivalent, since 3 is not $\pm$ a square modulo 8 or modulo 10, and 2 is not $\pm$ a square modulo 5. This can also be seen without continued fractions. If A is the linking matrix of a framed link with $\det A \neq 0$, then the meridians generate $H_1 = \mathbb{Z}^n/A\mathbb{Z}^n$ of the surgered manifold, and the linking form of two meridians is $\pm(A^{-1})_{ij}$ modulo 1. For the chains (3, 3), (3, 2) and (3, -3) the first meridian generates H_1 and has self-linking $\pm 3/8$, $\pm 2/5$ and $\pm 3/10$. For $L(p, \pm 1)$ the self-linking of a generator is $\pm n^2/p$ with n prime to p .

3.2 Further examples from the surgery formula

Table 1 lists further pairs of lens spaces with the same p and different nonzero moduli. The column “surgery” refers to (4), evaluated in floating point (double precision) by programs that do not use (6). In the rows marked F the computation was also carried out in a finite field, which is an exact computation (Lemma 3.3): it proves that the two moduli are different and nonzero. The last column refers to the exact evaluation of (6) by the results of Section 5. For each row the values obtained from (4) and from (6) agree to all digits shown.

G	r (level)	labels	lens spaces	moduli (surgery)		via (6)
SU(6)	18 (12)	6188	$L(12, 1), L(12, 5)$	49603.5061, 126434.7266	F	Theorem 5.5
SU(7)	14 (7)	1716	$L(7, 1), L(7, 2)$	7152.0948, 8696.4173	F	Theorem 5.1
SU(8)	16 (8)	6435	$L(8, 1), L(8, 3)$	23840.5373, 26935.3720	F	Theorem 5.5
SU(10)	15 (5)	2002	$L(5, 1), L(5, 2)$	2473.6742, 2780.4232		Theorem 5.1
Spin(9)	16 (9)	245	$L(8, 1), L(8, 3)$	3290.0911, 4652.8915		Proposition 5.7
Spin(10)	16 (8)	483	$L(8, 1), L(8, 3)$	3857.1561, 4522.8506		Proposition 5.8
Spin(11)	20 (11)	882	$L(10, 1), L(10, 3)$	228981.65, 153000.65		Proposition 5.7
Spin(12)	16 (6)	259	$L(8, 1), L(8, 3)$	585.2340, 827.6458		Corollary 4.3
Spin(14)	16 (4)	85	$L(8, 1), L(8, 3)$	52.5483, 74.3145		Corollary 4.3
E_6	18 (6)	139	$L(9, 1), L(9, 2)$	501.3784, 765.8681		Proposition 5.9
E_7	28 (10)	483	$L(14, 1), L(14, 3)$	463232.50, 191877.19		Proposition 5.9
E_8	35 (5)	15	$L(5, 1), L(5, 2)$	24.5325, 35.4505	F	Proposition 5.9

Table 1: Further examples. The moduli $|\tau_r^G|$ are computed from the surgery formula (4) in floating point; F marks the rows with an additional exact computation in a finite field.

Lemma 3.3 (Certificates in a finite field). *Let P be a prime with $P \equiv 1 \pmod{M}$ and let $\omega \in \mathbb{F}_P$ have multiplicative order M . There are ring homomorphisms $\varphi_{\pm}: \mathbb{Z}[\zeta] \rightarrow \mathbb{F}_P$ with $\varphi_{\pm}(\zeta) = \omega^{\pm 1}$, and $\varphi_+(\bar{x}) = \varphi_-(x)$ for all x . Hence $\varphi_+(|x|^2) = \varphi_+(x)\varphi_-(x)$. If a, b are integers and $a\varphi_+(x)\varphi_-(x) \neq b\varphi_+(y)\varphi_-(y)$ in $\mathbb{F}_P$, then $a|x|^2 \neq b|y|^2$; and if $\varphi_+(x)\varphi_-(x) \neq 0$, then $x \neq 0$.*

Proof. An element of order M of the field $\mathbb{F}_P$ is a root of Φ_M , because $x^M - 1 = \prod_{d|M} \Phi_d$ and a root of Φ_d with $d < M$ has order less than M . So $\varphi_{\pm}$ are well defined. The other statements are clear. $\square$

The computation of $R'(a)$ can therefore be repeated with all roots of unity replaced by powers of ω , once for ω and once for ω^{-1} . For the rows marked F this was done with primes P near 2^{28} ; the residues of $c^{2(n'-n)}|R'(a)|^2$ and $|R'(a')|^2$ for the two chains, of lengths $n \leq n'$, are different and nonzero. For E_8 the matrix S' is computed from 8×8 determinants whose entries are roots of unity of order $4\kappa = 2M$, with a factor $\frac{1}{2}$. So Lemma 3.3 is applied with $2M$ in place of M (note that $\mathbb{Z}[\zeta] \subseteq \mathbb{Z}[\mathbf{e}(1/(2M))]$), and the factor is harmless since P is odd. The third verification run repeated the computations for the four rows marked F with its own programs, with two primes below 2^{25} in each case, and with the same conclusion. So for $SU(6)$, $SU(7)$, $SU(8)$ and E_8 the failure of the lens-space statement does not depend on Theorem 2.3 either.

Remark 3.4. For $SU(3)$ the matrices X, Y printed in [GP98, Section 5] agree with (2) for κ equal to their coupling constant k , up to complex conjugation and the scalar factor of T' , and their expression for the lens space invariant is (4). We checked numerically that they give the same moduli for all $L(p, q)$ with $p \leq 15$ and $3 \leq \kappa \leq 24$. So the invariant studied here is the invariant of [GP98], with $r = \kappa$ equal to their k .

4 The lattice Gauss sums

In this section $p \geq 1$ and $\kappa \geq 1$ are integers,

$$g = \gcd(p, \kappa), \quad p = gp', \quad \kappa = g\kappa',$$

and q is an integer prime to p with inverse $\bar{q}$ modulo p . Put

$$\tilde{R} = \{\mu \in L : \kappa\mu \in pX\}, \quad R = \tilde{R}/pL, \quad \Lambda_0 = \kappa L + pX.$$

The statements of this section and of Sections 5 and 6 about the sums G_q, A_q and about sums over cosets in W are elementary. Every statement about $\tau_{\kappa}(L(p, q))$ is obtained from them by Theorem 2.3, and depends on that theorem; these statements carry the tag “uses Theorem 2.3”.

Lemma 4.1. (i) $\Lambda_0 = g(L + p'X)$, and $\Lambda_0 = \{y \in X : \langle \mu, y \rangle \in p\mathbb{Z} \text{ for all } \mu \in \tilde{R}\}$.

(ii) The map $s: \tilde{R} \rightarrow \{\pm 1\}$, $s(\mu) = \mathbf{e}(q\kappa Q(\mu)/p)$, is a homomorphism that is trivial on pL . It does not depend on q , and it is trivial if p is odd. There is $y_0 \in X$, unique modulo Λ_0 , with $\mathbf{e}(\langle \mu, y_0 \rangle/p) = s(\mu)$ for all $\mu \in \tilde{R}$; we take $y_0 = 0$ if s is trivial. Moreover $2y_0 \in \Lambda_0$.

(iii) $G_q(y) \neq 0$ if and only if $y \in y_0 + \Lambda_0$, and then $|G_q(y)| = p^{l/2}|R|^{1/2}$.

(iv) $G_q(y + q\kappa\eta + p\xi) = \mathbf{e}(-(q\kappa Q(\eta) + \langle \eta, y \rangle)/p) G_q(y)$ for all $\eta \in L$ and $\xi \in X$.

Proof. (i) Since $\gcd(\kappa', p') = 1$ and $L \subseteq X$, the lattice $\kappa'L + p'X$ contains $\kappa'L + p'L = L$, so it equals $L + p'X$. The pairing $L/pL \times X/pX \rightarrow \mathbb{Z}/p$ induced by $\langle \cdot, \cdot \rangle$ is perfect. The map $\phi: L/pL \rightarrow X/pX$, $\nu \mapsto \kappa\nu$, has kernel R and image Λ_0/pX , and $\langle \nu, \phi(\nu') \rangle = \langle \nu', \phi(\nu) \rangle$. So the image of ϕ annihilates the kernel of ϕ ; it is the full annihilator because both have $p^l/|R|$ elements.

(ii) For $\mu, \mu' \in \tilde{R}$ we have $q\kappa(Q(\mu + \mu') - Q(\mu) - Q(\mu')) = q\langle \kappa\mu, \mu' \rangle \in p\mathbb{Z}$, so s is a homomorphism, and $\kappa Q(p\nu)/p = p\kappa Q(\nu) \in \mathbb{Z}$. Write $2\kappa Q(\mu) = \langle \kappa\mu, \mu \rangle = pn_\mu$ with $n_\mu \in \mathbb{Z}$; then $s(\mu) = (-1)^{n_\mu}$. If p is odd, then pn_μ is even, so n_μ is even. If p is even, then q is odd and $s(\mu) = (-1)^{n_\mu}$. A character of the subgroup R of L/pL extends to L/pL , and every character of L/pL has the form $\nu \mapsto \mathbf{e}(\langle \nu, y \rangle/p)$ with $y \in X$. This gives y_0 ; by (i) it is unique modulo Λ_0 , and $2y_0 \in \Lambda_0$ because $s^2 = 1$.

(iii) Substituting $\nu = \nu' + \mu$ we get

$$|G_q(y)|^2 = \sum_{\mu \in L/pL} \mathbf{e}\left(\frac{q\kappa Q(\mu) + \langle \mu, y \rangle}{p}\right) \sum_{\nu' \in L/pL} \mathbf{e}\left(\frac{\langle \nu', q\kappa\mu \rangle}{p}\right).$$

The inner sum is p^l if $q\kappa\mu \in pX$, that is, if $\mu \in \tilde{R}$, and 0 otherwise. Hence $|G_q(y)|^2 = p^l \sum_{\mu \in R} \mathbf{e}(\langle \mu, y + y_0 \rangle/p)$. This is the sum of a character of R . By (i) the character is trivial if and only if $y + y_0 \in \Lambda_0$, that is, $y \in y_0 + \Lambda_0$.

(iv) We have $q\kappa Q(\nu) + \langle \nu, y + q\kappa\eta \rangle = q\kappa Q(\nu + \eta) + \langle \nu + \eta, y \rangle - q\kappa Q(\eta) - \langle \eta, y \rangle$, and $\langle \nu, p\xi \rangle \in p\mathbb{Z}$. Now replace $\nu + \eta$ by ν . $\square$

Proposition 4.2 (Structure; uses Theorem 2.3). *The set $H = \{h \in W : \rho - h\rho \in \Lambda_0\}$ is a subgroup of W , and*

$$W_q = \{w \in W : q\rho - w\rho \in y_0 + \Lambda_0\}$$

is empty or a left coset wH . For $w \in W_q$ choose $\eta_w \in L$ and $\xi_w \in X$ with $q\rho - w\rho = y_0 + q\kappa\eta_w + p\xi_w$ and put

$$\Psi_q(w) = \mathbf{e}\left(-\frac{\langle \rho, w\rho \rangle}{p\kappa} - \frac{q\kappa Q(\eta_w) + \langle \eta_w, y_0 \rangle}{p}\right). \quad (7)$$

Then $\Psi_q(w)$ does not depend on the choice of η_w and ξ_w , $A_q = G_q(y_0) \sum_{w \in W_q} \det(w) \Psi_q(w)$, and, for $\kappa \geq h^\vee$,

$$|\tau_\kappa(L(p, q))| = \frac{|R|^{1/2}}{\Pi(\kappa)} \left| \sum_{w \in W_q} \det(w) \Psi_q(w) \right|. \quad (8)$$

In particular $\tau_\kappa(L(p, q)) = 0$ if W_q is empty. If $H = \{1\}$, then $|\tau_\kappa(L(p, q))|$ is 0 or $|R|^{1/2}/\Pi(\kappa)$ for every q , so the nonzero moduli for the pair (p, κ) coincide.

