

On a Question of Benjamini on First-Passage Percolation in Hierarchical Graphs: The Recursive Constant Exceeds the Branching Random Walk Constant

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Abstract

In Question 9.1 of his survey “Euclidean vs. Graph Metric” (2013), Benjamini considers first-passage percolation on the hierarchical graphs G_n obtained by replacing, recursively, every edge by one edge followed by two parallel edges. With independent positive edge weights of law ξ , the source–sink distance satisfies $D_n \stackrel{d}{=} D_{n-1} + \min(D'_{n-1}, D''_{n-1})$. The question asks to compute $\gamma_{\text{rec}} = \lim 2^{-n} \mathbb{E} D_n$ in terms of ξ , and in particular to show, if true, that $\gamma_{\text{rec}} > \gamma_{\text{brw}}$, the speed of the minimal position of the binary branching random walk with step law ξ , which is a lower bound for γ_{rec} . We give a partial answer. We prove $\gamma_{\text{rec}} > \gamma_{\text{brw}}$ for every non-degenerate law ξ that is bounded below and has finite mean, hence for every non-degenerate law of positive weights with finite mean. The proof is a standard first-moment (union) bound taken one level up the hierarchy, combined with the inequality $e^{-t \min(x,y)} < e^{-tx} + e^{-ty}$; at level zero the first-moment bound is γ_{brw} , by the classical theorem of Hammersley, Kingman and Biggins. We also show that γ_{rec} is the increasing limit of explicit first-moment bounds and the decreasing limit of $2^{-k} \mathbb{E} D_k$, that $D_n/2^n \rightarrow \gamma_{\text{rec}}$ in L^1 , as anticipated in the source, and that $2^{-k} \mathbb{E} D_k - \gamma_{\text{rec}} \leq \frac{1+\sqrt{2}}{2} \|\xi - \text{ess inf } \xi\|_2 2^{-k/2}$. Certified computer-assisted computations give $\gamma_{\text{rec}} \in [0.268785599, 0.268808710]$ for weights uniform on $[0, 1]$, where $\gamma_{\text{brw}} = 0.18482\dots$, and $\gamma_{\text{rec}} \in [0.379282228, 0.379412906]$ for exponential weights of mean one, where $\gamma_{\text{brw}} = 0.23196\dots$; in particular, the remark in the UnsolvedMath collection that the two constants coincide in the exponential case is incorrect. A closed formula for γ_{rec} in terms of ξ , the “compute” part of the question, remains open. The method is standard; no answer to the question was found in the literature accessible to us, and no priority is claimed. This is an unrefereed note.

1 Introduction

1.1 The question

Let G_1 be the graph with three edges on the left of Figure 1: an edge e_1 from the *source* s to a vertex v , followed by two parallel edges e_2, e_3 from v to the *sink* t , all edges being oriented from the source towards the sink. Following Benjamini [Ben13, §9.1], the graph G_n is obtained from G_{n-1} by replacing each oriented edge by a copy of G_1 whose source and sink are placed at the origin and at the target of the edge; we also put $G_0 :=$ a single edge from s to t . In [Ben13, §9.2] every edge of G_n receives a positive weight, for example uniform on $[0, 1]$, the weights being independent with a common law ξ , and

$D_n :=$ the weight of a minimal path linking the source and the sink of G_n .

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The pattern G_1 is the third of the three examples of [Ben13, Figure 1], and for it the source records the recursive distributional equation

$$D_n \stackrel{d}{=} D_{n-1} + \min(D'_{n-1}, D''_{n-1}), \quad (1)$$

where $D_{n-1}, D'_{n-1}, D''_{n-1}$ are independent copies of D_{n-1} ; it holds for all $n \geq 1$ with $D_0 = \xi$ (Lemma 2.1).

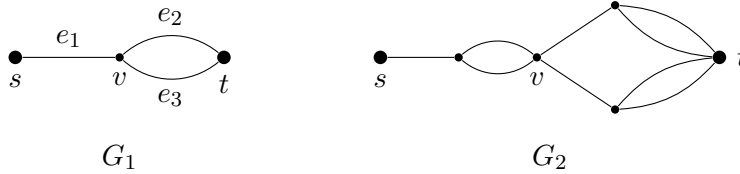


Figure 1: The pattern G_1 (the third example of [Ben13, Figure 1]) and G_2 ; edges are oriented from left to right. G_2 has 9 edges and 8 source–sink paths of 4 edges.

