

On a Problem of Ferrand and Stoimenow Answered in Print by Rutherford and Ito: A Sharper Signature Bound for the HOMFLY Polynomial of Alternating Links, and a Short Proof that Quasipositive Alternating Links Are Positive

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Abstract

Problem 1.16 of Ohtsuki's list (E. Ferrand, A. Stoimenow) asks whether every alternating link L satisfies $\sigma(L) \geq \min \deg_l P_L \geq \min \deg_a F_L(a^{-1}, z)$, where σ is the signature, $P_L(l, m)$ the HOMFLY polynomial and $F_L(a, z)$ the Kauffman polynomial. Both inequalities are in print: the second is a corollary of Rutherford (2006), based on a proposition of Ng, and the first is a theorem of Ito (2026). This note does not solve the problem; it gives a short proof of the first inequality in a sharper form. For a connected reduced alternating diagram D of L with writhe w , n_- negative crossings, s Seifert circles and s_A circles in the all- A state, put $d_-(D) = \frac{1}{2}(n_- + s - s_A)$. Then $d_-(D)$ is a non-negative integer, positive if D has a negative crossing, and

$$w - s + 1 \leq \min \deg_l P_L \leq \sigma(L) - d_-(D) = w - s + 1 + d_-(D).$$

In particular $\sigma(L) = \min \deg_l P_L$ if and only if D is positive. The proof compares degrees under the substitution which turns P into the Jones polynomial; its ingredients are classical (Morton's bound for the degree in m , the degree formula of Kauffman, Murasugi and Thistlethwaite, Traczyk's formula for the signature). The case of equality, and with it part (i) of a conjecture of Ito for alternating links and part (ii) for reduced alternating diagrams, also follows from Ito's theorem and an inequality in a paper of Stoimenow (2026), as we explain; the bound with $d_-(D)$ is stronger when $d_-(D) \geq 3$, and it gives a positive answer to a question of Orevkov (2020). With an elementary lemma on the signature and the nullity under a band move it yields a short proof that every reduced alternating diagram of a quasipositive alternating link is positive. This is the alternating case of the statement that homogeneous quasipositive links are positive, for which Stoimenow has announced a proof in a text on his home page; the question goes back to Baader (2005). We present a short independent proof of that case, not a new theorem. Exact computations, among them all prime alternating links with at most 14 crossings in all orientations and more than 50 000 quasipositive braid words, are tests of the statements; no proof depends on them. This is an unrefereed note.

1 Introduction

1.1 The problem and its status

Problem 1.16 of Ohtsuki's list of problems on invariants of knots and 3-manifolds [Oht02], posed by E. Ferrand and A. Stoimenow, asks whether every alternating link L satisfies

$$\sigma(L) \geq \min \deg_l P_L(l, m) \geq \min \deg_a F_L(a^{-1}, z). \quad (1)$$

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Here σ is the signature, P the HOMFLY (skein) polynomial and F the Kauffman polynomial, in the conventions of the list (Section 2). *The answer is yes, and both inequalities are in print; they are not results of this note.* The second inequality is Corollary 5.4 of Rutherford [Rut06], which rests on a proposition of Ng [Ng05]; it also follows from a theorem of Tanaka [Tan06] (Section 2.5). The first inequality is a theorem of Ito [Ito26] (Theorem 5 of the arXiv version), who states it as the solution of this half of Problem 1.16 and of a conjecture of Stoimenow [Sto02]. The problem is record AMR-103-0016 of the UnsolvedMath collection [ula26], where it is listed as open; this label is out of date. The present note gives a short proof of the first inequality in a sharper form, and two applications.

1.2 Results

All links are oriented and lie in S^3 . For a diagram D we write n_+ , n_- for the numbers of positive and negative crossings, $n = n_+ + n_-$, $w = n_+ - n_-$ for the writhe, s for the number of Seifert circles, and s_A , s_B for the numbers of circles of the all- A and the all- B state of the Kauffman bracket (Section 2.1). We put

$$d_-(D) = \frac{1}{2}(n_- + s - s_A), \quad d_+(D) = \frac{1}{2}(n_+ + s - s_B).$$

Theorem 1.1. *Let D be a connected reduced alternating diagram of an oriented link L .*

(a) *$d_-(D)$ is a non-negative integer, and $d_-(D) \geq 1$ if D has a negative crossing.*

(b) *We have*

$$w - s + 1 \leq \min \deg_l P_L \leq \sigma(L) - d_-(D) = w - s + 1 + d_-(D). \quad (2)$$

(c) *Consequently $\sigma(L) \geq \min \deg_l P_L$, with equality if and only if D is positive, that is, $n_- = 0$. Moreover $\min \deg_l P_L \equiv \sigma(L) \pmod{2}$, so that $\min \deg_l P_L \leq \sigma(L) - 2$ if D is not positive.*

(d) *Mirror form: $\sigma(L) + d_+(D) \leq \max \deg_l P_L \leq w + s - 1$, and $d_+(D) + d_-(D) = s - 1$.*

The left inequality of (2) and the right one of (d) are the Morton–Franks–Williams inequalities [Mor86, FW87] and are only recalled. The upper bound in (2) says that $\min \deg_l P_L$ lies in the lower half of the interval $[w - s + 1, \sigma(L)]$, whose length is $2d_-(D)$. By Proposition 3.6, $d_-(D)$ is the number of negative edges in any spanning tree of the Seifert graph of D , and $d_-(D) = s_+(D) - 1$, where $s_+(D)$ is the number of connected components of D after all negative crossings have been smoothed; in these terms Ito’s bound for alternating diagrams is $w - s + 2s_+(D) - 1 = \sigma(L)$. Part (a) holds for all diagrams, but the upper bound in (2) fails for diagrams which are not reduced (Remark 3.7). The case of equality in (c), and the bound $\min \deg_l P_L \leq \sigma(L) - 2$ for diagrams which are not positive, can also be derived from statements in print (Section 6.4); the bound with $d_-(D)$ is stronger when $d_-(D) \geq 3$. The next statement contains the first inequality of (1) in the form of [Ito26, Theorem 5].

Corollary 1.2. *Let L be an alternating link and D a reduced alternating diagram of L with connected components $D_1, \dots, D_r$. Then*

$$\min \deg_l P_L \leq \sigma(L) - (r - 1) - \sum_{i=1}^r d_-(D_i).$$

In particular $\sigma(L) \geq \min \deg_l P_L$, with equality if and only if D is connected and positive; and $\min \deg_l P_L + r - 1 \leq \sigma(L)$, with equality if and only if D is positive. The number r is the number of split components of L .

A link is *quasipositive* if it is the closure of a braid $\prod_{j=1}^k w_j \sigma_{i_j} w_j^{-1}$, a product of conjugates of the standard generators $\sigma_1, \dots, \sigma_{N-1}$ of the braid group B_N , the closure of σ_i having a positive crossing. A link is *positive* if it has a diagram without negative crossings.

Corollary 1.3 (a short proof of the alternating case of [Sto26a, Theorem 5.24]). *Let L be an alternating link which is quasipositive. Then every reduced alternating diagram of L is positive. More precisely, if L is the closure of a product of k conjugates of standard generators in B_N , and D is a reduced alternating diagram of L with r connected components, then D is positive, $\sigma(L) = k - N + r$ and $\min \deg_i P_L = k - N + 1$.*

Since positive links are strongly quasipositive (Nakamura, Rudolph; see [Ore20]), for an alternating link the properties “quasipositive”, “strongly quasipositive”, “positive” and “every reduced alternating diagram is positive” are thus equivalent. The proof of Corollary 1.3 uses Theorem 1.1, the Morton–Franks–Williams inequality for a closed braid, and the fact that a band move changes exactly one of the signature and the nullity, by one (Lemma 4.1); it uses no four-dimensional topology.

Corollary 1.4. (a) *Let D be a connected reduced alternating diagram of a link L and let β be a braid on b strands with closure L and exponent sum $w(\beta)$. Then $w(\beta) - b + 1 \leq w(D) - s(D) + 1 + d_-(D)$. For b the braid index of L this is the inequality $2r^-(D) \leq d_-(D)$ of Orevkov [Ore20, (4)]; so the answer to [Ore20, Question 8] is positive.*

(b) *Part (i) of [Ito26, Conjecture 1] holds for alternating links: an alternating link L satisfies $\min \deg_i P_L = 1 - \chi_4(L)$ if and only if it is positive.*

(c) *Part (ii) of [Ito26, Conjecture 1] holds for reduced alternating diagrams: for such a diagram D of L the inequality of [Ito26, Theorem 2] is an equality if and only if D is positive.*

The “if” directions in (b) and (c) are known facts about positive diagrams. Part (ii) of the conjecture concerns all homogeneous diagrams of L ; we prove it only for the reduced alternating ones. Parts (b) and (c) also follow from the statements in print which are discussed in Section 6.4; part (a) needs the bound with $d_-(D)$.