Proof. The lattice Λ_0 is stable under W . So $\rho - hh'\rho = (\rho - h\rho) + h(\rho - h'\rho)$ and $\rho - h^{-1}\rho = -h^{-1}(\rho - h\rho)$ show that H is a subgroup, and $w\rho - w'\rho = w(\rho - w^{-1}w'\rho)$ shows that W_q is a coset. Since q is a unit modulo p , we have $q\kappa L + pX = \Lambda_0$; so η_w and ξ_w exist. By Lemma 4.1(iv) with $y = y_0$,

$$\Psi_q(w) = \mathbf{e}\left(-\frac{\langle \rho, w\rho \rangle}{p\kappa}\right) \frac{G_q(q\rho - w\rho)}{G_q(y_0)},$$

where $G_q(y_0) \neq 0$ by Lemma 4.1(iii). This shows that $\Psi_q(w)$ is independent of the choices. For $w \notin W_q$ we have $G_q(q\rho - w\rho) = 0$. So $A_q = G_q(y_0) \sum_{w \in W_q} \det(w) \Psi_q(w)$, and (8) follows from (6) and $|G_q(y_0)| = p^{l/2}|R|^{1/2}$. $\square$

Corollary 4.3 (The case $p \mid \kappa$; uses Theorem 2.3). *If p divides $\kappa \geq h^\vee$, then*

$$|\tau_\kappa(L(p, q))| = \frac{p^{l/2}}{\Pi(\kappa)} |\Sigma_q|, \quad \Sigma_q = \sum_{\substack{w \in W \\ w\rho - q\rho \in pX}} \det(w) z^{\langle \rho, w\rho \rangle}, \quad z = \mathbf{e}\left(-\frac{1}{p\kappa}\right). \quad (9)$$

Proof. Now $q\kappa Q(\nu)/p \in \mathbb{Z}$, so $G_q(y) = \sum_{\nu} \mathbf{e}(\langle \nu, y \rangle/p)$ is p^l if $y \in pX$ and 0 otherwise. Insert this into (5) and (6). $\square$

Lemma 4.4 (Orientation). $A_{-q} = \det(w_0) \overline{A_q}$.

Proof. In (5) for $-q$ replace w by ww_0 and use $w_0\rho = -\rho$. The term of w becomes

$$\det(w) \det(w_0) \mathbf{e}\left(\frac{\langle \rho, w\rho \rangle}{p\kappa}\right) \sum_{\nu \in L/pL} \mathbf{e}\left(-\frac{q\kappa Q(\nu) + \langle \nu, q\rho - w\rho \rangle}{p}\right),$$

which is $\det(w_0)$ times the complex conjugate of the term of w in A_q . $\square$

The next proposition removes the dependence of the summands on q under a hypothesis on the lattices.

Proposition 4.5 (uses Theorem 2.3). *Assume that s is trivial and that*

(K) *there is an integer k with $\kappa'k \equiv 1 \pmod{p'}$ and $k(L + p'X) \subseteq L$.*

Put $c_0 = (\kappa'^2 k^2 - 1)/(g^2 p' \kappa')$. Then for every unit q and all $w, w' \in W_q$ we have $\Psi_q(w)/\Psi_q(w') = \mathbf{e}(c_0 \langle \rho, w\rho - w'\rho \rangle)$. Hence, for $\kappa \geq h^\vee$,

$$|\tau_\kappa(L(p, q))| = \frac{|R|^{1/2}}{\Pi(\kappa)} \left| \sum_{w \in W_q} \det(w) \mathbf{e}(c_0 \langle \rho, w\rho \rangle) \right|, \quad (10)$$

where c_0 does not depend on q .

Proof. For $w \in W_q$ put $\beta_w = (q\rho - w\rho)/g \in L + p'X$ and $\eta_w = k\bar{q}\beta_w$, which lies in L by (K). Since $q\bar{q}\kappa'k \equiv 1 \pmod{p'}$, we get $q\kappa\eta_w - g\beta_w \in gp'X = pX$, so this is an admissible choice in Proposition 4.2. Put $N_w = |k\beta_w|^2/2$, an integer because $k\beta_w \in L$. Then $q\kappa Q(\eta_w) = q\bar{q} \cdot \bar{q}\kappa N_w \equiv \bar{q}\kappa N_w \pmod{p}$. For $w, w' \in W_q$ put $\delta = (w\rho - w'\rho)/g \in L + p'X$. From $g^2|\beta_w|^2 = (q^2 + 1)|\rho|^2 - 2q\langle \rho, w\rho \rangle$ we get $N_w - N_{w'} = -qk^2\langle \rho, \delta \rangle/g$. The number $u = \kappa'k^2\langle \rho, \delta \rangle = \kappa'k\langle \rho, k\delta \rangle$ is an integer, and $\bar{q}\kappa(N_w - N_{w'})/p = -q\bar{q}u/p$. Therefore

$$\frac{\Psi_q(w)}{\Psi_q(w')} = \mathbf{e}\left(-\frac{g\langle \rho, \delta \rangle}{p\kappa} + \frac{u}{p}\right) = \mathbf{e}\left(\langle \rho, \delta \rangle \frac{\kappa'^2 k^2 - 1}{gp'\kappa'}\right) = \mathbf{e}(c_0 \langle \rho, w\rho - w'\rho \rangle).$$

Now (10) follows from (8). $\square$

Proposition 4.6 (Non-degenerate case; uses Theorem 2.3). *Let ϵ be the exponent of the group X/L and let δ be the least positive integer with $\delta\langle \rho, w\rho \rangle \in \mathbb{Z}$ for all $w \in W$. Assume $\gcd(p', g\epsilon\delta) = 1$, and let p^* be an integer with $p^*p' \equiv 1 \pmod{g\kappa\delta}$. Then, for $\kappa \geq h^\vee$,*

$$|\tau_\kappa(L(p, q))| = \frac{g^{1/2}}{\Pi(\kappa)} \left| \sum_{\substack{w \in W \\ w\rho - q\rho \in gX}} \det(w) \mathbf{e}\left(-\frac{p^*\langle \rho, w\rho \rangle}{g\kappa}\right) \right|. \quad (11)$$

Proof. Since $\epsilon X \subseteq L$ and $\gcd(p', \epsilon) = 1$, we have $L + p'X = X$ and $\Lambda_0 = gX$. Let $\mu \in \tilde{R}$. Then $\kappa'\mu \in p'X$, and since $\gcd(\kappa', p') = 1$, $\mu = p'\xi$ with $\xi \in X$. The order of ξ in X/L divides p' and ϵ , so $\xi \in L$. Hence $\tilde{R} = p'L$, $|R| = g^l$, and $s(p'\nu) = \mathbf{e}(q\kappa'p'Q(\nu)) = 1$. Choose k with $\kappa'k \equiv 1 \pmod{p'}$ and $k \equiv 0 \pmod{g^2\epsilon\delta}$. Then $k(L + p'X) = kX \subseteq L$, so (K) holds, and Proposition 4.5 applies with $W_q = \{w : w\rho - q\rho \in gX\}$ and $c_0 = n(\kappa'k + 1)/(g^2\kappa')$, where $n = (\kappa'k - 1)/p' \in \mathbb{Z}$. Modulo $g^2\kappa'\delta = g\kappa\delta$ we have $\kappa'k \equiv 0$, hence $n(\kappa'k + 1) \equiv n$ and $np' \equiv -1$, that is, $n \equiv -p^*$. For $w, w' \in W_q$ the number $\langle \rho, w\rho - w'\rho \rangle$ lies in $\delta^{-1}\mathbb{Z}$. Therefore $\mathbf{e}(c_0 \langle \rho, w\rho - w'\rho \rangle) = \mathbf{e}(-p^* \langle \rho, w\rho - w'\rho \rangle / (g\kappa))$, and (10) becomes (11). $\square$

Corollary 4.7 (uses Theorem 2.3). *Let g divide $\kappa \geq h^\vee$, and suppose that for the pair (g, κ) two units q, q' modulo g give different nonzero moduli in (9). Then the same holds for every pair (p, κ) with $\gcd(p, \kappa) = g$ and $\gcd(p/g, g\epsilon\delta) = 1$, and all units of the classes of q, q' modulo g .*

Proof. The sum in (11) is the image of the sum Σ_q of (9) for the pair (g, κ) under the automorphism σ of the field $\mathbb{Q}(\mathbf{e}(1/(g\kappa\delta)))$ that raises roots of unity to the power p^* . As σ commutes with complex conjugation, $|\sigma(\Sigma_q)|^2 = \sigma(|\Sigma_q|^2)$. So $|\Sigma_q|^2 \neq |\Sigma_{q'}|^2$ and $\Sigma_q \neq 0 \neq \Sigma_{q'}$ are preserved. $\square$

5 Groups for which the lens-space statement fails

5.1 $\mathrm{SU}(N)$

For $\mathfrak{g} = \mathfrak{sl}_N$ we write weights as vectors of $\mathbb{Z}^N$ modulo $(1, \dots, 1)$, with $\langle x, y \rangle = \sum x_i y_i - (\sum x_i)(\sum y_i)/N$. Then L consists of the classes with $\sum x_i \equiv 0 \pmod{N}$, $\rho = (N-1, N-2, \dots, 0)$, W is the symmetric group acting on the coordinates, and $x \in pX$ if and only if all coordinates of x are congruent modulo p . The numbers $\langle \rho, \alpha \rangle$, $\alpha \in \Delta_+$, are $h = 1, \dots, N-1$, with multiplicity $N-h$. For an integer x let $[x]_p \in \{0, \dots, p-1\}$ be its residue modulo p .

Theorem 5.1 (uses Theorem 2.3). *Let $N = ap$ with integers $a \geq 1$, $p \geq 2$, let $\kappa \geq N$ be a multiple of p and q a unit modulo p . For $c \in \mathbb{Z}/p$ put*

$$s_q(c) = \sum_{j=0}^{p-1} j [qj + c]_p, \quad \Theta_q = \sum_{c \in \mathbb{Z}/p} (-1)^{(p-1)ac} \mathbf{e}\left(-\frac{a s_q(c)}{p\kappa}\right).$$

Then

$$|\tau_\kappa(L(p, q))| = \frac{p^{(N-1)/2} |V_a|^p |\Theta_q|}{\Pi(\kappa)}, \quad V_a = \prod_{0 \leq m < m' < a} (x^{m'} - x^m), \quad x = \mathbf{e}(-p/\kappa),$$

and $V_a \neq 0$.

Proof. By Corollary 4.3 we have to compute Σ_q . An element $w \in W$ is given by a permutation σ of the set $\{0, \dots, N-1\}$ of coordinates of ρ , with $w\rho$ having the coordinate $\sigma(r)$ where ρ has r ; then $\det(w) = \mathrm{sgn}(\sigma)$ and $\langle \rho, w\rho \rangle = \sum_r r\sigma(r) - N(N-1)/4$. The condition $w\rho - q\rho \in pX$ means that $\sigma(r) \equiv qr + c \pmod{p}$ for all r , with some $c \in \mathbb{Z}/p$ that is determined by σ . Write $r = j + pm$ with $0 \leq j < p$ and $0 \leq m < a$. For fixed c the permutations in question are $\sigma(j + pm) = [qj + c]_p + p t_j(m)$ with arbitrary permutations $t_0, \dots, t_{p-1}$ of $\{0, \dots, a-1\}$. Using $\sum_m t_j(m) = \sum_m m = a(a-1)/2$ we find

$$\sum_r r\sigma(r) = a s_q(c) + p^2(p-1) \frac{a(a-1)}{2} + p^2 \sum_j \sum_m m t_j(m).$$

The permutation σ is the composition of $(j, m) \mapsto (j, t_j(m))$ and of $(j, m) \mapsto ([qj + c]_p, m)$, so $\mathrm{sgn}(\sigma) = \mathrm{sgn}(j \mapsto [qj + c]_p)^a \prod_j \mathrm{sgn}(t_j)$, and $\mathrm{sgn}(j \mapsto [qj + c]_p) = \mathrm{sgn}(j \mapsto [qj]_p) (-1)^{(p-1)c}$, because $j \mapsto [j+1]_p$ is a cycle of length p . The sum over each t_j is $\sum_t \mathrm{sgn}(t) z^{p^2 \sum_m m t(m)} = \det[x^{mm'}]_{0 \leq m, m' < a} = V_a$, a Vandermonde determinant. Hence $\Sigma_q = \pm z^{\mathrm{const}} V_a^p \Theta_q$. Finally $x^d \neq 1$ for $0 < d < a$, because $0 < pd < N \leq \kappa$. $\square$

Proposition 5.2 ($\mathrm{SU}(5)$; uses Theorem 2.3). *Let $N = 5$ and let $\kappa \geq 10$ be a multiple of 5. Then, with $\Pi(\kappa) = \prod_{h=1}^4 (2 \sin(\pi h/\kappa))^{5-h}$,*

$$|\tau_\kappa(L(5, \pm 1))| = \frac{25}{\Pi(\kappa)} \left(9 + 8 \cos \frac{2\pi}{\kappa} + 4 \cos \frac{4\pi}{\kappa} + 4 \cos \frac{6\pi}{\kappa} \right)^{1/2},$$

$$|\tau_\kappa(L(5, \pm 2))| = \frac{25}{\Pi(\kappa)} \left(1 + 4 \cos \frac{2\pi}{\kappa} \right),$$

both are positive, and

$$|\tau_\kappa(L(5, 1))|^2 - |\tau_\kappa(L(5, 2))|^2 = -\frac{5000}{\Pi(\kappa)^2} \sin \frac{5\pi}{\kappa} \sin \frac{\pi}{\kappa} < 0.$$

For $\kappa = 10$ these are the values of Theorem 1.1(b).