For comparison the source uses the branching random walk. Let T_n be the binary tree that consists of a root edge followed by binary branching, up to level n (so that every path from the root to level n has n edges), with independent weights of law ξ , and let M_n be the weight of the lightest path from the root to level n . Then $M_n \stackrel{d}{=} \xi + \min(M'_{n-1}, M''_{n-1})$, M_n is, up to the root edge, the minimal position of a branching random walk, and M_n/n converges to a constant γ_{brw} which is given explicitly in terms of ξ (Section 3). The height of G_n is 2^n : every source–sink path has 2^n edges. The source shows by induction that D_n dominates M_{2^n} stochastically, so that $\gamma_{\text{brw}} 2^n \leq \mathbb{E} D_n$, and $\mathbb{E} D_n \leq 2^n$ for weights in $[0, 1]$; it then states that a monotonicity argument shows that, for non-degenerate ξ , the limit

$$\gamma_{\text{rec}} := \lim_{n \rightarrow \infty} 2^{-n} \mathbb{E} D_n$$

exists and lies in $[\gamma_{\text{brw}}, 1)$, and asks:

Question 9.1 [Ben13]. “Compute γ_{rec} in terms of ξ in particular show (if true) $\gamma_{\text{rec}} > \gamma_{\text{brw}}$.”

Right after the question the authors say that they expect the convergence in mean of D_n to imply, through (1), its convergence in probability. Their Question 9.2, on the concentration of D_n and its lower-order terms, is not treated here. The source states that its Section 9 is based on an ongoing project with N. Curien. The question is record AMR-099-0065 of the UnsolvedMath collection [ula26], where it is classified as partially solved; the summary attached to the record says that the constant equals the minimal rate of the branching random walk and that γ_{rec} equals the branching random walk constant in the solvable exponential case. Neither is so (Theorem A, Theorem C(b)).

Conventions. The following readings are needed; none of them changes the answer. (i) As in the source, distances in G_n are divided by the height 2^n , and γ_{brw} is a speed per level of a binary tree with a root edge. Then G_n and T_{2^n} both have 2^{2^n-1} paths with 2^n edges each from the source (root) to the sink (level 2^n) (Lemma 2.1), and the first-moment bound for D_n at the finest level is the first-moment bound for M_{2^n} . (ii) The source takes a minimal path linking the source and the sink and does not say whether paths must follow the orientation of the edges; for nonnegative weights this does not matter, since a minimal path in the unoriented graph can be taken oriented (Lemma 2.1(a)). (iii) For weights that are bounded below but may be negative we define D_n as the minimum over oriented source–sink paths. Since all these paths have 2^n edges and all root–leaf paths of T_n have n edges, replacing ξ by $\xi + c$ adds c to both γ_{rec} and γ_{brw} . So every statement below for laws bounded below is equivalent to

the same statement for positive laws. In particular, the weights with values in $\{0, 1\}$ of Theorem C(c) are equivalent to the positive weights with values in $\{c, 1+c\}$, $c > 0$, for which all constants are shifted by c . (iv) If $\xi \equiv c$ is constant, then $D_n = c2^n$ and $M_n = cn$, so $\gamma_{\text{rec}} = \gamma_{\text{brw}} = c$; the source assumes that ξ is non-degenerate. (v) γ_{rec} is defined through means. By Theorem B(ii) also $D_n/2^n \rightarrow \gamma_{\text{rec}}$ in probability, so the other natural reading gives the same constant. (vi) The mirror pattern, two parallel edges followed by one edge, gives the same law of D_n (reverse all orientations).