1.3 The history of the positivity question

The question answered by Corollary 1.3 has the following history, as far as we could trace it. Baader asked whether alternating quasipositive knots have positive diagrams, noting that this is so up to ten crossings [Baa05b, question 4 of the list closing Section 4 of the arXiv version]. The version for homogeneous links is attributed to [Baa05a] in [Ito22, Ito26]. Boileau, Boyer and Gordon proved that an alternating link is strongly quasipositive if and only if it is special alternating [BBG19, Corollary 7.3], and asked whether there is a quasipositive alternating link which is not strongly quasipositive [BBG19, Question 7.5]. Orevkov [Ore20] proved that an alternating diagram D of a quasipositive link is positive if its number of Seifert circles equals the braid index of the link (his Theorem 2; by a theorem of Diao, Heteyi and Liu quoted there, this means that no two Seifert circles of D are joined by exactly one crossing) and, more generally, that a reduced alternating diagram is positive if $2r^-(D) \leq d_-(D)$ (his Theorem 6; see Section 6.1); he asked whether this inequality holds for every reduced alternating diagram (his Question 8). Ito [Ito22] proved that a homogeneous diagram of a quasipositive link is positive if it realises the maximal self-linking number, and remarked that a conjecture of Murasugi and Przytycki on the braid index would imply a positive answer for alternating links. Ozbagci [Ozb24] characterised the quasipositive two-bridge knots. Ito [Ito26] showed that each part of his Conjecture 1 would imply that quasipositive homogeneous links are positive, and wrote that the question “remains open even for alternating links”. Finally, the version of Stoimenow’s paper [Sto26b] on his home page [Sto26a], dated 11 March 2026, contains a Section 5 which is not in the published version; its Theorem 5.24 states, with a proof, that every homogeneous quasipositive link is positive. We did not check the proof. Its alternating case is Corollary 1.3.

1.4 What is claimed, and what is not

The two inequalities of (1) are due to Ito and to Rutherford, Ng and Tanaka. Corollary 1.3 is the alternating case of a statement announced, with a proof, by Stoimenow; we claim a short independent proof, not the statement. The case of equality in Theorem 1.1(c), the bound $\min \deg_i P_L \leq \sigma(L) - 2$ for diagrams which are not positive, and Corollary 1.4(b), (c) are not stated in the sources which we read, but they follow from two statements in print: Ito's theorem on the coefficient of the highest power of m [Ito26] and an inequality in the proof of [Sto26b, Corollary 4.16]; with [Ito26, Theorem 4] these also give the statement of Corollary 1.3 (Section 6.4). We do not claim them. The upper bound in (2) with $d_-(D)$, which is stronger when $d_-(D) \geq 3$, and Corollary 1.4(a) were not found in the sources which we read; the nearest statements are discussed in Section 6. The proof of (2) takes half a page and uses only classical facts, so the bound may well be known to specialists; we make no claim of priority (see "Scope and priority" at the end). The computations of Section 7 are tests.

2 Conventions

Signs matter throughout: with another sign of the signature, or with l^{-1} for l , or $F_L(a, z)$ for $F_L(a^{-1}, z)$, the chain (1) fails already for a trefoil. We therefore fix all conventions and normalise them on small examples (Table 1).

2.1 Diagrams, smoothings and states

A *diagram* is an oriented link diagram on the sphere S^2 ; it may be disconnected and may contain circles without crossings. A crossing is *positive* if the over-strand, turned counterclockwise by less than π onto the under-strand, arrives with the direction of the under-strand, and *negative* otherwise (Figure 1). The two corners of a crossing which the over-strand sweeps during this turn are its *A-corners*, the other two its *B-corners*. The *A-smoothing* replaces the crossing by two arcs so that the *A-corners* are joined, each arc cutting off one *B-corner*; the *B-smoothing* joins the *B-corners*. A *state* S is a choice of a smoothing at every crossing; it turns D into $|S|$ disjoint circles. The all-*A* state has $s_A(D)$ circles, the all-*B* state $s_B(D)$. The *oriented smoothing* respects the orientation; applied to all crossings it gives the *Seifert state*, whose circles are the $s(D)$ *Seifert circles*. We use three elementary facts.

- (S1) *At a positive crossing the oriented smoothing is the A-smoothing, at a negative crossing it is the B-smoothing.* Let the over-strand of a positive crossing point east and the under-strand north. The oriented smoothing joins the west end to the north end and the south end to the east end, so it cuts off the north-west and the south-east corner; and the over-strand, turned counterclockwise, sweeps the north-east and the south-west corner. A negative crossing is the mirror image.
- (S2) *Changing the smoothing at one crossing of a state changes the number of circles by exactly one:* two circles are merged if the two arcs of the state at this crossing lie on different circles, and one circle is split into two otherwise. In the second case let $p_1, \dots, p_4$ be the ends of the two arcs p_1p_2, p_3p_4 on a small circle around the crossing, in cyclic order. Outside the small disc the circle through them consists of two disjoint arcs; they cannot join p_1 to p_3 and p_2 to p_4 , so they join p_2 to p_3 and p_4 to p_1 , and the other smoothing, with arcs p_2p_3 and p_4p_1 , gives two circles.
- (S3) *The two arcs of the Seifert state at a crossing lie on different Seifert circles.* They are parallel and equally oriented, and the point where the crossing was lies on the right of one and on the

left of the other. If both were on one Seifert circle C , this point would lie in the disc on the right of C and in the disc on its left.

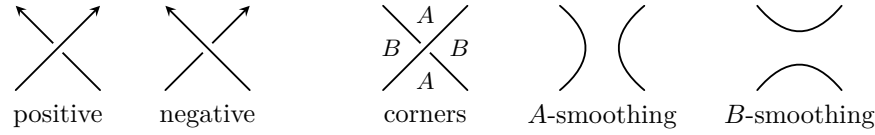


Figure 1: The sign of a crossing; the A - and B -corners of a crossing and its two smoothings. If both strands of the third crossing point upwards, it is positive and its oriented smoothing is the A -smoothing.

The *Seifert graph* $\Gamma(D)$ has the Seifert circles as vertices and one edge for every crossing, joining the two Seifert circles at this crossing and carrying its sign; by (S3) it has no loops, and it is connected if and only if D is. A crossing c is *nugatory* if some simple closed curve on S^2 meets D only in c and passes there through two opposite corners. A diagram is *reduced* if it has no nugatory crossing, *positive* if $n_- = 0$, and *alternating* if every edge between two crossings passes over at one end and under at the other. A nugatory crossing of an alternating diagram can be removed by a twist, and the result is alternating; so every alternating link has a reduced alternating diagram.

Lemma 2.1. *Let D be a connected alternating diagram with $n \geq 1$ crossings.*

- (a) *Every face of D has only A -corners or only B -corners. The circles of the all- A state correspond bijectively to the faces with B -corners, those of the all- B state to the faces with A -corners. Hence $s_A + s_B = n + 2$.*
- (b) *If D is reduced, then at every crossing the two arcs of the all- A state lie on different circles, and so do the two arcs of the all- B state.*
- (c) *The oriented smoothing of a crossing of an alternating diagram yields an alternating diagram.*

Proof. (a) Let e be an edge, traversed from a crossing c_1 to a crossing c_2 , and f the face on its left. The over-strand of a crossing, turned counterclockwise, sweeps the corner on the left of each of its two halves, seen from the crossing. Hence the corner of f at c_1 is an A -corner if and only if e leaves c_1 on the over-strand, and the corner of f at c_2 is an A -corner if and only if e arrives at c_2 on the under-strand. For an alternating diagram the two conditions are equivalent. So consecutive corners of f are of the same kind, and all corners of f are, because the boundary of a face of a connected diagram is one closed walk. The arcs of the all- A state cut off the B -corners; thus every circle of this state runs inside one face with B -corners, parallel to its boundary walk, and every such face contains exactly one circle. The same holds for the all- B state. Finally D has $n + 2$ faces, by Euler's formula for a connected 4-valent graph on the sphere with n vertices and $2n$ edges.

(b) The two arcs of the all- A state at a crossing c cut off the two B -corners of c . If they lie on one circle, these corners belong to the same face, and a simple closed curve through c which otherwise runs inside this face shows that c is nugatory.

(c) The oriented smoothing at c joins the edge which arrives at c on the over-strand to the edge which leaves c on the under-strand. The first left its initial crossing on the under-strand and the second arrives at its final crossing on the over-strand, so the new edge alternates (or it is a circle without crossings). The same holds for the other pair of edges. $\square$

For a diagram consisting of one circle without crossings, $s = s_A = s_B = 1$, and the formula $s_A + s_B = n + 2$ remains true.