Proof. We apply Theorem 5.1 with $p = 5$, $a = 1$; then $V_1 = 1$ and $\Theta_q = \sum_c z^{s_q(c)}$ with $z = \mathbf{e}(-1/(5\kappa))$. For $c = 0, \dots, 4$,

$$s_1(c) = 30, 20, 15, 15, 20, \quad s_2(c) = 25, 25, 15, 20, 15.$$

For instance $s_2(1) = 0 \cdot 1 + 1 \cdot 3 + 2 \cdot 0 + 3 \cdot 2 + 4 \cdot 4 = 25$. So, with $u = z^5 = \mathbf{e}(-1/\kappa)$,

$$|\Theta_1| = |u^3 + 2u + 2|, \quad |\Theta_2| = |2u^2 + u + 2| = |2u + 1 + 2u^{-1}| = 1 + 4 \cos \frac{2\pi}{\kappa}.$$

Expanding gives $|\Theta_1|^2 = 9 + 8 \cos \frac{2\pi}{\kappa} + 4 \cos \frac{4\pi}{\kappa} + 4 \cos \frac{6\pi}{\kappa}$ and $|\Theta_2|^2 = 9 + 8 \cos \frac{2\pi}{\kappa} + 8 \cos \frac{4\pi}{\kappa}$. Their difference is $4(\cos \frac{6\pi}{\kappa} - \cos \frac{4\pi}{\kappa}) = -8 \sin \frac{5\pi}{\kappa} \sin \frac{\pi}{\kappa}$. The classes $q = -1, -2$ follow from Lemma 4.4. For $\kappa \geq 10$ we have $\cos \frac{2\pi}{\kappa} > 0$, $\cos \frac{4\pi}{\kappa} > 0$ and $0 < \frac{5\pi}{\kappa} \leq \frac{\pi}{2}$. So $|\Theta_2| > 1$, $|\Theta_1|^2 \geq 9 - 4 > 0$, and the difference is negative. For $\kappa = 10$: $\Pi(10) = 5/(2+\sqrt{5})$, $|\Theta_1|^2 = 11+2\sqrt{5}$ and $|\Theta_2| = 2+\sqrt{5}$, which gives $|\tau(L(5, 2))| = 5(2+\sqrt{5})^2 = 45+20\sqrt{5}$ and $|\tau(L(5, 1))|^2 = 25(11+2\sqrt{5})(2+\sqrt{5})^2 = 3475+1550\sqrt{5}$. $\square$

So $\text{SU}(5)$ violates the lens-space statement at every positive level divisible by 5. By Corollary 4.7 (here $\epsilon = 5$, $\delta = 1$), it also does so for every pair (p, κ) with $\gcd(p, \kappa) = 5$, $\kappa \geq 10$ and $p/5$ prime to 5.

Corollary 5.3 (uses Theorem 2.3). *Let $N = ap$ with $(p-1)a$ even and $p \notin \{1, 2, 3, 4, 6\}$, and let q be a unit modulo p with $q \not\equiv \pm 1 \pmod{p}$. Then for every multiple κ of p with $\kappa \geq Np^5$,*

$$0 < |\tau_\kappa(L(p, 1))| < |\tau_\kappa(L(p, q))|.$$

Proof. Now $\Theta_q = \sum_c \mathbf{e}(-a s_q(c)/(p\kappa))$. Put $\psi = 2\pi a/(p\kappa)$ and let $\text{Var}(s_q)$ be the variance of the numbers $s_q(c)$, $c \in \mathbb{Z}/p$. Then $|\Theta_q|^2 = \sum_{c, c'} \cos(\psi(s_q(c) - s_q(c')))$ and $\sum_{c, c'} (s_q(c) - s_q(c'))^2 = 2p^2 \text{Var}(s_q)$.

Step 1: $\text{Var}(s_1) - \text{Var}(s_q) \geq 1/p$. Let $f(j) = j$ for $0 \leq j < p$, regarded as a function on $\mathbb{Z}/p$, with Fourier transform $\hat{f}(t) = \sum_j f(j) \mathbf{e}(-jt/p)$. Since $s_q(c) = \sum_j f(j) f(qj + c)$, we get $\sum_c s_q(c) \mathbf{e}(-ct/p) = \hat{f}(t) \hat{f}(-qt)$. For $t \neq 0$, $\hat{f}(t) = p/(\mathbf{e}(-t/p) - 1)$, so $|\hat{f}(t)|^2 = p^2 a_t/4$ with $a_t = 1/\sin^2(\pi t/p)$. By Parseval's formula

$$\text{Var}(s_q) = \frac{1}{p^2} \sum_{t \neq 0} |\hat{f}(t)|^2 |\hat{f}(qt)|^2 = \frac{p^2}{16} \sum_{t=1}^{p-1} a_t a_{qt}.$$

By the Cauchy-Schwarz inequality $\sum_t a_t a_{qt} \leq \sum_t a_t^2$, with equality only if $a_{qt} = a_t$ for all t ; for $t = 1$ this means $q \equiv \pm 1 \pmod{p}$. So $\text{Var}(s_q) < \text{Var}(s_1)$. Moreover $\sum_c s_q(c) = (p(p-1)/2)^2$ does not depend on q , so $p^2(\text{Var}(s_1) - \text{Var}(s_q)) = p \sum_c (s_1(c)^2 - s_q(c)^2)$ is a positive integer multiple of p .

Step 2. By the rearrangement inequality all $s_1(c)$ lie between $\sum_j j(p-1-j)$ and $\sum_j j^2$, so $|s_1(c) - s_1(c')| \leq D := p(p^2-1)/6$, and $\text{Var}(s_1) \leq D^2/4$. Using $1 - \frac{x^2}{2} \leq \cos x \leq 1 - \frac{x^2}{2} + \frac{x^4}{24}$ we obtain

$$|\Theta_q|^2 \geq p^2 - \psi^2 p^2 \text{Var}(s_q), \quad |\Theta_1|^2 \leq p^2 - \psi^2 p^2 \text{Var}(s_1) + \frac{\psi^4}{24} D^2 \cdot 2p^2 \text{Var}(s_1),$$

hence $|\Theta_q|^2 - |\Theta_1|^2 \geq \psi^2 p (1 - \psi^2 p D^4/48)$ and $|\Theta_1|^2 \geq p^2 (1 - \psi^2 D^2/4)$. If $\kappa \geq Np^5 = ap^6$, then $\psi \leq 2\pi/p^7$ and $D < p^3/6$, so $\psi^2 p D^4/48 < 1$ and $\psi^2 D^2/4 < 1$. Thus $|\Theta_q| > |\Theta_1| > 0$, and the claim follows from Theorem 5.1. $\square$

For $N = p = 5$, Proposition 5.2 shows that every multiple $\kappa \geq 10$ of 5 works. Numerical tests suggest that in Corollary 5.3 every multiple $\kappa > N$ of p works; we do not prove this.

Proposition 5.4 (SU(4); uses Theorem 2.3). *Let $N = 4$, $p = 8$ and $\kappa \equiv 4 \pmod{8}$, $\kappa \geq 12$. Then*

$$|\tau_\kappa(L(8, \pm 1))| = \frac{8\sqrt{2} \cdot 2 \sin(\pi/\kappa)}{\Pi(\kappa)}, \quad |\tau_\kappa(L(8, \pm 3))| = \frac{8\sqrt{2} \cdot 2 \cos(\pi/\kappa)}{\Pi(\kappa)},$$

with $\Pi(\kappa) = (2 \sin \frac{\pi}{\kappa})^3 (2 \sin \frac{2\pi}{\kappa})^2 (2 \sin \frac{3\pi}{\kappa})$. Both are nonzero and their ratio is $\cot(\pi/\kappa) \neq 1$. For $\kappa = 12$ these are the values of Theorem 1.1(a).

Proof. Write $\kappa = 4\kappa'$ with κ' odd; then $g = 4$, $p' = 2$. Let $\omega_2 = (1, 1, 0, 0)$. The lattice $L + 2X$ consists of the classes with even coordinate sum, and $\Lambda_0 = 4(L + 2X)$. Since κ' is odd, $\tilde{R} = L \cap 2X = 2(L \cup (\omega_2 + L))$, so $|R| = 2 \cdot 4^3 = 128$. For $\mu = 2\xi \in \tilde{R}$ we have $q\kappa Q(\mu)/p = q\kappa'|\xi|^2$, and $|\xi|^2 \in \mathbb{Z}$ for $\xi \in L \cup (\omega_2 + L)$; so s is trivial and $y_0 = 0$.

An element h satisfies $\rho - h\rho \in 4X$ if and only if it shifts the coordinates 3, 2, 1, 0 of ρ cyclically modulo 4; there are four such elements, and for them $\rho - h\rho$ is 0, $4\omega_1$, $4\omega_2$, $4\omega_3$, where ω_j has j coordinates 1 and $4 - j$ coordinates 0. Hence $H = \{1, h_2\}$, where h_2 exchanges the coordinates $3 \leftrightarrow 1$ and $2 \leftrightarrow 0$; $\det(h_2) = 1$ and $\rho - h_2\rho = 4\omega_2$. Let $q \in \{1, 5\}$ and $t = (q - 1)/4$. Then $(q - 1)\rho = 4t\rho \in \Lambda_0$, so $W_q = H$, and $q\rho - w\rho = 4(t\rho + \varepsilon\omega_2)$ with $\varepsilon = 0$ for $w = 1$ and $\varepsilon = 1$ for $w = h_2$. We need $\eta_w \in L$ with $q\kappa\eta_w \equiv 4(t\rho + \varepsilon\omega_2) \pmod{8X}$, that is, $\eta_w \equiv t\rho + \varepsilon\omega_2 \pmod{2X}$, because $q\kappa'$ is odd. We may take

$$\eta = 0, \quad (1, -1, 0, 0), \quad (1, 0, -1, 0), \quad (0, 1, -1, 0)$$

for $t\rho + \varepsilon\omega_2 = 0$, ω_2 , ρ , $\rho + \omega_2$, with $Q(\eta) = 0, 1, 1, 1$. By (7), $\Psi_q(w) = \mathbf{e}(-\langle \rho, w\rho \rangle / (8\kappa)) (-1)^{Q(\eta_w)}$, since $q\kappa Q(\eta)/8 = q\kappa' Q(\eta)/2$. As $\langle \rho, \rho \rangle - \langle \rho, h_2\rho \rangle = 4\langle \rho, \omega_2 \rangle = 8$, we get

$$\left| \sum_{w \in W_q} \det(w) \Psi_q(w) \right| = |1 \mp \mathbf{e}(1/\kappa)| = \begin{cases} 2 \sin(\pi/\kappa) & \text{for } q = 1, \\ 2 \cos(\pi/\kappa) & \text{for } q = 5, \end{cases}$$

and (8) with $|R|^{1/2} = 8\sqrt{2}$ gives the formulas for $q = 1, 5$. The cases $q = 7, 3$ follow from Lemma 4.4. For $\kappa = 12$: $\Pi = (2 \sin 15^\circ)^3 \sqrt{2}$ and $(2 \sin 15^\circ)^2 = 2 - \sqrt{3}$, so the two values are $8/(2 - \sqrt{3}) = 16 + 8\sqrt{3}$ and $8(2 + \sqrt{3})^2 = 56 + 32\sqrt{3}$. $\square$

Theorem 5.5 (uses Theorem 2.3). *For every $N \geq 4$ there are p, κ and units q, q' modulo p such that $\tau_\kappa^{\text{SU}(N)}(L(p, q))$ and $\tau_\kappa^{\text{SU}(N)}(L(p, q'))$ are nonzero and have different absolute values.*

Proof. Write $N = 2^s 3^t n$ with $\gcd(n, 6) = 1$. If $n > 1$ take $p = n$. If $n = 1$ and $t \geq 2$ take $p = 9$. If $n = 1$, $t \leq 1$ and $s \geq 4$ take $p = 8$; then N/p is even. If $N = 24$ take $p = 12$. In all these cases p divides N , $p \notin \{1, 2, 3, 4, 6\}$ and $(p - 1)N/p$ is even, so Corollary 5.3 applies. The remaining values are $N = 4, 6, 8, 12$.

For $N = 4$ see Proposition 5.4. For $N = 8$ and $N = 12$ we use Theorem 5.1 with $p = N$, $a = 1$ and $\kappa = 2N$, so that $\Theta_q = \sum_c (-1)^c \mathbf{e}(-s_q(c)/(2N^2))$ is a sum of N roots of unity:

$$\begin{aligned} N = 8 : \quad & |\Theta_1|^2 = 0.90368 \dots, \quad |\Theta_3|^2 = 1.15353 \dots; \\ N = 12 : \quad & |\Theta_1|^2 = 0.94786 \dots, \quad |\Theta_5|^2 = 1.66572 \dots \end{aligned}$$

For $N = 6$ we take $(p, \kappa) = (12, 18)$ and evaluate (5) directly: $|A_1|^2 = 4803353755.04 \dots$ and $|A_5|^2 = 31207039519.09 \dots$. In all three cases the two numbers were also shown to be different and nonzero by exact computation in the cyclotomic ring. $\square$

5.2 Spin(7)

For type B_n we use the standard coordinates: $\langle \cdot, \cdot \rangle$ is the dot product on $\mathbb{R}^n$, $X = \mathbb{Z}^n \cup (\mathbb{Z} + \frac{1}{2})^n$, L consists of the integer vectors with even coordinate sum, $\rho = (n - \frac{1}{2}, \dots, \frac{3}{2}, \frac{1}{2})$, and W consists of the signed permutations of the coordinates.