1.2 Results

Throughout, ξ is a real random variable with

$$m := \text{ess inf } \xi > -\infty \quad \text{and} \quad \mathbb{E}|\xi| < \infty,$$

which covers the positive weights with finite mean of the source. Let $D_0 := \xi$ and let the laws of $D_1, D_2, \dots$ be defined by (1). For $k \geq 0$ and $t > 0$ put

$$g_k(t) := -\frac{\log(2 \mathbb{E} e^{-tD_k})}{2^k t}, \quad \gamma_k := \sup_{t>0} g_k(t), \quad a_k := 2^{-k} \mathbb{E} D_k, \quad X_k := 2^{-k} D_k. \quad (2)$$

By the theorem of Hammersley, Kingman and Biggins (Theorem 3.1),

$$\gamma_{\text{brw}} = \gamma_0 = \sup_{t>0} -\frac{\log(2 \mathbb{E} e^{-t\xi})}{t}.$$

Theorem A. *Let ξ be non-degenerate and bounded below, with $\mathbb{E}|\xi| < \infty$. Then $\gamma_{\text{rec}} > \gamma_{\text{brw}}$. More precisely, let $k \geq 0$ be the least integer with $\mathbb{P}(D_k = 2^k m) < \frac{1}{2}$ (it exists, and $k = 0$ if and only if $\mathbb{P}(\xi = m) < \frac{1}{2}$). Then g_k attains its supremum at some $t_k^* \in (0, \infty)$, and*

$$\gamma_{\text{brw}} = \gamma_0 \leq \gamma_k = g_k(t_k^*) < g_{k+1}(t_k^*) \leq \gamma_{k+1} \leq \gamma_{\text{rec}}. \quad (3)$$

In particular $\gamma_{\text{rec}} > \gamma_{\text{brw}}$ for every non-degenerate law of positive weights with finite mean.

For laws without an atom at the minimum, such as the uniform or the exponential law, $k = 0$ (see Remark 4.7 for the gap). If $\mathbb{P}(\xi = m) \geq \frac{1}{2}$, then $\gamma_{\text{brw}} = m$, the supremum defining γ_0 is not attained, and the proof works at the first level where the atom of D_k at its minimum has mass below $\frac{1}{2}$ (Lemma 4.5).

Theorem B. *Let ξ be bounded below with $\mathbb{E}|\xi| < \infty$ (degenerate laws are allowed).*

- (i) *The sequence (γ_k) is non-decreasing and $\gamma_{\text{rec}} = \lim_{k \rightarrow \infty} \gamma_k = \sup_{k \geq 0} \sup_{t > 0} -\frac{\log(2 \mathbb{E} e^{-tD_k})}{2^k t}$.*
- (ii) *The sequence (a_k) is non-increasing, $\gamma_{\text{rec}} = \lim_k a_k = \inf_k 2^{-k} \mathbb{E} D_k$, and $D_n/2^n \rightarrow \gamma_{\text{rec}}$ in L^1 . In particular $\gamma_k \leq \gamma_{\text{rec}} \leq a_k$ for every k .*
- (iii) *If moreover $\mathbb{E}\xi^2 < \infty$, then $\text{Var } D_n \leq 2^n \mathbb{E}(\xi - m)^2$ and $0 \leq a_n - \gamma_{\text{rec}} \leq \frac{1+\sqrt{2}}{2} \|\xi - m\|_2 2^{-n/2}$ for every $n \geq 0$.*

Part (ii) proves the expectation stated after Question 9.1: $D_n/2^n \rightarrow \gamma_{\text{rec}}$ in probability, indeed in L^1 , whenever $\mathbb{E}|\xi| < \infty$. Parts (i)–(iii) are a convergent characterisation of γ_{rec} , not a closed formula.

Theorem C (computer-assisted). *The following enclosures hold.*

(a) If ξ is uniform on $[0, 1]$, then

$$\gamma_{\text{brw}} \in [0.184827520905, 0.184827520906], \quad \gamma_{\text{rec}} \in [0.268785599, 0.268808710].$$

(b) If ξ is exponential with mean 1, then $\gamma_{\text{brw}} = 1/u^*$, where u^* is the root in $(2, \infty)$ of $\log(u/2) = 1 - 1/u$, and

$$\gamma_{\text{brw}} \in [0.231960952986, 0.231960952987], \quad \gamma_{\text{rec}} \in [0.379282228, 0.379412906];$$

the weaker bound $\gamma_{\text{rec}} \geq g_1(\frac{11}{5}) = \frac{\log 3.36}{4.4} > 0.27544$ can be checked without a computer.