2.2 The HOMFLY polynomial

The HOMFLY polynomial $P_L(l, m) \in \mathbb{Z}[l^{\pm 1}, m^{\pm 1}]$ is determined by

$$l^{-1}P_{L_+}(l, m) - lP_{L_-}(l, m) = mP_{L_0}(l, m), \quad P_O = 1, \quad (3)$$

where L_+ , L_- , L_0 have diagrams which differ at one crossing, which is positive, negative and smoothed in the oriented way, and O is the unknot. This is the convention of [Oht02, §1.2], and that of [Ito26] with $(v, z) = (l, m)$. For a crossing of sign $\varepsilon = \pm 1$ of a diagram D , (3) reads

$$P(D) = l^{2\varepsilon}P(D^\times) + \varepsilon l^\varepsilon m P(D^0), \quad (4)$$

where $D^\times$ is D with this crossing switched and D^0 is D with this crossing smoothed. Put $\delta = (l^{-1} - l)/m$. Applying (3) to a kink gives $P_{L \sqcup O} = \delta P_L$; hence the unlink with μ components has polynomial $\delta^{\mu-1}$, and $P_{L \sqcup L'} = \delta P_L P_{L'}$ for a split union, since both sides satisfy (3) in L' and agree when L' is an unlink. The mirror image L^* of L has $P_{L^*}(l, m) = P_L(l^{-1}, -m)$, because the right-hand side satisfies (3) for L^* .

Theorem 2.2 (Morton [Mor86]; Franks and Williams [FW87]). *For every diagram D of a link L , with writhe w and s Seifert circles,*

$$w - s + 1 \leq \min \deg_l P_L \quad \text{and} \quad \max \deg_l P_L \leq w + s - 1.$$

For closed braid diagrams this is due to both sources, for arbitrary diagrams to Morton. We use the general case for the left inequality in (2) and for the “if” part of Theorem 1.1(c), and the case of closed braids in Sections 5 and 6. The directions of the inequalities are those which belong to (3): for $k \geq 2$ the closure of σ_1^k in B_2 has $\min \deg_l P = k - 1$ and $\max \deg_l P = k + 1$ (Table 1 for $k = 2, 3$).

2.3 The Jones polynomial and the Kauffman bracket

We define $V_L(t) = P_L(t, t^{1/2} - t^{-1/2})$; this is the Jones polynomial [Oht02, §1.2], with $V = t + t^3 - t^4$ for the positive trefoil. The Kauffman bracket of a diagram D is $\langle D \rangle = \sum_S A^{a(S)-b(S)} (-A^2 - A^{-2})^{|S|-1}$, the sum being over all states, where $a(S)$ and $b(S)$ are the numbers of A - and B -smoothings of S . Kauffman [Kau87] proved that $(-A^3)^{-w(D)} \langle D \rangle$ is an invariant of oriented links. In our conventions

$$V_L(t) = (-A^3)^{-w(D)} \langle D \rangle \quad \text{at } A = t^{-1/4}. \quad (5)$$

Indeed, $\langle D \rangle = A \langle D_A \rangle + A^{-1} \langle D_B \rangle$ for the two smoothings of a crossing, and switching the crossing exchanges them. For a positive crossing, (S1) gives $A \langle D_+ \rangle - A^{-1} \langle D_- \rangle = (A^2 - A^{-2}) \langle D_0 \rangle$; for $f(D) = (-A^3)^{-w(D)} \langle D \rangle$ this is $-A^4 f(D_+) + A^{-4} f(D_-) = (A^2 - A^{-2}) f(D_0)$, which at $A = t^{-1/4}$ is (3) with $(l, m) = (t, t^{1/2} - t^{-1/2})$; and $f(O) = 1$.

2.4 Signature and nullity

A *Seifert surface* of L is a compact oriented surface $F \subset S^3$ without closed components and with $\partial F = L$; it need not be connected. Let $\theta(x, y) = \text{lk}(x, y^+)$ be its Seifert form on $H_1(F)$ and $W_F = \theta + \theta^T$. For connected F the signature and the nullity of W_F are invariants of L [Mur65]; we write $\sigma(L) = \pm \text{sign } W_F$ and $\eta(L) = \text{null } W_F$, the sign being fixed once and for all so that the positive trefoil has $\sigma = +2$. This is the convention of [Ng05, Ito26]; the tables [LM26] and the papers [Tra04, Ore20] use the opposite sign. The mirror image has $\sigma(L^*) = -\sigma(L)$.

Lemma 2.3. *If F is a Seifert surface of L with q components, then $\sigma(L) = \pm \text{sign } W_F$, with the same sign as above, and $\eta(L) = \text{null } W_F + q - 1$. Consequently the unlink with μ components has $(\sigma, \eta) = (0, \mu - 1)$, and for a split union $\sigma(L \sqcup L') = \sigma(L) + \sigma(L')$ and $\eta(L \sqcup L') = \eta(L) + \eta(L') + 1$.*

Proof. Let $q \geq 2$. The complement of F is connected, as F has no closed component. Join two components of F by an arc γ whose interior misses F , remove a small disc from F at each end of γ and add the boundary of a thin tube around γ ; the ends of γ can be chosen so that the new surface F' is oriented. It has $q - 1$ components, and $H_1(F') = H_1(F) \oplus \mathbb{Z}\mu$ for a meridian circle μ of the tube. The Seifert form of F' restricts to θ on $H_1(F)$. The circle μ and its push-offs are parallel circles around γ : they bound discs across the tube which miss the cycles of F and their push-offs, and two of them do not link. So $W_{F'} = W_F \oplus (0)$. After $q - 1$ steps the surface is connected, with form $W_F \oplus (0)^{q-1}$. For the consequences take discs for the unlink, and two connected Seifert surfaces separated by a sphere for $L \sqcup L'$. $\square$

2.5 The Kauffman polynomial and the second inequality

The Kauffman polynomial is $F_L(a, z) = a^{-w(D)}[D]$, where $[D]$ is the invariant of regular isotopy of unoriented diagrams with $[D_+] + [D_-] = z([D_0] + [D_\infty])$, $[O] = 1$, and such that a kink of writhe $+1$ multiplies $[D]$ by a [Oht02, §1.2]. Thus $\min \deg_a F_L(a^{-1}, z) = -\max \deg_a F_L(a, z)$.

Theorem 2.4 (Rutherford [Rut06], with Ng [Ng05]; Tanaka [Tan06]). *Every alternating link L satisfies $\min \deg_l P_L(l, m) \geq \min \deg_a F_L(a^{-1}, z)$.*

We only report this theorem and explain why it is the second inequality of (1). A Legendrian link with Thurston–Bennequin number tb and rotation number r satisfies $tb + |r| \leq \min \deg_l P_L - 1$ and $tb \leq \min \deg_a F_L(a^{-1}, z) - 1$, in the conventions of [Oht02]; see [Fer02], where Ferrand asked whether the second bound is at least as good as the first for alternating knots. Rutherford [Rut06] normalises P and F by these bounds, which he writes $\beta < -\deg_a P$ and $\beta < -\deg_a F$ for the Thurston–Bennequin number β . So the fourth corollary of his Section 5 (Corollary 5.4 in the numbering used in [Ito26]), $\deg_a P \leq \deg_a F$ for alternating links, is Theorem 2.4. Its proof: every alternating link has a Legendrian representative with a ruling (Ng [Ng05, Section 4]; quoted in [Rut06] as Proposition 11), which therefore attains the second bound [Rut06]; and it satisfies the first. The two trefoils confirm the translation: their maximal Thurston–Bennequin numbers are 1 and -6 , and the two bounds are 1, 1 and $-5, -6$ (Table 1). Tanaka [Tan06] proved the sharpness of the second bound for alternating links directly. By [Ng05, Theorem 4 and the remark after its proof], the maximal Thurston–Bennequin number of a non-split alternating link is $\sigma(L) - n_-(D) - 1$ for a reduced alternating diagram D . So, with Theorem 1.1, the chain (1) becomes

$$\min \deg_a F_L(a^{-1}, z) = \sigma(L) - n_- \leq \min \deg_l P_L \leq \sigma(L) - d_-(D) \leq \sigma(L) \quad (6)$$

for a connected reduced alternating diagram, and $d_-(D) \leq n_-$ by Lemma 3.5.