Proposition 5.6 (Spin(7); uses Theorem 2.3). *Let $\mathfrak{g}$ be of type B_3 , $p = 10$ and $\kappa \equiv 5 \pmod{10}$, $\kappa \geq 15$. Then*

$$|\tau_\kappa(L(10, \pm 1))| = \frac{10\sqrt{10} \cos(5\pi/(4\kappa))}{\Pi(\kappa)}, \quad |\tau_\kappa(L(10, \pm 3))| = \frac{10\sqrt{10} \sin(5\pi/(4\kappa))}{\Pi(\kappa)}.$$

Both are nonzero, and their ratio is $\tan(5\pi/(4\kappa)) < 1$. For $\kappa = 15$ these are the values of Theorem 1.1(c).

Proof. Write $\kappa = 5\kappa'$ with κ' odd; then $g = 5$, $p' = 2$. Here $L + 2X = \mathbb{Z}^3$, so $\Lambda_0 = 5\mathbb{Z}^3$. Since κ' is odd, $\tilde{R} = L \cap 2X = (2\mathbb{Z})^3$, and $|R| = [\mathbb{Z}^3 : 5L] = 250$. For $\mu = 2\xi \in \tilde{R}$ we have $q\kappa Q(\mu)/p = q\kappa'|\xi|^2 \in \mathbb{Z}$, so s is trivial and $y_0 = 0$. Modulo 5 the six numbers $\pm\frac{5}{2}, \pm\frac{3}{2}, \pm\frac{1}{2}$ are different except that $\frac{5}{2} \equiv -\frac{5}{2}$. Hence $H = \{1, h\}$, where h changes the sign of the first coordinate, $\det(h) = -1$; and $W_3 = \{w_+, w_-\}$ with $w_\pm \rho = (\pm\frac{5}{2}, -\frac{1}{2}, \frac{3}{2})$ and $\det(w_-) = -\det(w_+)$.

For $q = 1$ and $w = h$ we have $\rho - h\rho = (5, 0, 0) = \kappa\eta + 10\xi$ with $\eta = (0, 1, 1)$ and $\xi = (\frac{1}{2}, -\frac{\kappa'}{2}, -\frac{\kappa'}{2}) \in X$; $Q(\eta) = 1$ and $|\rho|^2 - \langle \rho, h\rho \rangle = \frac{25}{2}$. So $\Psi_1(h)/\Psi_1(1) = \mathbf{e}(\frac{5}{4\kappa} - \frac{\kappa'}{2}) = -\mathbf{e}(\frac{5}{4\kappa})$, and $|\Psi_1(1) - \Psi_1(h)| = 2\cos(5\pi/(4\kappa))$.

For $q = 3$ we have $3\rho - w_+\rho = (5, 5, 0) = 3\kappa\eta_+ + 10\xi_+$ with $\eta_+ = (1, 1, 0)$, and $3\rho - w_-\rho = (10, 5, 0) = 3\kappa\eta_- + 10\xi_-$ with $\eta_- = (1, 0, 1)$; in both cases $\xi_\pm \in X$ and $Q(\eta_\pm) = 1$. Since $\langle \rho, w_\pm \rho \rangle = \pm\frac{25}{4}$, we get $\Psi_3(w_-)/\Psi_3(w_+) = \mathbf{e}(\frac{5}{4\kappa})$ and $|\Psi_3(w_+) - \Psi_3(w_-)| = 2\sin(5\pi/(4\kappa))$.

Now (8) with $|R|^{1/2} = 5\sqrt{10}$ gives the formulas for $q = 1, 3$, and Lemma 4.4 gives them for $q = 9, 7$. For $\kappa = 15$ the angle is $\pi/12$. $\square$

The mechanism is the same as for $SU(4) = \text{Spin}(6)$: hypothesis (K) of Proposition 4.5 fails because $p' = 2$.

5.3 Spin($2n + 1$), $n \geq 4$

Let $O_n = \{1, 3, \dots, 2n - 1\}$, the set of coordinates of 2ρ , and $s_n = (-1)^{n(n+1)/2}$. For an odd integer x let $\text{sgn}_{4n}(x)$ be the sign of the residue of x modulo $4n$ in the interval $(-2n, 2n)$. For an integer q prime to $2n$ put

$$M_q = \sum_{r \in O_n} r \text{sgn}_{4n}(qr).$$

Since $(q + 2n)r \equiv qr + 2n \pmod{4n}$ for odd r , we have $M_{q+2n} = -M_q$; so $|M_q|$ depends only on q modulo $2n$.

Proposition 5.7 (uses Theorem 2.3). *Let $\mathfrak{g}$ be of type B_n with $n \geq 4$, let $p = 2n$ and let κ be a multiple of $2n$. Then for every unit q modulo $2n$,*

$$|\tau_\kappa(L(2n, q))|^2 = \frac{(2n)^n}{\Pi(\kappa)^2} \left(2 + 2s_n \cos \frac{\pi M_q}{2\kappa} \right).$$

Moreover $M_{\pm 1} = \pm n^2$, and $|M_q| < n^2$ for $q \not\equiv \pm 1 \pmod{2n}$; M_q is odd if n is odd, and $M_{n+1} = 2n$ if $n \equiv 2 \pmod{4}$. Consequently, for every multiple $\kappa > n^2/2$ of $2n$ there is a unit q with $0 \neq |\tau_\kappa(L(2n, q))| \neq |\tau_\kappa(L(2n, 1))| \neq 0$.

Proof. By Corollary 4.3 we need the signed permutations w for which all coordinates of $2w\rho - 2q\rho$ are $\equiv 0 \pmod{4n}$, or all are $\equiv 2n \pmod{4n}$. The numbers $\pm r$, $r \in O_n$, represent the odd residues modulo $4n$, and q is odd. The maps $x \mapsto qx$ and $x \mapsto qx + 2n$ permute the odd residues and commute with $x \mapsto -x$. So there are exactly two such elements, w_q and w'_q : the coordinate of $2w_q\rho$ at the place where 2ρ has r is the residue of qr in $(-2n, 2n)$, and for w'_q it is the residue of $qr + 2n$. The second residue equals the first minus $2n \operatorname{sgn}_{4n}(qr)$. Since $\langle \rho, w\rho \rangle = \frac{1}{4}(2\rho) \cdot (2w\rho)$, we get $\langle \rho, w'_q\rho \rangle - \langle \rho, w_q\rho \rangle = -\frac{n}{2}M_q$. For $q = 1$ the two elements are 1 and $h = w'_1$, and h reverses the order of the coordinates and changes all signs, so $\det(h) = (-1)^{n(n-1)/2}(-1)^n = s_n$. As in Proposition 4.2, the set $\{w_q, w'_q\}$ is a coset of the group $\{1, h\}$; hence $w'_q = w_q h$ and $\det(w'_q) = s_n \det(w_q)$. With $z = \mathbf{e}(-1/(2n\kappa))$ this gives $|\Sigma_q| = |1 + s_n z^{-nM_q/2}| = |1 + s_n \mathbf{e}(M_q/(4\kappa))|$, which is the formula.

Clearly $M_1 = n^2$ and $M_{-q} = -M_q$. Suppose that all $\operatorname{sgn}_{4n}(qr)$, $r \in O_n$, are $+$, and let $q_0 \in (0, 4n)$ be the residue of q ; then $q_0 < 2n$ (take $r = 1$), q_0 is odd, and $q_0 \neq n$ because q is prime to n . Assume $3 \leq q_0 \leq 2n - 3$. If $q_0 < n$, the numbers $q_0, 3q_0, \dots, (2n-1)q_0$ increase in steps of $2q_0 < 2n$, and the last one exceeds $2n$; the first one that is $\geq 2n$ lies in $(2n, 4n)$: a contradiction. If $q_0 > n$, put $d = 4n - 2q_0 \in (0, 2n)$; the numbers $(2j+1)q_0$ are congruent to $q_0 - jd$, $0 \leq j < n$, and $q_0 - (n-1)d < 0$ because $q_0 \leq 2n - 2$; the first negative one lies in $(-2n, 0)$: a contradiction. So the signs are all $+$ only if $q_0 \in \{1, 2n-1\}$, and they are all equal only for $q \equiv \pm 1 \pmod{2n}$. Also $M_q \equiv \sum_{r \in O_n} r = n^2 \pmod{2}$. For $n = 4t + 2$ and $q = n + 1$ we have $qr \equiv n + r$ or $3n + r \pmod{4n}$ according as $r \equiv 1$ or $3 \pmod{4}$; so the sign is $+$ exactly for $r \equiv 1$, $r < n$ and for $r \equiv 3$, $r > n$, and summing gives $M_{n+1} = 8t + 4 = 2n$.

Let $\kappa > n^2/2$. Then all angles $\pi|M_q|/(2\kappa)$ lie in $[0, \pi)$, where the cosine is strictly decreasing, and $|\Sigma_1|^2 = 2 + 2s_n \cos(\pi n^2/(2\kappa)) > 0$. If $s_n = 1$, every $q \not\equiv \pm 1$ gives $|\Sigma_q|^2 > |\Sigma_1|^2$. If $s_n = -1$, every $q \not\equiv \pm 1$ with $M_q \neq 0$ gives $0 < |\Sigma_q|^2 < |\Sigma_1|^2$; such q exist, because $s_n = -1$ means $n \equiv 1$ or $2 \pmod{4}$, and we may take any $q \not\equiv \pm 1$ if n is odd and $q = n + 1$ if $n \equiv 2 \pmod{4}$. Units $q \not\equiv \pm 1 \pmod{2n}$ exist because $2n \geq 8$. $\square$

5.4 Spin($2n$), $n \geq 5$

For type D_n we use the standard coordinates: X and L are as for B_n , $\rho = (n-1, \dots, 1, 0)$, and W consists of the signed permutations with an even number of sign changes; $\det(w)$ is the sign of the underlying permutation. Put $m = n - 1$ and $s'_n = (-1)^{m(m+1)/2}$. For an integer x not divisible by m let $\operatorname{sgn}_{2m}(x)$ be the sign of the residue of x modulo $2m$ in $(-m, m)$, and for an odd integer q prime to m put $M'_q = \sum_{v=1}^{m-1} v \operatorname{sgn}_{2m}(qv)$.

Proposition 5.8 (uses Theorem 2.3). *Let $\mathfrak{g}$ be of type D_n with $n \geq 5$, let $p = 2m = 2n - 2$ and let κ be a multiple of $2m$. Then for every unit q modulo $2m$,*

$$|\tau_\kappa(L(2m, q))|^2 = \frac{4(2m)^n}{\Pi(\kappa)^2} \left(1 + \gamma_m^2 + 2s'_n \gamma_m \cos \frac{\pi M'_q}{\kappa}\right), \quad \gamma_m = \cos \frac{\pi m}{\kappa}.$$

Moreover $|M'_q| = m(m-1)/2$ for $q \equiv \pm 1 \pmod{2m}$ and $|M'_q| < m(m-1)/2$ otherwise. Consequently, for every multiple κ of $2m$ with $\kappa > 2m$ and $\kappa > m(m-1)/2$, and every unit $q \not\equiv \pm 1 \pmod{2m}$, we have $0 \neq |\tau_\kappa(L(2m, q))| \neq |\tau_\kappa(L(2m, 1))| \neq 0$.

Proof. By Corollary 4.3 we need the $w \in W$ for which all coordinates of $w\rho - q\rho$ are $\equiv 0 \pmod{2m}$, or all are $\equiv m \pmod{2m}$. The coordinates of ρ are $0, 1, \dots, m$, and the numbers $\pm v$, $0 \leq v \leq m$, represent all residues modulo $2m$, with 0 and m equal to their negatives. An element of W is determined by $w\rho$, that is, by a permutation of the coordinates and signs on the nonzero ones. The map $x \mapsto qx$ fixes the residues 0 and m , and $x \mapsto qx + m$ exchanges them. So there are exactly four such elements. Two have shift 0 : the coordinate m goes to $+m$ or to $-m$, and $\langle \rho, w\rho \rangle = E_0 \pm m^2$.

Two have shift m : the coordinate $\pm m$ is put at the place where ρ has 0, and 0 at the place where ρ has m ; both have $\langle \rho, w\rho \rangle = E_m$. Here $E_c = \sum_{0 < v < m} v \cdot (\text{residue of } qv + c \text{ in } (-m, m))$. The residue of $qv + m$ is that of qv minus $m \operatorname{sgn}_{2m}(qv)$, so $E_0 - E_m = mM'_q$. With $z = \mathbf{e}(-1/(2m\kappa))$ we get

$$\Sigma_q = \pm z^{E_0} (z^{m^2} + z^{-m^2} + 2s z^{-mM'_q}),$$

where s is the ratio of the signs of the two underlying permutations. As in Proposition 4.2, the four elements form a coset wH of the group H of the four elements for $q = 1$. A signed permutation maps a vector whose coordinates are all $\equiv c \pmod{2m}$, $c \in \{0, m\}$, to a vector of the same kind; so the shift of wh is the sum of the shifts of w and h , and $s = \det(h)$ for an element $h \in H$ with shift m . Hence s does not depend on q . For $q = 1$ the two permutations are the identity and the reversal $v \mapsto m - v$ of n coordinates, so $s = s'_n$. Finally $z^{m^2} + z^{-m^2} = 2\gamma_m$ and $z^{-mM'_q} = \mathbf{e}(M'_q/(2\kappa))$, which gives the formula.

Clearly $M'_1 = m(m-1)/2$ and $M'_{-q} = -M'_q$. If all $\operatorname{sgn}_{2m}(qv)$, $0 < v < m$, are $+$, let $q_0 \in (0, 2m)$ be the residue of q ; then $q_0 < m$, and q_0 is odd. If $q_0 \geq 3$, the multiples $q_0, 2q_0, \dots, (m-1)q_0$ increase in steps of $q_0 < m$, the last one is larger than m , and the first one that is $\geq m$ lies in $(m, 2m)$, a contradiction (it is not m , because m does not divide qv for $0 < v < m$). So $q \equiv 1 \pmod{2m}$; if all signs are $-$, then $q \equiv -1 \pmod{2m}$.

For $\kappa > 2m$ we have $0 < \gamma_m < 1$, hence $1 + \gamma_m^2 + 2s'_n \gamma_m \cos x \geq (1 - \gamma_m)^2 > 0$ for all x . For $\kappa > m(m-1)/2$ the angles $\pi|M'_q|/\kappa$ lie in $[0, \pi)$, where the cosine is injective. Units $q \not\equiv \pm 1 \pmod{2m}$ exist because $2m \geq 8$. $\square$

For $n = 4$ we have $2m = 6$, and there is no unit other than ± 1 ; this is in accordance with Theorem 6.1.

5.5 The exceptional groups E_6, E_7, E_8

Proposition 5.9 (computer-assisted; uses Theorem 2.3). *In the notation of Corollary 4.3:*

- (a) $E_6, (p, \kappa) = (9, 18), z = \mathbf{e}(-1/162)$: $\Sigma_1 = \pm(z^{78} + 2z^{24} - z^{-3} - 2z^{-57})$ and $\Sigma_2 = \pm(z^{57} + 2z^{30} - z^{-24} - 2z^{-51})$, so that $|\Sigma_1|^2 = 12$ and $|\Sigma_2|^2 = 28$.
- (b) $E_7, (p, \kappa) = (14, 28), y = \mathbf{e}(-1/784)$: $\Sigma_1 = \pm(y^{399} - y^{105} - y^7 + y^{-287})$ and $\Sigma_3 = \pm(y^{273} - y^{175} - y^{-119} + y^{-217})$, so that $|\Sigma_1|^2 = 8 + 4\sqrt{2}$ and $|\Sigma_3|^2 = 8 - 4\sqrt{2}$.
- (c) $E_8, (p, \kappa) = (5, 35)$: Σ_1 and Σ_2 are sums of 14400 roots of unity each, with $|\Sigma_1|^2 = 336821.635\dots$ and $|\Sigma_2|^2 = 703332.750\dots$

In each case the two moduli are nonzero and different. So $\operatorname{LS}(G)$ fails for $G = E_6, E_7, E_8$.

Proof. For E_6 and E_7 the elements w with $w\rho - q\rho \in pX$ were found by enumerating the orbit $W\rho$ (51840 and 2903040 elements). Given the terms, the moduli follow by a short calculation: with $u = z^{27} = \mathbf{e}(-1/6)$ we have $|\Sigma_1| = |u^5 + 2u^3 - u^2 - 2| = |-3 + i\sqrt{3}|$ and $|\Sigma_2| = |u^4 + 2u^3 - u - 2| = |-5 + i\sqrt{3}|$; with $v = y^{98} = \mathbf{e}(-1/8)$ we have $|\Sigma_1| = |v^7 - v^4 - v^3 + 1| = |2 + \sqrt{2} + i\sqrt{2}|$ and $|\Sigma_3| = |v^5 - v^4 - v + 1| = |2 - \sqrt{2} + i\sqrt{2}|$.

For E_8 we have $X = L$, the root lattice, and the group $H = \{h : h\rho - \rho \in pL\}$ is the image in W of the stabilizer of ρ in the affine Weyl group $W \ltimes pL$. This stabilizer is generated by the reflections that it contains [Hum90, Section 4.8], so H is generated by the reflections in the roots α with $p \mid \langle \rho, \alpha \rangle$. For $p = 5$ there are 20 such positive roots and $|H| = 14400$. (The order of H is confirmed by a count: the orbit of ρ in $L/5L$ under W has $48384 = |W|/14400$ elements.) The program enumerates the orbit $H\rho$, finds an element w_q with $w_q\rho - q\rho \in pL$ by moving ρ and $q\rho$ into the same alcove, and evaluates $|\Sigma_q|^2$ in $\mathbb{Z}[\mathbf{e}(1/175)]$. The two values are nonzero and different as elements of this ring. $\square$

The values of $|\tau|$ obtained from Proposition 5.9 and (9) are those of Table 1, which were computed from the surgery formula. The same exact evaluation of (9) gives the rows $\text{Spin}(12)$ and $\text{Spin}(14)$ of the table ($|\Sigma_1|^2 = 8$, $|\Sigma_3|^2 = 16$, and $|\Sigma_1|^2 = 32$, $|\Sigma_3|^2 = 64$).

Proof of Theorem 1.2(b). Each simply connected compact simple group other than those of Theorem 1.2(a) is one of $\text{SU}(N)$, $N \geq 4$; $\text{Spin}(2n+1)$, $n \geq 3$; $\text{Spin}(2n)$, $n \geq 5$; E_6 , E_7 , E_8 . (Recall $\text{Spin}(6) = \text{SU}(4)$.) The statements (i)–(vi) are Propositions 5.2, 5.4 and 5.6 to 5.9, Corollary 5.3, and Theorem 5.5. $\square$

6 Groups for which the lens-space statement holds

6.1 $\text{SU}(3)$, G_2 , $\text{Spin}(8)$ and F_4

Theorem 6.1 (uses Theorem 2.3). *Let $\mathfrak{g}$ be of type A_2 , G_2 , D_4 or F_4 . For all $p \geq 1$ and $\kappa \geq h^\vee$ the nonzero numbers among $|\tau_\kappa(L(p, q))|$, q prime to p , coincide.*

In these four cases the group X/L has prime exponent ϵ : $\epsilon = 3$ for A_2 and G_2 , where L is the lattice A_2 and X its dual, and $\epsilon = 2$ for D_4 and F_4 , where L is the lattice D_4 and $X = \mathbb{Z}^4 \cup (\mathbb{Z} + \frac{1}{2})^4$. (For G_2 the coroot lattice is spanned by the long roots; for F_4 it is spanned by the long roots $\pm e_i \pm e_j$ and the vectors $2e_i$, $(\pm 1, \pm 1, \pm 1, \pm 1)$.)

For a sublattice $\Lambda \subseteq X$ that is stable under W and an integer q put

$$W_q(\Lambda) = \{w \in W : qp - w\rho \in \Lambda\}, \quad P_{q,\Lambda} = \sum_{w \in W_q(\Lambda)} \det(w) x^{\langle \rho, w\rho \rangle}.$$

Here $P_{q,\Lambda}$ is a Laurent polynomial in the indeterminate $x^{1/\delta}$, with δ as in Proposition 4.6. As in Proposition 4.2, $W_1(\Lambda)$ is a subgroup of W and $W_q(\Lambda)$ is empty or a coset of it.

Lemma 6.2 (computer-assisted for D_4 and F_4). *Let $\mathfrak{g}$ be of type A_2 , G_2 , D_4 or F_4 , let $g \geq 1$, and let $\Lambda = gX$ or $\Lambda = gL$. Let q be an integer prime to g if $\Lambda = gX$, and prime to ϵg if $\Lambda = gL$. If $W_q(\Lambda)$ is not empty, then there are signs $\varepsilon, \varepsilon' \in \{\pm 1\}$ and an exponent t with*

$$P_{q,\Lambda}(x) = \varepsilon x^t P_{1,\Lambda}(x^{\varepsilon'}).$$

Proof. If $W_1(\Lambda) = \{1\}$, both sides are monomials. If $(q-1)\rho \in \Lambda$, then $W_q(\Lambda) = W_1(\Lambda)$ and $P_{q,\Lambda} = P_{1,\Lambda}$. If $(q+1)\rho \in \Lambda$, then $W_q(\Lambda) = w_0 W_1(\Lambda)$ and $P_{q,\Lambda}(x) = \det(w_0) P_{1,\Lambda}(x^{-1})$, because $\langle \rho, w_0 h\rho \rangle = -\langle \rho, h\rho \rangle$.

The group $W_1(\Lambda)$ is not trivial only if g divides the content of $\rho - h\rho$ in X , resp. in L , for some $h \neq 1$. This leaves the following lattices Λ :

type	$\Lambda = gX$	$\Lambda = gL$
A_2	$g \leq 3$	$g \leq 2$
G_2	$g \leq 5$	$g \leq 3$
D_4	$g \leq 6$	$g \leq 5$
F_4	$g \leq 11$	$g \leq 8$

For these, $W_q(\Lambda)$ depends only on q modulo g , resp. modulo ϵg . A program computed, with integer arithmetic over the whole Weyl group (6, 12, 192 and 1152 elements), the sets $W_q(\Lambda)$ and the polynomials $P_{q,\Lambda}$ for all these Λ and all units q . The result is as follows. For A_2 , G_2 and D_4 , whenever

$W_q(\Lambda)$ is not empty, $(q-1)\rho$ or $(q+1)\rho$ lies in Λ . For A_2 and G_2 this can be read off from Table 2. For F_4 the same is true except in the following four cases, where $y = x^{1/2}$:

$$\begin{aligned} \Lambda = 8X : \quad P_1 &= y^{78} - y^{-50}; & P_3 &= y^{66} - y^{-62}, \quad P_5 = -y^{62} + y^{-66}; \\ \Lambda = 9X : \quad P_1 &= y^{78} - y^{-3}; & P_2 = P_5 &= -y^{57} + y^{-24}, \quad P_4 = P_7 = y^{24} - y^{-57}; \\ \Lambda = 4L : \quad P_1 &= y^{78} - 2y^{46} + 2y^{-18} - y^{-50}; & P_3 &= y^{-12}P_1, \quad P_5 = -y^{-16}P_1; \\ \Lambda = 6L : \quad P_1 &= y^{78} - y^6; & P_5 &= y^{30} - y^{-42}, \quad P_7 = -y^{42} + y^{-30}. \end{aligned}$$

(The indices are residues modulo 8, 9, 8 and 12.) In each case $P_q = \pm y^t P_1(y)$. $\square$

A_2				G_2			
$w\rho$	$\det$	$\langle \rho, w\rho \rangle$	$\rho - w\rho$	$w\rho$	$\det$	$3\langle \rho, w\rho \rangle$	$\rho - w\rho$
(1, 1)	+	2	(0, 0)	(1, 1)	+	14	(0, 0)
(-1, 2)	-	1	(2, -1)	(-1, 2)	-	13	(2, -1)
(2, -1)	-	1	(-1, 2)	(4, -1)	-	11	(-3, 2)*
(1, -2)	+	-1	(0, 3)	(5, -2)	+	7	(-4, 3)
(-2, 1)	+	-1	(3, 0)	(-4, 3)	+	7	(5, -2)
(-1, -1)	-	-2	(2, 2)	(-5, 3)	-	2	(6, -2)*
				(5, -3)	-	-2	(-4, 4)
				(4, -3)	+	-7	(-3, 4)*
				(-5, 2)	+	-7	(6, -1)*
				(-4, 1)	-	-11	(5, 0)
				(1, -2)	-	-13	(0, 3)*
				(-1, -1)	+	-14	(2, 2)

Table 2: The orbit $W\rho$ for A_2 and G_2 , in coordinates with respect to the fundamental weights (for G_2 the first simple root is the short one). A vector (a, b) lies in gX if and only if g divides a and b . For A_2 , $(a, b) \in gL$ if and only if g divides $(2a+b)/3$ and $(a+2b)/3$. For G_2 , $(a, b) \in gL$ if and only if g divides $(2a+3b)/3$ and $a+2b$; the nonzero vectors $\rho - w\rho$ that lie in L are marked with *.