3 Five lemmas and the proof of the signature bound

Lemma 3.1 (degree comparison). *For every link L , $\min \deg_t V_L \geq \min \deg_l P_L - \frac{1}{2} \max \deg_m P_L$.*

Proof. Put $x = t^{1/2}$. The substitution $(l, m) \mapsto (x^2, x - x^{-1})$ sends a monomial $l^i m^j$ to $x^{2i}(x - x^{-1})^j = (-1)^j x^{2i-j}(1 - x^2)^j$. For every integer j , positive or negative (negative powers of m occur for links with several components), $(1 - x^2)^j$ is a power series in x with constant term 1. So in the field $\mathbb{Q}((x))$ of Laurent series the image of $l^i m^j$ has order exactly $2i - j$, and V_L , a finite sum of such images, has order at least the minimum of $2i - j$ over the monomials of P_L . This minimum is at least $2 \min \deg_l P_L - \max \deg_m P_L$, and the order of V_L in x is $2 \min \deg_t V_L$. $\square$

Lemma 3.2 (Morton’s bound). *Let D be a diagram of a link L with μ components, with n crossings and s Seifert circles. Then $\max \deg_m P_L \leq n - s + 1$, and every monomial $l^i m^j$ of P_L has $i \equiv j \equiv \mu - 1 \pmod{2}$.*

link, diagram	(n_+, n_-)	(s, s_A, s_B)	σ	d_-	$w - s + 1$	e_P	$\sigma - d_-$	e_F
positive trefoil T_+ (closure of σ_1^3)	(3, 0)	(2, 2, 3)	2	0	2	2	2	2
negative trefoil T_-	(0, 3)	(2, 3, 2)	-2	1	-4	-4	-3	-5
positive Hopf link H_+ (closure of σ_1^2)	(2, 0)	(2, 2, 2)	1	0	1	1	1	1
negative Hopf link H_-	(0, 2)	(2, 2, 2)	-1	1	-3	-3	-2	-3
figure-eight knot 4_1	(2, 2)	(3, 3, 3)	0	1	-2	-2	-1	-2
6_1	(4, 2)	(5, 5, 3)	0	1	-2	-2	-1	-2
$T(2, 4)$, antiparallel, negative	(0, 4)	(4, 2, 4)	-1	3	-7	-5	-4	-5
unknot with one negative kink	(0, 1)	(2, 1, 2)	0	1	-2	0	-1	0

Table 1: Normalisation of the conventions; $e_P = \min \deg_l P_L(l, m)$ and $e_F = \min \deg_a F_L(a^{-1}, z)$. The first seven diagrams are connected, reduced and alternating; all links have nullity 0. The polynomials P are $2l^2 - l^4 + l^2 m^2$ for T_+ , $2l^{-2} - l^{-4} + l^{-2} m^2$ for T_- , $lm^{-1} + lm - l^3 m^{-1}$ for H_+ , $l^{-3} m^{-1} - l^{-1} m^{-1} - l^{-1} m$ for H_- , $l^{-2} - 1 - m^2 + l^2$ for 4_1 , $l^{-2} - m^2 - l^2 - l^2 m^2 + l^4$ for 6_1 and $l^{-5} m^{-1} - l^{-3} m^{-1} - l^{-3} m - l^{-1} m$ for $T(2, 4)$; and $V_{T_+} = t + t^3 - t^4$. With the opposite sign of σ the row T_+ of (1) would read $-2 \geq 2$, with l replaced by l^{-1} it would read $2 \geq -4 \geq 2$, and with $F_L(a, z)$ in place of $F_L(a^{-1}, z)$ the row T_- would read $-4 \geq 2$.

The bound is due to Morton [Mor86]; it holds for all diagrams, connected or not, reduced or not. As we could not consult [Mor86] itself, we give a proof.

Proof. Order the components of D and choose a base point on each. Call D *descending* if, when the components are traversed in this order from their base points, every crossing is first met on its over-strand; a descending diagram is a diagram of an unlink. If D is not descending, apply (4) to the first crossing which is first met on its under-strand. The diagram $D^\times$ has one such crossing less, and D^0 , with any order and base points, has one crossing less. By induction on the number of crossings and on the number of crossings first met on the under-strand we obtain finitely many descending diagrams D_λ , each arising from D by switching some crossings and smoothing $j(\lambda)$ others, such that

$$P_L = \sum_{\lambda} \pm l^{w(D)-w(D_\lambda)} m^{j(\lambda)} \delta^{\mu(\lambda)-1}, \quad \delta = (l^{-1} - l)/m, \quad (7)$$

where $\mu(\lambda)$ is the number of components of D_λ . The exponent of l is as stated because a switch contributes $l^{2\varepsilon}$ and lowers the writhe by 2ε , and a smoothing contributes l^ε and lowers it by ε . Switching a crossing and smoothing it in the oriented way do not change the Seifert state; so D_λ has $n - j(\lambda)$ crossings and the same s Seifert circles as D . An oriented smoothing of a crossing merges two components of a link or splits one into two. Applied to the $n - j(\lambda)$ crossings of D_λ this gives $s \leq \mu(\lambda) + n - j(\lambda)$, so the degree of the λ -th term in m is $j(\lambda) - \mu(\lambda) + 1 \leq n - s + 1$. Applied to the $j(\lambda)$ smoothings leading from D to D_λ it gives $\mu(\lambda) \equiv \mu + j(\lambda)$, and $w(D) - w(D_\lambda) \equiv j(\lambda)$; so all exponents of l and of m in the λ -th term are congruent to $\mu - 1$ modulo 2. $\square$

Lemma 3.3 (Kauffman, Murasugi, Thistlethwaite). *Let D be a diagram of L such that at every crossing the two arcs of the all- A state lie on different circles of that state. Then $\max \deg_A \langle D \rangle = n + 2s_A - 2$, with coefficient ± 1 , and*

$$\min \deg_t V_L = \frac{1}{4}(3w - n - 2s_A + 2).$$

By Lemma 2.1(b) this holds for every connected reduced alternating diagram.

For reduced alternating diagrams this is contained in [Kau87, Mur87, Thi87]; diagrams with the stated property are called A -adequate.

Proof. The all- A state contributes $A^n(-A^2 - A^{-2})^{s_A-1}$, whose term of highest degree is $\pm A^{n+2s_A-2}$. A state S with $b \geq 1$ B -smoothings is obtained from the all- A state by changing b smoothings one after the other. By hypothesis and (S2) the first change merges two circles, and each further change adds at most one circle; so $|S| \leq s_A + b - 2$, and the degree of the contribution of S is at most $(n - 2b) + 2(s_A + b - 3) = n + 2s_A - 6$. Hence the term $\pm A^{n+2s_A-2}$ is not cancelled. By (5) the highest power of A in $(-A^3)^{-w}\langle D \rangle$ is $n + 2s_A - 2 - 3w$, and $A = t^{-1/4}$ turns it into the lowest power of t . $\square$

Lemma 3.4 (Traczyk [Tra04]). *Let D be a connected reduced alternating diagram of an oriented link L . Then*

$$\sigma(L) = n_+ - s_A + 1 = s_B - n_- - 1, \quad (8)$$

and $\eta(L) = 0$. In particular L is not a split link.

This is [Tra04, Theorem 2, formula (2)], stated there for connected reduced alternating diagrams of oriented links, with any number of components, in the form $\sigma = -\frac{1}{2}w + \frac{1}{2}(W - B)$. Here W and B are the numbers of white and black regions of a chessboard colouring; they enter Traczyk's proof through Kauffman's formulas for the extreme degrees of the bracket, and comparison with Lemma 3.3 shows $W = s_A$ and $B = s_B$. His signature has the opposite sign (-2 for the positive trefoil). With $s_A + s_B = n + 2$ his formula becomes (8); the rows of Table 1 confirm the signs. Formula (1) of the same theorem is misprinted (a factor $\frac{1}{2}$ in front of the writhe must be deleted), as noted in [Ore20, Ito26]; corrected, it is Proposition 3.6(c). That $\eta(L) = 0$ follows from the appendix of [Tra04], where the determinant is shown to be non-zero for every connected alternating diagram; and a split link has positive nullity by Lemma 2.3.

Lemma 3.5 (circle count). *For every diagram D we have $s_A \equiv s + n_- \pmod{2}$ and $|s_A - s| \leq n_-$; if $n_- \geq 1$, then $s_A \leq s + n_- - 2$. Hence $d_-(D)$ is an integer with $0 \leq d_-(D) \leq n_-$, and $d_-(D) \geq 1$ if $n_- \geq 1$.*

Proof. By (S1) the Seifert state and the all- A state differ exactly at the n_- negative crossings, and by (S2) each change of a smoothing changes the number of circles by one. This gives the first two claims. If $n_- \geq 1$, pass from the Seifert state to the all- A state and change a negative crossing first: by (S3) this merges two circles, and the remaining $n_- - 1$ changes add at most one circle each, so $s_A \leq (s - 1) + (n_- - 1)$. $\square$

Proof of Theorem 1.1. (a) is contained in Lemma 3.5.