For example, for A_2 and $\Lambda = 3X$ the group $W_1(\Lambda)$ consists of the three elements with $\rho - w\rho \in \{(0, 0), (0, 3), (3, 0)\}$, so $P_1 = x^2 + 2x^{-1}$; the other three elements form $W_2(\Lambda)$, and $P_2 = -2x - x^{-2} = -P_1(x^{-1})$. The complete output of the program, with all polynomials for the four algebras, is part of the source files (Section 9). As a control, for A_4 and $\Lambda = 5X$ the program finds $P_1 = x^{10} + 2 + 2x^{-5}$ and $P_2 = P_3 = -(2x^5 + 1 + 2x^{-5})$, which are not related in this way; this is the example of Proposition 5.2.

Proof of Theorem 6.1. Fix p and κ and use the notation of Section 4.

Step 1: s is trivial. Let $\mu \in \tilde{R}$. Then $\kappa'\mu \in p'X$, and since $\gcd(\kappa', p') = 1$ we have $\mu = p'\xi$ with $\xi \in X$; so $\kappa Q(\mu)/p = \kappa'p'|\xi|^2/2$. If $\xi \in L$ this is an integer. Otherwise ϵ divides p' , because $p'\xi \in L$ and ξ has order ϵ in X/L . For $\epsilon = 3$ we have $|\xi|^2 \in \frac{2}{3}\mathbb{Z}$, since $3|\xi|^2 = \langle 3\xi, \xi \rangle \in \mathbb{Z}$ and $9|\xi|^2 = |3\xi|^2$ is even; so $p'|\xi|^2/2$ is an integer. For $\epsilon = 2$, $|\xi|^2$ is an integer for every $\xi \in \mathbb{Z}^4 \cup (\mathbb{Z} + \frac{1}{2})^4$, so $(p'/2)|\xi|^2$ is an integer.

Step 2: hypothesis (K). If $\epsilon \nmid p'$, then $L + p'X = X$, $\Lambda_0 = gX$, and (K) holds with $k \equiv 0 \pmod{\epsilon}$ and $\kappa'k \equiv 1 \pmod{p'}$. If $\epsilon \mid p'$, then $L + p'X = L$, $\Lambda_0 = gL$, and (K) holds with any k such that $\kappa'k \equiv 1 \pmod{p'}$; in this case ϵg divides p .

Step 3. By Proposition 4.5, for every unit q modulo p ,

$$|\tau_\kappa(L(p, q))| = \frac{|R|^{1/2}}{\Pi(\kappa)} |P_{q, \Lambda_0}(\mathbf{e}(c_0))|,$$

where $P(\mathbf{e}(c_0))$ is the value obtained by replacing each x^a by $\mathbf{e}(c_0 a)$, and c_0 does not depend on q . A unit q modulo p is prime to g , and in the case $\Lambda_0 = gL$ it is prime to ϵg , which divides p ; so Lemma 6.2 applies to $\Lambda = \Lambda_0$. If $\tau_\kappa(L(p, q)) \neq 0$, then $W_q(\Lambda_0)$ is not empty, and Lemma 6.2 gives $|P_{q, \Lambda_0}(\mathbf{e}(c_0))| = |P_{1, \Lambda_0}(\mathbf{e}(c_0))|$, because P_{1, Λ_0} has real coefficients. So every nonzero $|\tau_\kappa(L(p, q))|$ equals $|\tau_\kappa(L(p, 1))|$. $\square$

6.2 $\mathrm{Sp}(2n)$

Theorem 6.3 (uses Theorem 2.3). *Let $\mathfrak{g} = \mathfrak{sp}_{2n}$, $n \geq 1$. For all $p \geq 1$ and $\kappa \geq n + 1$ the nonzero numbers among $|\tau_\kappa(L(p, q))|$, q prime to p , coincide.*

For $n = 1$ this is the theorem of Guadagnini and Pilo for $\mathrm{SU}(2)$, and for $n = 2$ it is the statement for $\mathrm{Spin}(5)$.

Proof. We use the coordinates $X = \mathbb{Z}^n$, $L = 2\mathbb{Z}^n$, $\langle x, y \rangle = x \cdot y/2$, $\rho = (n, n-1, \dots, 1)$, and W the group of signed permutations. Units q modulo p are represented by positive integers. For $\nu = 2m$ with $m \in \mathbb{Z}^n$ we have $Q(\nu) = m \cdot m$ and $\langle \nu, y \rangle = m \cdot y$. So the inner sum of (5) is a product over the coordinates, and summing over the signs and then over the permutations we obtain, with $z = \mathbf{e}(-1/(p\kappa))$,

$$A_q = \det[\mu(a, b)]_{a, b=1}^n, \quad \mu(a, b) = \Phi(a, b) - \Phi(a, -b), \quad \Phi(a, b) = z^{ab/2} \gamma(qa - b), \quad (12)$$

where $\gamma(t) = \sum_{m \bmod p} \mathbf{e}((q\kappa m^2 + tm)/p)$ and $z^{ab/2} = \mathbf{e}(-ab/(2p\kappa))$; here a, b are arbitrary integers. Note that $\gamma(-t) = \gamma(t)$ and $\Phi(-a, -b) = \Phi(a, b)$.

Step 1: the sum γ . Put $g_2 = \gcd(2\kappa, p)$, $p = g_2 p''$, $2\kappa = g_2 k''$, and let $t_0 = 0$ if $k'' p''$ is even and $t_0 = g_2/2$ if $k'' p''$ is odd. (In the second case g_2 and p are even and q is odd.) As in Lemma 4.1,

$$|\gamma(t)|^2 = p \sum_{j \bmod g_2} \mathbf{e}\left(\frac{qk'' p'' j^2}{2} + \frac{tj}{g_2}\right),$$

so $\gamma(t) \neq 0$ if and only if $t \equiv t_0 \pmod{g_2}$, and then $|\gamma(t)| = \sqrt{p g_2}$. Moreover $\gamma(t + 2q\kappa\eta) = \mathbf{e}(-(q\kappa\eta^2 + t\eta)/p) \gamma(t)$ for $\eta \in \mathbb{Z}$. If u is an integer with $qk'' u \equiv 1 \pmod{p''}$, then $2q\kappa u s \equiv g_2 s \pmod{p}$, and therefore

$$\gamma(t_0 + g_2 s) = \gamma(t_0) \mathbf{e}\left(-\frac{q\kappa u^2 s^2 + ut_0 s}{p}\right) \quad (s \in \mathbb{Z}). \quad (13)$$

Step 2: factorization. Let q_1, q_2 be two units. Choose an integer v with $v \equiv 1 \pmod{p''}$ and $v \equiv 0 \pmod{q_1 q_2 k''}$; this is possible because $q_1 q_2 k''$ is prime to p'' . The integer v depends on the pair (q_1, q_2) . For $q \in \{q_1, q_2\}$ take $u = v/(qk'')$ in (13). Let a, b be integers, of any sign, with $qa - b \equiv t_0 \pmod{g_2}$, and put $s = (qa - b - t_0)/g_2$. Expanding the exponent of $\Phi(a, b) = z^{ab/2} \gamma(t_0 + g_2 s)$ as a polynomial in a and b , with $q\kappa u^2/p = v^2/(2qk'' p'')$, one finds

$$\Phi(a, b) = \gamma(t_0) F_1(a) F_2(b) \xi^{ab}, \quad \xi = \mathbf{e}\left(\frac{v^2 - 1}{g_2^2 p'' k''}\right), \quad (14)$$

where $|F_1| = |F_2| = 1$ and

$$F_2(b) = \mathbf{e}\left(-\frac{v^2(b + t_0)^2}{2qk'' g_2^2 p''} + \frac{t_0 v(b + t_0)}{qk'' g_2^2 p''}\right).$$

The coefficient of ab is $-1/(g_2^2 p'' k'') + v^2/(g_2^2 p'' k'')$; so ξ does not depend on q . Further $F_2(-b)/F_2(b) = \mathbf{e}(2t_0 v(v-1)b/(qk'' g_2^2 p''))$. This is 1 if $t_0 = 0$. If $t_0 = g_2/2$ it equals $\mathbf{e}(u j_v b/g_2)$ with $j_v = (v-1)/p'' \in \mathbb{Z}$;

for $b \equiv 0 \pmod{g_2}$ this is 1, and for $b \equiv g_2/2 \pmod{g_2}$ it is the sign $(-1)^{vj_v}$, because $u \equiv v \pmod{2}$ (qk'' is odd in this case). So for $2b \equiv 0 \pmod{g_2}$ the quotient $\sigma(b) = F_2(-b)/F_2(b)$ is a sign that does not depend on q .

Step 3: blocks. By Step 1, $\mu(a, b) = 0$ unless $b \equiv \pm(qa - t_0) \pmod{g_2}$; note $2t_0 \equiv 0$. For a residue r modulo g_2 let

$$\text{Row}(r) = \{a \leq n : a \equiv \pm r\}, \quad \text{Col}(r) = \{b \leq n : b \equiv \pm(qr - t_0)\} \pmod{g_2}.$$

The sets $\text{Row}(r)$ partition $\{1, \dots, n\}$. The map $r \mapsto qr - t_0$ is a bijection of $\mathbb{Z}/g_2$ that commutes with $r \mapsto -r$, so the sets $\text{Col}(r)$ partition $\{1, \dots, n\}$ as well, and the rows in $\text{Row}(r)$ vanish outside the columns in $\text{Col}(r)$. Hence $A_q = 0$ unless $|\text{Row}(r)| = |\text{Col}(r)|$ for all r , and then $|A_q|$ is the product of the moduli of the determinants of the blocks $\mu[\text{Row}(r), \text{Col}(r)]$.

Step 4: generic blocks, $2r \not\equiv 0 \pmod{g_2}$. Let $I = \{x \in \mathbb{Z} : x \equiv r, 1 \leq |x| \leq n\}$ and $J = \{y \in \mathbb{Z} : y \equiv qr - t_0, 1 \leq |y| \leq n\}$. For $a \in \text{Row}(r)$ exactly one of $\pm a$ lies in I ; call it $\tilde{a}$, and define $\tilde{b} \in J$ for $b \in \text{Col}(r)$ in the same way. Only one of the two terms of $\mu(a, b)$ can be nonzero, and using $\Phi(-a, -b) = \Phi(a, b)$ one checks that $\mu(a, b) = \text{sgn}(\tilde{a}) \text{sgn}(\tilde{b}) \Phi(\tilde{a}, \tilde{b})$, where $q\tilde{a} - \tilde{b} \equiv t_0$. So (14) applies to $(\tilde{a}, \tilde{b})$; here negative indices are used. The sets I and J consist of $k = |\text{Row}(r)|$ consecutive terms of arithmetic progressions with difference g_2 (they do not contain 0). Hence, by (14) and the Vandermonde determinant,

$$|\det \mu[\text{Row}(r), \text{Col}(r)]| = (pg_2)^{k/2} |\det[\xi^{xy}]_{x \in I, y \in J}| = (pg_2)^{k/2} \prod_{0 \leq i < j < k} |\chi^j - \chi^i|, \quad \chi = \xi^{g_2^2}.$$

This depends only on $k = |\text{Row}(r)|$ and on (p, κ, v) .

Step 5: special blocks, $r = 0$ or $r = g_2/2$. Then $\text{Col}(r) = \{b \leq n : b \equiv r - t_0 \pmod{g_2}\}$, which does not depend on q (for $r = g_2/2$, q is odd). For $a \in \text{Row}(r)$ and $b \in \text{Col}(r)$ both terms of $\mu(a, b)$ satisfy the condition of Step 2, and $\mu(a, b) = \gamma(t_0)F_1(a)F_2(b)(\xi^{ab} - \sigma(b)\xi^{-ab})$. So the modulus of the determinant of this block is $(pg_2)^{k/2} |\det[\xi^{ab} - \sigma(b)\xi^{-ab}]|$, which does not depend on q .

Step 6. The sets $\text{Row}(r)$ do not depend on q . If A_{q_1} and A_{q_2} are both nonzero, then by Steps 3–5 they are products of the same block moduli, so $|A_{q_1}| = |A_{q_2}|$. The claim follows from (6). $\square$

The case $t_0 = g_2/2$ occurs exactly when the 2-adic valuation of p exceeds that of κ by one. The proof does not use $\kappa \geq n + 1$.

Proof of Theorem 1.2(a). This is Theorems 6.1 and 6.3. $\square$

7 From lens spaces to all closed oriented 3-manifolds

Proposition 7.1. *Let G be a simply connected compact simple Lie group and $\kappa \geq h^\vee$. The following are equivalent.*

- (a) *For all closed oriented 3-manifolds M, N with isomorphic fundamental groups and $\tau_\kappa(M) \neq 0 \neq \tau_\kappa(N)$ we have $|\tau_\kappa(M)| = |\tau_\kappa(N)|$.*
- (b) *For every $p \geq 1$ the nonzero numbers among $|\tau_\kappa(L(p, q))|$, q prime to p , coincide.*

Proof. (a) implies (b) because $\pi_1(L(p, q)) \cong \mathbb{Z}/p$. For the converse we use three facts.

(1) *Multiplicativity.* For closed connected oriented 3-manifolds M_1, M_2 and $\tau = \tau_{(\mathcal{V}, \mathcal{D})}$ one has $\tau(M_1 \# M_2) = \mathcal{D} \tau(M_1) \tau(M_2)$ [Tur10, (II.2.3.a)]; see also [BK01, Remark 4.1.14]. In our normalization this reads $\tau_\kappa(M_1 \# M_2) = \tau_\kappa(M_1) \tau_\kappa(M_2)$.

(2) *Orientation.* For a Hermitian modular category with real rank, $\tau(-M) = \overline{\tau(M)}$ [Tur10, Theorem II.5.4]. The category $\mathcal{C}(\mathfrak{g}, \kappa)$ is Hermitian by [Kir96, Theorem 4.2 and the paragraph after it]; it is even unitary [Wen98]. We use only $|\tau_\kappa(-M)| = |\tau_\kappa(M)|$.

(3) *Topology.* Let M and N be closed oriented 3-manifolds with isomorphic fundamental groups. By [AFW15, Theorem 2.3] (in the numbering of the third arXiv version), which rests on the Geometrization Theorem, there are integers p_i, q_i, q'_i and oriented manifolds N_j, N'_j such that

$$M \cong L(p_1, q_1) \# \cdots \# L(p_m, q_m) \# N_1 \# \cdots \# N_s, \quad N \cong L(p_1, q'_1) \# \cdots \# L(p_m, q'_m) \# N'_1 \# \cdots \# N'_s,$$

and N'_j is homeomorphic to N_j , possibly with the opposite orientation. (If the homeomorphisms do not preserve the orientations, or if the convention for $L(p, q)$ differs from ours in the orientation, the absolute values below are not affected, by (2) and Proposition 2.1.)

Assume (b) and $\tau_\kappa(M) \neq 0 \neq \tau_\kappa(N)$. By (1), $\tau_\kappa(M)$ and $\tau_\kappa(N)$ are the products of the invariants of the summands, so all these are nonzero. By (2), $|\tau_\kappa(N'_j)| = |\tau_\kappa(N_j)|$, and by (b), $|\tau_\kappa(L(p_i, q'_i))| = |\tau_\kappa(L(p_i, q_i))|$. Hence $|\tau_\kappa(M)| = |\tau_\kappa(N)|$. $\square$

Corollary 1.3 follows from Proposition 7.1 and Theorems 1.1 and 1.2. Nothing in Proposition 7.1 is special to the groups of Theorem 1.2(a), and the argument applies to the invariant of any Hermitian modular category.

8 Remarks

Remark 8.1 (SU(2) and two formulas of [GP98]). For $n = 1$, Theorem 6.3 is the lens-space theorem of Guadagnini and Pilo. Theorem 2 of [GP98] also gives closed formulas for $|\tau_r(L(p, q))|^2$. We compared them with the surgery formula for all $p \leq 60$ and $r \leq 80$ and found that, in the arXiv version, two of them are inaccurate. (We could not consult the journal version.)

- (a) For $p = 4$ and $r \equiv 3 \pmod{4}$ the value is $(1 + \sin \frac{\pi}{2r}) / \sin^2 \frac{\pi}{r}$, whereas the formula for p prime to r gives $(1 - \sin \frac{\pi}{2r}) / \sin^2 \frac{\pi}{r}$, which is the correct value for $r \equiv 1 \pmod{4}$. For example $|\tau_3(L(4, 1))|^2 = 2$.
- (b) For $\gcd(r, p) = 2$ with $p/2$ odd and $p \geq 6$ the value is $2 \sin^2(\frac{\pi b}{2r}) / \sin^2 \frac{\pi}{r}$ for every q , where b is the inverse of $p/2$ modulo $2r$, whereas formula (44) of [GP98] gives $1 / \sin^2 \frac{\pi}{r}$. The two expressions agree for $r = 2$ (level 0) and differ for every $r \geq 4$. For example $|\tau_4(L(6, 1))|^2 = 2 + \sqrt{2}$.

Both values also follow from Section 4. For type A_1 we have $\epsilon = \delta = 2$, $\langle \rho, \rho \rangle = \frac{1}{2}$ and $\Pi(r) = 2 \sin \frac{\pi}{r}$. In case (b), (11) applies with $g = 2$ and $W_q = W$, and gives the value stated. In case (a) we have $|R| = 2$, $y_0 = 0$ and $W_q = W$, and (8) is evaluated as in the proof of Proposition 5.4; the relative phase of the two terms is $-\mathbf{e}(\frac{1}{4r} - \frac{r}{4})$ for $q \equiv 1 \pmod{4}$.

In both cases the correct value does not depend on q . So the conclusion of [GP98], that for SU(2) the nonzero values of $|\tau_r(L(p, q))|$ depend only on p and r , is not affected; it also follows from Theorem 6.3, whose proof is independent of these formulas. In the range of the comparison all other formulas of that theorem agree with the surgery formula.

Remark 8.2 (Homotopy equivalent lens spaces). The spaces $L(7, 1)$ and $L(7, 2)$ are homotopy equivalent ($2 \equiv 3^2 \pmod{7}$) and not homeomorphic. For SU(7) and $\kappa = 14$, Theorem 5.1 gives $|\tau(L(7, q))| = 7^3 |\Theta_q| / \Pi(14)$ with $|\Theta_1|^2 = 21.768 \dots$ and $|\Theta_2|^2 = |\Theta_3|^2 = 32.183 \dots$, that is, the values 7152.09 $\dots$ and 8696.41 $\dots$ of Table 1. So the absolute value of a quantum invariant can separate homotopy equivalent manifolds on which it does not vanish; by the finite-field computation for the row SU(7) of Table 1, this does not depend on Theorem 2.3. For SU(2) and $r = 7$ the two spaces are separated only by the vanishing of the invariant on $L(7, 2)$. The pairs of Theorem 1.1 are not homotopy equivalent (Remark 3.2).

Remark 8.3 (Levels). The lowest levels at which we found failures of the lens-space statement are 8 for $SU(4)$, 5 for $SU(5)$, 10 for $Spin(7)$, 6 for E_6 and $Spin(12)$, 5 for E_8 and 4 for $Spin(14)$. These come from floating-point scans over limited ranges of p and κ (Section 9). We do not claim that they are minimal.

Remark 8.4 (Projective versions and the centre). All examples for $SU(N)$ in this note have $\gcd(N, r) > 1$. The projective invariants, such as the $SO(3)$ invariant of [KM91] and the invariants of type $PSU(N)$ [KT93, Tak96] (see [Oht02, Section 7.1]), are different invariants. Our results do not decide the conjecture for them. On the other hand the centre of G is not the cause of the failures: E_8 , F_4 and G_2 have trivial centre, and the statement fails for E_8 and holds for F_4 and G_2 .

9 Computations and reproducibility

The programs and their outputs are in the source files, in five folders: **author** (the author's programs), **independent_run_neg** and **independent_run_pos** (the programs of the first and of the second independent verification run), **independent_run_2** (the programs of the third run, which checked the complete text of this paper), and **independent_check_short** (a short stand-alone program). They are written in Python, with numpy and mpmath. We list what the statements of this note rest on, and separately the tests.

Exact computations. These use integer arithmetic only.

- Theorem 3.1 and the statements made after the proof of Theorem 1.1. The author's programs are `verify_counterexamples.py` and `closed_forms_exact.py`; they work with integer vectors of length M and reduction modulo Φ_M , and run in a few seconds. The programs `vexact_main.py` and `vexact2.py` of the first verification run were written independently, with root systems built from Euclidean coordinates; the first one takes between 8 and 20 minutes per group. The programs `r2_exact_thm11.py` and `r2_exact_modular.py` of the third run, again written independently, use explicit coordinates for $SU(N)$ and $Spin(7)$, and 64-bit integers as long as an a priori bound on the coefficients allows it; they take a few seconds, resp. about one minute, per group.
- Theorem 5.5 for $N = 6, 8, 12$: `exceptional_N.py` (author), `vexact2.py` and `r2_exact_misc.py`.
- Proposition 5.9 and the values of $|\Sigma_q|^2$ quoted after it: `theorem1_exact.py` and `exceptional_terms.py` (author), and `r2_orbits_E.py` (third run).
- Lemma 6.2: the author's program `laurent_tables.py` computes all sets $W_q(\Lambda)$ and all polynomials; it prints the number of elements of each set, every $P_{1,\Lambda}$, and every $P_{q,\Lambda}$ or its relation to $P_{1,\Lambda}$. An earlier program, `criterion_tables.py`, checks an equivalent criterion. The second verification run recomputed the tables from explicit coordinates (`tablesC.py`). The third run computed all sets and polynomials again (`r2_laurent.py`); they agree with the output of `laurent_tables.py` in all 126 cases (Λ, q) .
- The rows marked F in Table 1. For $SU(6)$, $SU(7)$ and $SU(8)$ the program `sun_modular_det.py` computes the matrix S' from $N \times N$ determinants; it takes up to 13 minutes. For E_8 the program `e8_level15.py` organizes the sum over the Weyl group by the 135 cosets of the Weyl group of D_8 . The program `certificate_selftest.py` compares the finite-field computation with the reduction of the exact element for $SU(4)$ and $SU(5)$. The third run repeated the finite-field computations for the four rows (`r2_sun_det.py`, `r2_coset_surgery.py`).

Floating-point values. The moduli of Table 1 were computed from (4) in double precision by the programs `vsun.py`, `vsurg.py` and `vcoset.py` of the first verification run; for E_7 and E_8 the sum over the Weyl group is organized by cosets of the Weyl groups of A_7 and A_8 . The rows marked F were also computed by the author’s programs named above. The third run computed all rows once more (`r2_sun_det.py`, `r2_table1.py`, `r2_coset_surgery.py`; for E_7 and E_8 with cosets of the Weyl groups of $D_6 \times A_1$ and D_8). The values for $SU(4)$, $SU(5)$ and $SU(7)$ were computed once more by the short programs `minimal_check.py` and `lead_sun.py`. Different chains for the same lens space give the same value, and the relations $S^2 = C$ and $(ST)^3 \sim S^2$ hold up to rounding errors in each case. The statements of Remarks 3.4 and 8.1 about the comparison with [GP98] rest on `gp_su3_check.py` and `su2_gp_check.py`; the third run repeated both comparisons (`r2_gp_su3.py`, `r2_su2_gp.py`).

Tests. The following computations are tests; no statement of this note depends on them.

(1) Formula (6) was compared with (4) in floating point for 15549 triples (p, q, κ) by the author’s program, and for 32850, 586073 and 91772 triples in the three verification runs. Together they cover fifteen types of rank at most 5, among them B_2 , B_3 , B_4 , C_3 , C_4 , C_5 , G_2 and F_4 , which are not simply laced. The largest relative deviation was below $2 \cdot 10^{-10}$.

(2) The intermediate formulas (Lemma 4.1, Propositions 4.2, 4.5 and 4.6, Theorem 5.1 and Propositions 5.2, 5.4 and 5.6 to 5.8, and each step of the proof of Theorem 6.3) were compared with direct evaluations of (5) on ranges of parameters. In particular, `printed_formulas_check.py` evaluates the closed formulas of Sections 4 and 5 and the determinant (12) as they are printed here; the third run repeated these comparisons with its own programs (`r2_closed_forms.py`, `r2_sp2n.py`).

(3) Scans for pairs (p, κ) with two different nonzero moduli, with thresholds for “zero” and “equal” in floating point. The author’s scans use the second formula of [HT04, Theorem 5.1]. They found no such pair for A_1 ($p \leq 300$, $\kappa \leq 400$), A_2 ($p, \kappa \leq 240$), B_2 and G_2 ($p \leq 160$, $\kappa \leq 200$), C_3 ($p \leq 80$, $\kappa \leq 100$), C_4 ($p \leq 32$, $\kappa \leq 60$), D_4 ($p \leq 44$, $\kappa \leq 60$) and F_4 ($p \leq 40$, $\kappa \leq 60$ for $p \geq 31$ and $\kappa \leq 50$ otherwise), and they found the pairs described in Section 5 for the other types. The second verification run scanned the groups of Theorem 1.2(a) with the surgery formula: about $2.2 \cdot 10^8$ triples in floating point and $3 \cdot 10^7$ triples in exact arithmetic modulo two primes, without finding a pair with two different nonzero moduli. Floating-point thresholds are not reliable when $S_{\rho\rho}$ is very small. This occurred for F_4 at $\kappa \geq 21$ in the scan of the second verification run, and these levels were redone in modular arithmetic. The same caution applies to the author’s scans at the larger levels.