(b) The left inequality is Theorem 2.2. By Lemmas 3.1 to 3.3,

$$4 \min \deg_l P_L \leq 4 \min \deg_t V_L + 2 \max \deg_m P_L \leq (3w - n - 2s_A + 2) + 2(n - s + 1) = 3w + n - 2s_A - 2s + 4.$$

Since $3w + n = 4n_+ - 2n_-$, the right-hand side equals $4(n_+ - s_A + 1) - 2(n_- + s - s_A)$, which is $4\sigma(L) - 4d_-(D)$ by Lemma 3.4. Likewise $\sigma(L) - (w - s + 1) = (n_+ - s_A + 1) - (n_+ - n_- - s + 1) = n_- + s - s_A = 2d_-(D)$, which is the equality in (2).

(c) If $n_- \geq 1$, then $d_-(D) \geq 1$ and $\min \deg_l P_L < \sigma(L)$. If $n_- = 0$, then the all- A state is the Seifert state, so $d_-(D) = 0$ and the two outer terms of (2) coincide. For the parity, let μ be the number of components of L . Smoothing the n crossings of D one after the other changes the number of components by one each time and ends with s circles, so $\mu \equiv n + s$; and $\sigma(L) = n_+ - s_A + 1 \equiv n_+ + n_- + s + 1$ by Lemma 3.5. Hence $\sigma(L) \equiv \mu - 1$, and $\min \deg_l P_L \equiv \mu - 1$ by Lemma 3.2.

(d) The mirror image D^* is a connected reduced alternating diagram of L^* with $n_\pm(D^*) = n_\mp(D)$, $w(D^*) = -w(D)$, $s(D^*) = s(D)$ and $s_A(D^*) = s_B(D)$, so $d_-(D^*) = d_+(D)$; moreover $\sigma(L^*) = -\sigma(L)$ and $\min \deg_l P_{L^*} = -\max \deg_l P_L$. So (d) is (b) for D^* . Finally $d_+ + d_- = \frac{1}{2}(n + 2s - s_A - s_B) = s - 1$ by Lemma 2.1(a). $\square$

Proof of Corollary 1.2. The diagram D presents the split union of the links L_i presented by its components D_i . Each D_i is alternating, and it is reduced: a simple closed curve which meets D_i only in a crossing c runs otherwise in one face of D_i ; the other components of D in this face lie in a closed disc inside the face, and an isotopy of the sphere supported in the face moves this disc off the curve, so c would be nugatory in D . (If D_i is a circle without crossings, then L_i is the unknot and all statements of Theorem 1.1 hold trivially, with $d_-(D_i) = 0$.) By Section 2.2, $P_L = \delta^{r-1} \prod_i P_{L_i}$, and $\min \deg_l \delta = -1$; as $\mathbb{Z}[l^{\pm 1}, m^{\pm 1}]$ is an integral domain, $\min \deg_l P_L = \sum_i \min \deg_l P_{L_i} - (r - 1)$. The signature is additive (Lemma 2.3). So Theorem 1.1(b) for the D_i gives the inequality. If all D_i are positive, then $\min \deg_l P_{L_i} = \sigma(L_i)$ for all i by Theorem 1.1(c), and $\min \deg_l P_L + r - 1 = \sigma(L)$; otherwise $\sum_i d_-(D_i) \geq 1$. Each L_i is non-split by Lemma 3.4, so r is the number of split components of L . $\square$

Proposition 3.6. *Let D be a connected alternating diagram of L . Let $s_+(D)$ and $s_-(D)$ be the numbers of connected components of the graphs obtained from the Seifert graph $\Gamma(D)$ by deleting all negative edges, respectively all positive edges; equivalently, of the diagrams obtained from D by smoothing all negative, respectively all positive, crossings.*

- (a) $d_-(D) = s - s_-(D) = s_+(D) - 1$ and $d_+(D) = s - s_+(D) = s_-(D) - 1$.
- (b) Every spanning tree of $\Gamma(D)$ has exactly $d_+(D)$ positive and $d_-(D)$ negative edges.
- (c) If D is reduced, then $\sigma(L) = w - d_+(D) + d_-(D) = w - s + 2s_+(D) - 1$.

Proof. (a) Let D' be the diagram obtained from D by smoothing all positive crossings. By Lemma 2.1(c) it is alternating; it has the n_- negative crossings of D , the same Seifert circles, and $s_-(D)$ connected components. By (S1) the all- A state of D' is the all- A state of D , and the all- B state of D' is the Seifert state. Adding the formula of Lemma 2.1(a) over the components of D' gives $s_A + s = n_- + 2s_-(D)$, that is, $d_-(D) = s - s_-(D)$. For the mirror image this reads $d_+(D) = s - s_+(D)$. As $d_+ + d_- = s - 1$ (the proof of this in Theorem 1.1(d) does not use that D is reduced), we get $s_+ + s_- = s + 1$.

(b) A spanning tree has $s - 1$ edges. Its negative edges form a forest in a graph with s vertices and $s_-(D)$ components, so there are at most $s - s_-(D) = d_-(D)$ of them; likewise there are at most $d_+(D)$ positive ones. As $d_+ + d_- = s - 1$, both bounds are attained.

(c) By Lemma 3.4, $\sigma(L) = w - s + 1 + 2d_- = w - (d_+ + d_-) + 2d_-$, and $2d_- = 2s_+ - 2$. $\square$

Thus $d_-(D)$ is the number $d^-(D)$ of [Ore20] and d_- of [Tra04, Ito26], and $s_+(D)$ is as in [Ito26]. Statement (b) is usually deduced from the fact that alternating diagrams are homogeneous [Cro89]; the proof above does not use it.

Remark 3.7 (the hypotheses). Lemmas 3.1, 3.2 and 3.5 hold for all links and diagrams. The diagram must be alternating for Lemma 3.4 and, through Lemma 2.1, for Lemma 3.3. Lemma 3.4 holds for all connected alternating diagrams, reduced or not; this follows from the theorem of Gordon and Litherland [GL78], by a derivation which each of the three verification runs (see “Verification” below) carried out and which is not included here. But the diagram must be reduced for Lemma 3.3: for the diagram of the unknot with one negative kink, $\sigma = 0$ and $d_- = 1$, but $\min \deg_l P = 0 > \sigma - d_-$ (Table 1). For an A -adequate diagram which is not alternating the three lemmas still give $4 \min \deg_l P_L \leq 3w + n - 2s_A - 2s + 4$, but the right-hand side is no longer $4\sigma - 4d_-$, and the first inequality of (1) fails for some non-alternating knots (Section 7).

Remark 3.8 (examples). Figure 2 shows an example with crossings of both signs. For the antiparallel torus link $T(2, 4)$ with four negative crossings the Seifert graph is a cycle of four negative edges, so $d_- = 3$; here $\sigma = -1$, the bound is -4 and $\min \deg_l P = -5$, while $w - s + 1 = -7$ is smaller than the Kauffman degree $\sigma - n_- = -5$ of (6).

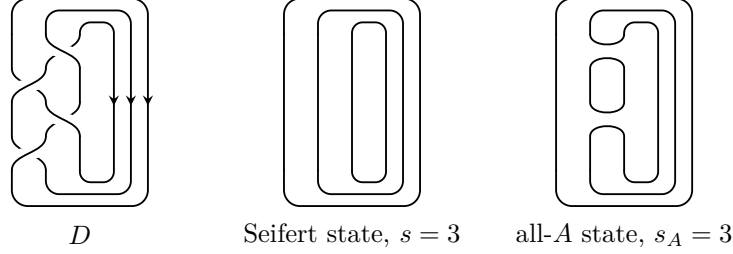


Figure 2: The figure-eight knot as the closure of the braid $\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}$ (read from bottom to top, strands oriented upwards), its Seifert state and its all-A state. Here $n_+ = n_- = 2$ and $s = s_A = s_B = 3$, so $\sigma = n_+ - s_A + 1 = 0$ and $d_- = 1$, and (2) reads $-2 \leq \min \deg_l P \leq -1$; indeed $P = l^{-2} - 1 - m^2 + l^2$.

4 A band move changes the signature or the nullity, by one

Lemma 4.1 (band move). *Let L' be a link with a diagram D' , and let L_0 be the link presented by the diagram obtained from D' by smoothing one crossing in the oriented way. Then exactly one of the two numbers σ and η changes, and it changes by one:*

$$|\sigma(L') - \sigma(L_0)| + |\eta(L') - \eta(L_0)| = 1.$$

That the signature changes by at most one is classical: it is [Mur65, Lemma 7.1], as recalled in [MV20], and it is used in the appendix of [Tra04]. We could not consult [Mur65] and do not know whether the statement with the nullity is formulated there; in any case the proof is short.

Proof. Let F' be the surface which Seifert's algorithm produces from D' : a disc for every Seifert circle and a half-twisted band for every crossing. It has no closed component. Removing the band of the crossing in question gives the surface F_0 of Seifert's algorithm for the smoothed diagram, a Seifert surface of L_0 ; thus F' is F_0 with one band attached. Let q' and q_0 be the numbers of components of F' and F_0 . We use Lemma 2.3 for both surfaces.

If the band joins two components of F_0 , then $q' = q_0 - 1$ and the inclusion $F_0 \subset F'$ induces an isomorphism of H_1 which respects the Seifert forms. So $W_{F'} = W_{F_0}$: the signature does not change, and $\eta(L') = \eta(L_0) - 1$.

Otherwise $q' = q_0$ and $H_1(F') = H_1(F_0) \oplus \mathbb{Z}$, and $W_{F'}$ restricts to W_{F_0} on the first summand. Let p, ν, z be the numbers of positive, negative and zero eigenvalues of W_{F_0} , and p', ν', z' those of $W_{F'}$. The number p is the largest dimension of a subspace on which the form is positive definite, and a subspace of $H_1(F'; \mathbb{R})$ meets the hyperplane $H_1(F_0; \mathbb{R})$ in a subspace of codimension at most one; hence $p \leq p' \leq p + 1$, and likewise $\nu \leq \nu' \leq \nu + 1$. As $p' + \nu' + z' = p + \nu + z + 1$, the triple $(p' - p, \nu' - \nu, z' - z)$ is one of $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(1, 1, -1)$. In the first two cases the signature changes by one and the nullity does not change; in the last two cases the signature does not change and the nullity changes by one. $\square$

Corollary 4.2. *Let L be the closure of $\beta = \prod_{j=1}^k w_j \sigma_{i_j} w_j^{-1} \in B_N$. Then $|\sigma(L)| \leq k - N + 1 + \eta(L)$.*

This is the Murasugi–Tristram inequality for quasipositive braids in the form of [Ore20, inequality (2)], where it is quoted from [Mur65] and [Ore02, Corollary 3.2].

Proof. Let β_j be the product of the first j factors and L_j its closure in B_N ; so L_0 is the unlink with N components, with $(\sigma, \eta) = (0, N - 1)$ by Lemma 2.3, and $L_k = L$. In the closed braid diagram of the word $\beta_{j-1} w_j \sigma_{i_j} w_j^{-1}$ all strands are oriented in the same direction, and smoothing the crossing σ_{i_j} gives the closed braid diagram of $\beta_{j-1} w_j w_j^{-1}$, a diagram of L_{j-1} . By Lemma 4.1 each of the k

steps from L_0 to L_k changes either η or σ , by one. At least $N - 1 - \eta(L)$ steps are needed to bring the nullity from $N - 1$ to $\eta(L)$, so at most $k - (N - 1 - \eta(L))$ steps change the signature. $\square$

5 Quasipositive alternating links

Proof of Corollary 1.3. Let L be the closure of $\beta = \prod_{j=1}^k w_j \sigma_{i_j} w_j^{-1} \in B_N$, and let D be a reduced alternating diagram of the oriented link L , with connected components $D_1, \dots, D_r$ presenting $L_1, \dots, L_r$. By Lemmas 2.3 and 3.4, $\eta(L_i) = 0$ for all i and $\eta(L) = r - 1$. We compare three numbers.

- (1) The closed braid diagram of the given word has writhe k , since the letters of w_j and w_j^{-1} cancel, and N Seifert circles. By Theorem 2.2 for closed braids, $k - N + 1 \leq \min \deg_l P_L$.
- (2) By Corollary 1.2, $\min \deg_l P_L \leq \sigma(L) - (r - 1) - \sum_i d_-(D_i)$.
- (3) By Corollary 4.2, $\sigma(L) \leq k - N + 1 + \eta(L) = k - N + r$.

Together,

$$k - N + 1 \leq \min \deg_l P_L \leq \sigma(L) - (r - 1) - \sum_i d_-(D_i) \leq k - N + 1 - \sum_i d_-(D_i).$$

Hence $d_-(D_i) = 0$ for all i , and all inequalities are equalities. By Theorem 1.1(a) no D_i has a negative crossing. $\square$

The argument has the pattern of Orevkov's proofs of his Theorems 2 and 6 in [Ore20], which also combine Traczyk's formula with the Murasugi–Tristram inequality for a quasipositive braid. There the alternating diagram and the quasipositive braid are compared through braids of minimal index, by theorems of Yamada, of Hayden, and of Dynnikov and Prasolov, which requires a hypothesis on the diagram. Here they are compared through the HOMFLY polynomial: step (1) holds for a quasipositive braid on any number of strands, and step (2) is Theorem 1.1.

6 Relation to the work of Orevkov, Ito and Stoimenow

6.1 Orevkov's inequality

Let D be a connected reduced alternating diagram of L with s Seifert circles, let b be the braid index of L and β a braid on b strands with closure L . Orevkov [Ore20, Section 3] defines $d^\pm(D)$ as the numbers of positive and negative edges of a spanning tree of the Seifert graph, and $r^\pm(D)$ by $r^+ + r^- = s - b$ and $r^+ - r^- = w(D) - w(\beta)$; the exponent sum $w(\beta)$ does not depend on β , by a theorem of Dynnikov and Prasolov. His Theorem 6 says that a reduced alternating diagram D of a quasipositive link with $2r^-(D) \leq d^-(D)$ is positive. He notes that the inequality holds whenever the braid index had been computed by Diao, Ernst, Heteyi and Liu, in particular for minimal diagrams of two-bridge links and of alternating Montesinos links, and asks whether it always holds (Question 8).

Proof of Corollary 1.4(a). By Theorem 2.2 for the closed braid β and by Theorem 1.1(b),

$$w(\beta) - b + 1 \leq \min \deg_l P_L \leq w(D) - s(D) + 1 + d_-(D),$$

for a braid β on any number b of strands with closure L . If b is the braid index, then $2r^- = (s - b) - (w(D) - w(\beta))$, so the inequality says $2r^-(D) \leq d_-(D)$, and $d_-(D) = d^-(D)$ by Proposition 3.6(b). $\square$

In the same way Theorem 1.1(d) gives $2r^+(D) \leq d^+(D)$. With [Ore20, Theorem 6] this is a second, longer proof of Corollary 1.3 for non-split links. For a reduced alternating diagram D with $r \geq 2$ components the same inequalities hold, with spanning forests in place of spanning trees: w , s and $d_{\pm}$ are additive, $\sigma(L) = w - s + r + 2d_-(D)$ by Theorem 1.1(b), and Corollary 1.2 reads $\min \deg_l P_L \leq w - s + 1 + d_-(D)$.

6.2 Ito's theorems and his conjecture

For a homogeneous diagram D of L , Ito [Ito26, Theorem 2] proves

$$\min \deg_l P_L \leq -s(D) + w(D) + 2s_+(D) + 1 - 2\#_{sp}(L), \quad (9)$$

where $\#_{sp}(L)$ is the number of split components, and deduces $\min \deg_l P_L \leq 1 - \chi_4(L)$ from a theorem of Cavallo and Collari, which he quotes as his Theorem 3; his Theorem 5 is $\min \deg_l P_L + \#_{sp}(L) - 1 \leq \sigma(L)$ for alternating links. The proofs follow Cromwell [Cro89]: a skein tree shows that the coefficient $h_L(l)$ of the highest power of m in P_L contains a certain monomial l^a , and for alternating diagrams a is identified with $\sigma(L)$ by Traczyk's formula. For a reduced alternating diagram with r components, the right-hand side of (9) is $\sigma(L) - (r - 1)$ by Proposition 3.6(c) and Corollary 1.2; so in this case (9) is Ito's Theorem 5, and Corollary 1.2 improves it by $\sum_i d_-(D_i) = \sum_i (s_+(D_i) - 1)$.

Ito remarks that for the knot 6_1 the bound for $\min \deg_l h_L$ is attained, so that his conjecture cannot be proved by looking at h_L alone. In our conventions the alternating diagram of 6_1 in Table 1 has $\sigma = 0$, $d_- = 1$ and $s_+ = 2$. The coefficient of the highest power m^2 of P is $-1 - l^2$, of lowest degree $0 = \sigma$; but $\min \deg_l P = -2$, and the monomial l^{-2} sits in the coefficient of m^0 . Theorem 1.1 gives the bound -1 , and -2 with the parity. So the substitution of Lemma 3.1 detects a term which the highest coefficient does not contain.