Verification. Theorem 1.1 rests on the exact computation of Theorem 3.1 and on Proposition 2.1. Theorem 1.2 and Corollary 1.3 are proved in Sections 4 to 7 on the basis of [HT04, Theorem 5.1]; Lemma 6.2 (for D_4 and F_4), Proposition 5.9 and the cases $N = 6, 8, 12$ of Theorem 5.5 are computer-assisted.

1. The author’s programs were run again for this version, with the same results as before; the long runs take up to 13 minutes.
2. A first independent verification run, which was AI-assisted, checked the counterexamples and the material of Sections 4 and 5 in the form of the author’s working notes. It re-derived every proof, reproduced Theorem 3.1 with programs written from scratch, and reproduced all rows of Table 1 from the surgery formula. It supplied the effective bound of Corollary 5.3, Proposition 5.6, and the complete proof of Proposition 4.6.
3. A second independent verification run, also AI-assisted, checked the material of Sections 6 and 7 in the same form. It re-derived the proofs of Theorems 6.1 and 6.3, recomputed the tables of Lemma 6.2 with its own code, proposed the polynomial form of that lemma, checked the

references used in Proposition 7.1 against the sources, and tried to find counterexamples for the groups of Theorem 1.2(a) with the surgery formula (test (3) above).

4. A third independent verification run, also AI-assisted, checked the complete text of this paper, which was written after the first two runs and follows their reports. It re-derived every proof, among them the parts written after the first two runs: the proof of Proposition 2.1 and Remark 2.2, Lemma 3.3, Proposition 4.6 and Corollaries 4.7 and 5.3, the proofs of Propositions 5.4 and 5.6, the explicit sums in Proposition 5.9, Lemma 6.2, and the derivation at the end of Remark 8.1. It compared the statements quoted from [GP98, Oht02, HT04, Kir96, AFW15] with the sources. With programs written from scratch it reproduced Theorem 3.1 exactly, all rows of Table 1 (the rows marked F also in finite fields), the finite cases of Theorem 5.5, Proposition 5.9, the tables of Lemma 6.2, Table 2, and the comparison of Remark 8.1. It repeated the literature search.
5. A short program written separately reproduces the values for $SU(4)$, $SU(5)$ and $SU(7)$ in floating point.

The author's programs and the programs of the first two verification runs were run again from the source files for this version, with the same results; only the long scans of the second run (test (3)) were not repeated.

Scope and priority. Theorem 1.1 refutes Conjecture 7.5 of [Oht02] as it is stated there and in [GP98], for an arbitrary gauge group; the dataset record [ula26] has the same statement. The classification of Theorem 1.2 and Corollary 1.3 concerns the simply connected compact simple groups and the full invariant; it depends on the published theorem [HT04, Theorem 5.1], which we did not reprove. The projective versions are not decided (Remark 8.4), and for the groups of Theorem 1.2(b) we did not determine all pairs (p, r) at which the statement fails.

Sources. We read [GP98] in the arXiv version, Section 7.1 and the beginning of Section 7.5 of [Oht02]; Sections 2, 4 and 5 of [HT04] in the arXiv version, whose numbering we use; Sections II.2 and II.5 of [Tur10]; Section 3.3 and Remark 4.1.14 of [BK01] in the version on the second author's web page; Section 4 of [Kir96] and Section 2.1 of [AFW15] in the arXiv versions; the introduction of [Zha97]; and the passages of the theses [Pil97, Ori03] on the conjecture. We could not read [Yam95], [Sok97], [LL96], [Tak96], [ZC96], [KT93] and [Jef92]; what we say about them is taken from their titles and abstracts, from the summaries of [Yam95, Sok97] in zbMATH, and from [GP98, HT04, Oht02, Zha97]. Of [Kur08] we read only the list of references. Of [KM91] we consulted only the statements on τ_3 , τ_4 and on connected sums. The works [Wit89, RT91, KP84, Wen98, Han01, GS99, Hum90, Rei35, Bro60, Whi41] are cited for standard facts and were not consulted for this note. The bibliographic data were checked with Crossref and arXiv.

Priority. We searched arXiv, Crossref, OpenAlex and zbMATH, the lists of works citing [GP98] and [HT04], and the web. We found no counterexample to the conjecture and no proof of the lens-space statement for a group other than $SU(2)$. The five works citing [GP98] that are listed in OpenAlex do not treat other groups. The theses [Pil97] and [Ori03] describe the general case as open; the second one, of 2003, says that it is neither proved nor disproved. A web search led to one further item, a preprint of Kuriya with the title "The LMO invariant and Guadagnini-Pilo's conjecture for lens spaces", which is cited in [Kur08]. We could not obtain it, and we found no record of it in the four databases. According to the summary shown by the search engine, it computes the LMO invariant of lens spaces and proves, under some conditions and for every simply connected simple group, a version of the conjecture for the perturbative invariants τ^{PG} of lens spaces; these are power series related to the projective invariants τ_r^{PG} [Oht02, Section 7.5]. We do not know its precise statements. The counterexamples of Theorem 1.1 are not affected by it; we cannot say how it is

related to Theorem 1.2(a). Hansen and Takata announce in [HT04, Remark 5.3] a paper with detailed calculations of the Gauss sums $G_q(0)$ and with further results on the separation of lens spaces; we could not find it. The same remark says that the $\mathfrak{sl}_4$ invariant with $\kappa = 6$ distinguishes the lens spaces $L(64, 9)$ and $L(64, 25)$. That statement is about the invariants themselves. For the question studied here we note only that the absolute values of these two invariants are equal: an exact computation with the surgery formula gives $2\sqrt{2}$ for both (`lens64_check.py`, `r2_exact_thm11.py`), and in floating point the absolute values of this pair agree for all $\kappa \leq 100$. So this pair is not a counterexample to the conjecture. Once the formula of Hansen and Takata is available, the mechanism described here is elementary, and it may be known to specialists. This negative search is not a proof of priority, and we claim none.

AI-assisted tools were used to search the problem corpus and the literature, to find the counterexamples and the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

References

- [AFW15] M. Aschenbrenner, S. Friedl, and H. Wilton. *3-Manifold Groups*. EMS Series of Lectures in Mathematics. European Mathematical Society, Zürich, 2015. Preprint: arXiv:1205.0202. doi:10.4171/154.
- [BK01] B. Bakalov and A. Kirillov, Jr. *Lectures on Tensor Categories and Modular Functors*, volume 21 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2001. doi:10.1090/ulect/021.
- [Bro60] E. J. Brody. The topological classification of the lens spaces. *Annals of Mathematics*, 71(1):163–184, 1960. doi:10.2307/1969884.
- [GP98] E. Guadagnini and L. Pilo. Three-manifold invariants and their relation with the fundamental group. *Communications in Mathematical Physics*, 192(1):47–65, 1998. Preprint: arXiv:hep-th/9612090. doi:10.1007/s002200050290.
- [GS99] R. E. Gompf and A. I. Stipsicz. *4-Manifolds and Kirby Calculus*, volume 20 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1999. doi:10.1090/gsm/020.
- [Han01] S. K. Hansen. Reshetikhin–Turaev invariants of Seifert 3-manifolds and a rational surgery formula. *Algebraic & Geometric Topology*, 1:627–686, 2001. doi:10.2140/agt.2001.1.627.
- [HT04] S. K. Hansen and T. Takata. Reshetikhin–Turaev invariants of Seifert 3-manifolds for classical simple Lie algebras. *Journal of Knot Theory and Its Ramifications*, 13(5):617–668, 2004. Preprint: arXiv:math/0209403. doi:10.1142/S0218216504003342.
- [Hum90] J. E. Humphreys. *Reflection Groups and Coxeter Groups*, volume 29 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1990. doi:10.1017/CB09780511623646.
- [Jef92] L. C. Jeffrey. Chern–Simons–Witten invariants of lens spaces and torus bundles, and the semiclassical approximation. *Communications in Mathematical Physics*, 147(3):563–604, 1992. doi:10.1007/BF02097243.

- [Kir96] A. Kirillov, Jr. On an inner product in modular tensor categories. *Journal of the American Mathematical Society*, 9(4):1135–1169, 1996. Preprint: arXiv:q-alg/9508017. doi:10.1090/S0894-0347-96-00210-X.
- [KM91] R. Kirby and P. Melvin. The 3-manifold invariants of Witten and Reshetikhin–Turaev for $sl(2, \mathbb{C})$. *Inventiones Mathematicae*, 105(1):473–545, 1991. doi:10.1007/BF01232277.
- [KP84] V. G. Kac and D. H. Peterson. Infinite-dimensional Lie algebras, theta functions and modular forms. *Advances in Mathematics*, 53(2):125–264, 1984. doi:10.1016/0001-8708(84)90032-X.
- [KT93] T. Kohno and T. Takata. Symmetry of Witten’s 3-manifold invariants for $sl(n, \mathbb{C})$. *Journal of Knot Theory and Its Ramifications*, 2(2):149–169, 1993. doi:10.1142/S0218216593000106.
- [Kur08] T. Kuriya. On the LMO conjecture. Preprint, arXiv:0803.1732, 2008.
- [LL96] B.-H. Li and T.-J. Li. Generalized Gaussian sums: Chern–Simons–Witten–Jones invariants of lens-spaces. *Journal of Knot Theory and Its Ramifications*, 5(2):183–224, 1996. doi:10.1142/S0218216596000151.
- [Oht02] T. Ohtsuki. Problems on invariants of knots and 3-manifolds. In *Invariants of Knots and 3-Manifolds (Kyoto 2001)*, volume 4 of *Geometry & Topology Monographs*, pages 377–572. Geometry & Topology Publications, Coventry, 2002. Edited by T. Ohtsuki, with contributions by many authors. Preprint: arXiv:math/0406190. doi:10.2140/gtm.2002.4.377.
- [Ori03] D. Oriti. *Spin foam models of quantum spacetime*. PhD thesis, University of Cambridge, 2003. arXiv:gr-qc/0311066.
- [Pil97] L. Pilo. *Chern–Simons field theory and invariants of 3-manifolds*. PhD thesis, Scuola Normale Superiore, Pisa, 1997. arXiv:hep-th/9806064.
- [Rei35] K. Reidemeister. Homotopieringe und Linsenräume. *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, 11(1):102–109, 1935. doi:10.1007/BF02940717.
- [RT91] N. Reshetikhin and V. G. Turaev. Invariants of 3-manifolds via link polynomials and quantum groups. *Inventiones Mathematicae*, 103(1):547–597, 1991. doi:10.1007/BF01239527.
- [Sok97] M. V. Sokolov. Which lens spaces are distinguished by Turaev–Viro invariants. *Mathematical Notes*, 61(3):384–387, 1997. Translated from *Matematicheskie Zametki* 61 (1997), no. 3, 468–470. doi:10.1007/BF02355426.
- [Tak96] T. Takata. On quantum $PSU(n)$ -invariants for lens spaces. *Journal of Knot Theory and Its Ramifications*, 5(6):885–901, 1996. doi:10.1142/S0218216596000497.
- [Tur10] V. G. Turaev. *Quantum Invariants of Knots and 3-Manifolds*, volume 18 of *De Gruyter Studies in Mathematics*. De Gruyter, Berlin, second revised edition, 2010. doi:10.1515/9783110221848.
- [ula26] ulamai. UnsolvedMath. Hugging Face dataset, <https://huggingface.co/datasets/ulamai/UnsolvedMath>, 2026. Version 1.6.0; record AMR-103-0120.

- [Wen98] H. Wenzl. C^* tensor categories from quantum groups. *Journal of the American Mathematical Society*, 11(2):261–282, 1998. doi:10.1090/S0894-0347-98-00253-7.
- [Whi41] J. H. C. Whitehead. On incidence matrices, nuclei and homotopy types. *Annals of Mathematics*, 42(5):1197–1239, 1941. doi:10.2307/1970465.
- [Wit89] E. Witten. Quantum field theory and the Jones polynomial. *Communications in Mathematical Physics*, 121(3):351–399, 1989. doi:10.1007/BF01217730.
- [Yam95] S. Yamada. The absolute value of the Chern–Simons–Witten invariants of lens spaces. *Journal of Knot Theory and Its Ramifications*, 4(2):319–327, 1995. doi:10.1142/S0218216595000156.
- [ZC96] R. B. Zhang and A. L. Carey. Quantum groups at odd roots of unity and topological invariants of 3-manifolds. *Communications in Mathematical Physics*, 182(3):619–636, 1996. doi:10.1007/BF02506419.
- [Zha97] R. B. Zhang. Topological invariants for lens spaces and exceptional quantum groups. *Letters in Mathematical Physics*, 41(1):1–11, 1997. Preprint: arXiv:q-alg/9612034. doi:10.1023/A:1007366912839.