Proof of Corollary 1.4(b), (c). Let D be a reduced alternating diagram of L with components $D_1, \dots, D_r$; then $r = \#_{sp}(L)$. By Corollary 1.2,

$$\min \deg_l P_L \leq \sigma(L) - (r - 1) - \sum_i d_-(D_i) \leq \sigma(L) - (r - 1),$$

and the last number is the right-hand side of (9), which is at most $1 - \chi_4(L)$ by [Ito26, Theorem 3]. If equality holds in (9), or in $\min \deg_l P_L \leq 1 - \chi_4(L)$, then $\sum_i d_-(D_i) = 0$, and D is positive by Theorem 1.1(a). Conversely, if D is positive, then equality holds in (9) by Corollary 1.2. If L is a positive link, with a positive diagram E with n crossings and s Seifert circles, then $n - s + 1 \leq \min \deg_l P_L \leq 1 - \chi_4(L) \leq n - s + 1$: the first inequality is Theorem 2.2, the second is [Ito26, Corollary 1], and the third holds because the surface of Seifert's algorithm for E has Euler characteristic $s - n$. $\square$

Part (ii) of [Ito26, Conjecture 1] is stated for all homogeneous diagrams of a homogeneous link. We prove it only for reduced alternating diagrams (for reduced homogeneous ones see Section 6.4). Some hypothesis like "reduced" is needed in (ii): the diagram of the unknot with one negative kink is homogeneous and not positive, and for it both sides of (9) are 0 (here $s = 2$, $w = -1$, $s_+ = 2$).

6.3 Stoimenow's Section 5

The text [Sto26a] is the version of [Sto26b] on the author's home page; a note there says that its Section 5, on quasipositivity and the four-genus, is omitted from the published version and is to become a separate paper. Its Question 5.14, attributed to [BBG19], asks whether every quasipositive alternating link is special alternating, and its Theorem 5.24 states that every homogeneous quasipositive

link is positive. As far as we can see, the proof combines an estimate of χ_4 by a modified index of Murasugi and Przytycki, the Morton–Franks–Williams inequality, the behaviour of the coefficient of the highest power of m under Murasugi sums, a comparison of degrees under the substitution which makes P trivial (similar in spirit to Lemma 3.1; see Section 6.4), and Cromwell’s theorem on the case of equality in his inequality [Cro89]; it depends on Sections 4 and 5.1–5.3 of that text, and we did not check it. Corollary 1.3 is the alternating case; its proof here uses neither the four-genus nor Murasugi sums nor Cromwell’s theorem.

The statements of [Sto26a] nearest to Theorem 1.1(c) are its Corollaries 5.28–5.30. Corollary 5.28 is a lower bound for the signature of an alternating diagram, proved with the τ -invariant of knot Floer homology. Corollary 5.29 says that $\sigma(L) \geq \min \deg_l P_L$ for a connected alternating diagram D of L which is “ind-optimal” (a condition involving the braid index of L and the index of D), with equality if and only if D is special alternating; the author remarks that one would expect a proof without Floer theory. Corollary 5.30 says that a quasipositive alternating link has equal genus, smooth four-genus and topological four-genus. He also recalls from [Sto02] the chain $2 \min \deg_t V_L \leq \min \deg_a F_L(a^{-1}, z) \leq \min \deg_l P_L \leq \sigma(L)$ for non-split alternating links, of which the two inequalities not involving P were proved there, with equality exactly for special alternating links; and he notes that for closures of alternating braids and for two-bridge links $\sigma(L) \geq \min \deg_l P_L$, with its case of equality, follows from explicit formulas (of [Sto02] and of Murasugi). So the relation $2 \min \deg_t V_L = \sigma(L) - 2n_-$, the first step of our proof of (2), is classical, and Theorem 1.1(c) is the statement of Corollary 5.29 for all connected reduced alternating diagrams. We found no openly accessible copy of [Sto02], where (1) was first studied; so we could not read it and cannot exclude that (2) is contained in it or follows from it at once.

6.4 The case of equality from statements in print

The substitution $m = l^{-1} - l$ makes P trivial: it gives $\delta = 1$, so the constant 1 satisfies (3) and has the value of P on all unlinks; as the expansion (7) uses nothing else, $P_L(l, l^{-1} - l) = 1$ for every link L (Lickorish and Millett [LM87], as quoted in [Sto26b]). As in Lemma 3.1, the image of a monomial $l^i m^j$ has order exactly $i - j$ in $\mathbb{Q}((l))$. Put $a = \min \deg_l P_L$ and $b = \max \deg_m P_L$, and let $h_L(l)$ be the coefficient of m^b in P_L . Every monomial of P_L has $i - j \geq a - b$, with equality only for $(i, j) = (a, b)$, and the sum of the images has order 0. Hence, with the parity in Lemma 3.2,

$$\min \deg_l P_L \leq \max \deg_m P_L, \quad \text{and} \quad \min \deg_l h_L \geq a + 2 \quad \text{if} \quad a < b. \quad (10)$$

Stoimenow uses this argument in the published paper [Sto26b], in the proofs of his Lemma 4.12 and Corollary 4.16. In the second he states, for a connected reduced homogeneous diagram D which is not positive, that

$$\min \deg_l P_L \leq \min \deg_l h_L - 2 \leq n_+ - s - 1 \quad (11)$$

(for such diagrams h_L is the coefficient of m^{n-s+1}); the strict inequality needed in (10) comes from a theorem of Cromwell [Cro89, Theorem 4], to which the first proof refers. By Cromwell’s theorem in the form of [Ito26] (Theorems 7 and 8 of the arXiv version), h_L contains the monomial $l^{w-s+2s_+(D)-1}$ if D is a connected homogeneous diagram, and this exponent is $\sigma(L)$ if D is reduced and alternating. Together,

$$\min \deg_l P_L \leq \sigma(L) - 2 \quad \text{for a connected reduced alternating diagram which is not positive.}$$

This is the case of equality of Theorem 1.1(c), with its sharpening by the parity; neither source states it, but it is immediate. In the same way (11) gives part (ii) of [Ito26, Conjecture 1] for connected reduced homogeneous diagrams, as far as we can see, hence Corollary 1.4(b), (c), and with [Ito26, Theorem 4], or with Corollary 4.2 as in Section 5, the statement of Corollary 1.3.

So, of the statements of this note, only the bound (2) with $d_-(D)$ and Corollary 1.4(a) do not follow from these statements. With the parity, (2) reads $\min \deg_i P_L \leq \sigma(L) - 2\lceil d_-(D)/2 \rceil$, which is stronger than $\sigma(L) - 2$ if $d_-(D) \geq 3$; for Orevkov’s inequality the bound $\sigma(L) - 2$ gives only $2r^-(D) \leq 2d^-(D) - 2$. The second bound in (11) equals $\sigma(L) - d_-(D) + (n_- - d_-(D)) - 2$ and is weaker than (2) if $n_- - d_-(D) \geq 3$. Among the 10 586 oriented census diagrams with at most 11 crossings which are not positive, (2) with the parity is equal to the better of the two bounds from the literature for 7 750 and stronger for 2 836 (run 3 of Section 7).

7 Tests and reproducibility

The statements of Sections 2 to 5 were tested by exact computations (integers and rationals only); no proof depends on them. There are four sets of programs, written independently and all contained in the reproducibility package with their recorded outputs: those with which the results were first obtained (`original/`), those of two verification runs (`verification_run_A/`, `verification_run_B/`), and those of a third verification run, made on the final text (`independent_run_2/`; “run 3” below). The programs of run A use no knot software: P comes from a skein recursion, (σ, η) from the Goeritz form of Gordon and Litherland [GL78] and, for braids, from a Seifert form. The original programs and those of run B compute P with Regina [BBP⁺25] and σ with their own Gordon–Litherland routines. Run 3 has its own library for all invariants (skein tree, state sum, Gordon–Litherland and Seifert forms, a Hecke algebra for closed braids) and uses SpheroGram only to simplify closed braid diagrams. The diagrams come from the census of Hoste and Thistlethwaite in SnapPy [CDGW25] and from the tables [LM26]. In every test below the number of failures is 0.

programs	diagrams (every orientation, both mirror images)	oriented diagrams
original	all 68 448 prime alternating links with at most 14 crossings	296 622
run B	the same, each oriented diagram built and computed separately	296 622
run B	those with at most 13 crossings, every invariant by a second method	72 838
run A	all with at most 12 crossings; a sample of 2 400 with 13	19 834 + 10 294
run 3	all with at most 12 crossings, each mirror image computed	39 668
original, run B	alternating entries of [LM26]: 6 729 knots, 2 242 links	17 942
original	torus, pretzel, two-bridge, Montesinos links, alternating braids, sums	13 894
original	1 380 random alternating diagrams with 15 to 60 crossings	9 014
run A	686 connected sums, 1 650 alternating braids, 270 others	3 456
run B	1 500 connected sums (up to 24 crossings)	1 500
run 3	shadows of 720 random closed braids (up to 16 crossings); 296 sums	4 276 + 296

Table 2: Tests of Theorem 1.1, Theorem 2.2, Proposition 3.6 and Lemmas 3.1 to 3.5 on connected reduced alternating diagrams. The census (knots are counted as links) has 26 265 knots and 42 183 links with up to 7 components. In the first row the mirror images are obtained by the rules in the proof of Theorem 1.1(d). In the sixth row run B treated the 1 851 knots with at most 12 crossings. Of the 4 276 diagrams of the last row, 3 832 have at least two components.

Theorem 1.1 and its lemmas were tested on the diagrams of Table 2. The original programs also reproduced the tabulated signatures and polynomials P , F of all 17 153 entries of [LM26] with at most 13 (knots) or 11 (links) crossings. Corollary 1.2 was tested on 321 (run A), 1 500 (run B) and 296 (run 3, with the last sentence of Section 6.1) split unions. Controls: among 600 connected alternating diagrams with nugatory crossings the upper bound of (2) fails for 183 (run A), among 3 824 others for 394 (run 3, which also removed the nugatory crossings by twists and obtained reduced alternating diagrams with the same P , σ and η); seven deliberately wrong variants of the statements fail on at least 634 of 5 610 census diagrams (run B); among the non-alternating table entries the first inequality

of (1) fails 260 times, for instance for 10_{139} with $\sigma = 6$ and $\min \deg_l P = 8$, and the second 11 times (original). An observation which we cannot prove: in the census, $\min \deg_l P_L = \sigma(L) - d_-(D)$ only for the 5977 positive diagrams (run B). *The second inequality* and the equality in (6) were tested with tabulated Kauffman polynomials on the 17942 oriented alternating table links (original) and with computed ones on the census diagrams with at most 10 crossings (run A, original; run 3 up to 9 crossings).

Lemma 4.1, Corollary 4.2 and Corollary 1.3. Lemma 4.1 was tested on 63790 band moves by run A, on 8118 by run B and on 8804 by run 3; all cases of the proof occur. Along the partial products of quasipositive braids, Lemma 4.1, Corollary 4.2 and step (1) of Section 5 were tested on 42177 braids (run A), 12770 (run B), 3000 (run 3) and 1222 (original). The tables [LM26] list quasipositive braids for 156 alternating knots and 228 alternating oriented links; in all cases the alternating diagram of the table is positive, or negative where the braid is one of the mirror image (runs A and B, and run 3 for the knots; run A missed one link). Run B simplified the closures of its 12770 words with Spherogram: 3237 unlinks, 7171 with a reduced alternating diagram, positive in all cases, 2063 certified not to be alternating, 299 undecided. Run 3 did the same for 1500 random words and for all 1100 words with two or three bands in B_3 and conjugating words of length at most one: 334 unlinks and 1834 reduced alternating diagrams with crossings, all positive, with $\sigma(L) = k - N + r$ and $\min \deg_l P_L = k - N + 1$ (on closed alternating braids both procedures find non-positive diagrams). Run 3 also tested Orevkov's inequality in his own terms, with the braids of minimal index of [LM26], on the 563 alternating knots with at most 11 crossings. These tests are not independent of the others: by steps (1) and (3) of Section 5, a counterexample to Corollary 1.3 would contradict Corollary 1.2.

The package. The exact program of this note, `writing_stage/check_paper.py`, recomputes every number printed in Table 1, in the captions and in the examples. The script `run_quick.sh` re-runs every program of the first three sets which needs at most about five minutes and compares the results with the recorded outputs; the README files list the other commands and the versions of the software.

Verification. The results were first obtained, with proofs of Theorem 1.1, Lemma 4.1 and Corollary 1.3 and with the programs in `original/`. Three independent verification runs, all AI-assisted, followed; each wrote its own programs, and the first two did not see each other. The first fixed the conventions anew, read [Tra04, Ito26], re-derived every proof and searched for counterexamples; it supplied the proof of Lemma 3.2, the argument of Lemma 2.1, a direct proof of $d_- = s_+ - 1$ and the parity in Theorem 1.1(c). The second examined the literature and the credits: it found the relation to Orevkov's inequality and the statements of [Sto26a] quoted in Section 6.3, corrected the attribution of the question, re-derived the proofs, repeated the census and re-ran the original programs. Neither found an error in a statement or a gap in a proof; the fourteen corrections which they required were made. When the note was written, the proof of Proposition 3.6 was simplified (it no longer uses that alternating diagrams are homogeneous), the parity in Theorem 1.1(c) was derived from (8), and two remarks were added (in the proof of Corollary 1.2; the last sentence of Section 6.1).

The third run examined the final text, and these four items first. It re-derived every proof line by line, compared Lemma 3.4 with the publisher's file of [Tra04] and with its own derivation from the theorem of Gordon and Litherland, checked the convention for F against an identity of [Oht02], and compared the citations with the sources listed below. It found no wrong statement and no gap in a proof. It found that the case of equality of Theorem 1.1(c) and Corollary 1.4(b), (c) follow from statements in print: Section 6.4 and the corresponding statements in the abstract and in Section 1.4 are due to this run. Its nine corrections (this one, citations, bibliographic data, one step in the proof of Lemma 2.3, the record of its tests) were made. Its programs, written from the statements of this note before the programs of the package were opened, confirmed every statement in the tests of Sections 6.4 and 7. It also ran `run_quick.sh` from an extracted copy of the archive: all comparisons

with the recorded outputs succeeded, up to the two differences documented in the `README`, which concern no statement.

Scope and priority. The note does not solve Problem 1.16: its two inequalities are due to Ito [Ito26] and to Rutherford [Rut06], Ng [Ng05] and Tanaka [Tan06]. The statement of Corollary 1.3 is the alternating case of [Sto26a, Theorem 5.24]; the question is due to Baader and to Boileau, Boyer and Gordon, and the proof has the pattern of Orevkov’s proofs in [Ore20]. Theorem 2.2, Lemmas 3.2 to 3.4, the signature part of Lemma 4.1 and Corollary 4.2 are classical. The case of equality in Theorem 1.1(c), the bound $\sigma(L) - 2$ for diagrams which are not positive, and Corollary 1.4(b), (c) follow from statements of [Ito26] and [Sto26b] (Section 6.4). Not found elsewhere: the upper bound in (2) with $d_-(D)$, hence Corollary 1.4(a), and the proofs given here of Theorem 1.1(c) and of Corollary 1.3.

Read, when the note was written or by the verification runs: [Ito26], [Ito22] and [Ore20] in the arXiv versions and [Tra04] in the publisher’s open file, completely; of [Baa05b] (arXiv) the list of questions; of [BBG19] (arXiv) Section 7; of [Sto26a] the note on the home page, the Introduction and Sections 5.2–5.5 with the proof of Theorem 5.24; of the published version [Sto26b], which is open access and has no such section, the contents and the part of Section 4.4 with Lemma 4.12 and Corollary 4.16; of [Oht02, Rut06, Ng05, Fer02, Ozb24], in the arXiv versions, the definitions and statements quoted; of [MV20] the sentence on Murasugi’s lemma; of [Baa05a] the abstract; of [Mor86, Sto02] the reviews in zbMATH. The proofs in these sources were not checked. Not read: [Sto02] (no openly accessible copy was found on the author’s home page, on arXiv or in the preprint archive of the Max Planck Institute for Mathematics), [Tan06, Mor86, FW87, Kau87, Mur87, Thi87, GL78, Mur65, Cro89, Ore02, LM87], the journal versions of the papers read on arXiv (that of [Ito26] is not openly accessible, so the number of the theorem there may differ), and the papers which are mentioned only through [Ore20, Ito22, Ito26, Sto26a, Sto26b].

The literature was searched on 10 October 2026, for the first version of the results and by the three verification runs: queries to arXiv (among them quasipositive with alternating, homogeneous, signature or HOMFLY; alternating with HOMFLY and signature), Crossref and zbMATH, the lists of works citing [Ore20, Ito22, Ito26, BBG19, Baa05b, Baa05a, Sto02, Sto26b, Tra04] in OpenAlex or Semantic Scholar, and eight web searches. No proof, announcement or refutation of the statement of Corollary 1.3 was found other than [Sto26a], and no statement of the upper bound in (2). A search which finds nothing is not a proof of novelty, the more so as several classical sources were not read; no priority is claimed.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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