(c) If $\mathbb{P}(\xi = 0) = p$ and $\mathbb{P}(\xi = 1) = 1 - p$ with $p \geq \frac{1}{2}$, then $\gamma_{\text{brw}} = 0$, and

$$\gamma_{\text{rec}} \in [0.12392609, 0.14126638] \text{ for } p = \frac{1}{2}, \quad \gamma_{\text{rec}} \in [0.00964880, 0.01735090] \text{ for } p = \frac{3}{4}.$$

In these computations every rounding is done in a known direction (exact, interval, or floating-point arithmetic with proven one-sided margins; Section 6). Table 1 adds first-moment bounds. Three independent verification runs computed narrower enclosures, which lie inside those of Theorem C (Section 7).

law of ξ	γ_{brw}	first-moment bounds $\gamma_k \leq \gamma_{\text{rec}}$	γ_{rec}
U[0, 1]	0.1848275209...	$\gamma_1 \geq 0.211023660047$	[0.268785599, 0.268808710]
Exp(1)	0.2319609529...	$\gamma_1 \geq 0.275441130448$	[0.379282228, 0.379412906]
		$\gamma_2 \geq 0.307222167962$	
		$\gamma_3 \geq 0.329920004276$	
$\{0, 1\}, p = \frac{1}{2}$	0	$\gamma_1 \geq 0.0372545589$	[0.12392609, 0.14126638]
		$\gamma_2 \geq 0.0666301146$	
$\{0, 1\}, p = \frac{3}{4}$	0	$\gamma_4 \geq 0.0014245853$	[0.00964880, 0.01735090]
		$\gamma_5 \geq 0.0046486167$	

Table 1: Certified values. The first-moment bounds are $g_k(t)$ at $t = 151/40$ (row 1); $11/5, 31/20, 11/10$ (row 2); $57/20, 19/10$ (row 3); $73/20, 21/10$ (row 4). For weights in $\{0, 1\}$, $p = \mathbb{P}(\xi = 0)$, and the last column is $[g_8(t), a_8]$ with $t = 31/100$ (row 3) and $t = 19/25$ (row 4).

1.3 What is new, what is not, and what stays open

The question is Benjamini's [Ben13, Question 9.1], from a section that the source describes as based on an ongoing project with N. Curien. The formula for γ_{brw} is the classical theorem of Hammersley [Ham74], Kingman [Kin75] and Biggins [Big76], which we use in the form of [Big10, Theorem 3.1]. The proof of Theorem A is a standard first-moment argument: a union bound over the paths of the coarser graph G_{n-k} , with the exact passage time inside each of its blocks, which are copies of G_k . At $k = 0$ this is the first-moment bound of the branching random walk, and the gain at the next level comes from the edge e_1 that the two paths of G_1 share. The argument may well be known to experts.

Related work, none of which settles the question. Khristoforov, Kleptsyn and Triestino [KKT16] construct stationary random metrics on the limits of hierarchical graphs, with edge lengths given by a multiplicative cascade and renormalised, via $(\min, +)$ -type recursive distributional equations; they do not consider independent weights on a fixed G_n , γ_{rec} , or a comparison with γ_{brw} . Hambly and Jordan [HJ04] study, among other things, first-passage percolation on random hierarchical graphs in which every edge is replaced at random by two edges in series or in parallel; there the graph is random, while Question 9.1 has a fixed pattern and random weights. Addario-Berry, Cairns, Devroye, Kerriou

and Mitchell [ABCD⁺20] prove distributional convergence for “hipster random walks”, related to Pemantle’s min-plus binary trees and to the lattice of [HJ04]; their recursions choose the operation at random. Dekking and Host [DH91] prove that the minimal displacement of a branching random walk is tight around its mean (the source sketches their argument for M_n), and Benamini and Zeitouni [BZ12] extend this tightness to lattices in hyperbolic spaces and to the lamplighter graph over $\mathbb{N}$; neither concerns hierarchical graphs or time constants. Lacoïn and Moreno [LM10] study directed polymers on hierarchical lattices at positive temperature, not the zero-temperature constant γ_{rec} .

No answer to Question 9.1 was found in the literature accessible to us; a search that finds nothing is not a proof of novelty, and no priority is claimed (see the paragraph “Scope and priority” in Section 7).

What stays open. (1) A closed formula for γ_{rec} in terms of the law of ξ , comparable to that for γ_{brw} : the “compute” part of Question 9.1; Theorem B reduces it to two monotone sequences with an explicit rate, so γ_{rec} is computable to any precision (Section 6). (2) Question 9.2 of the source, on the concentration and the lower-order terms of D_n , is not treated; Theorem B(iii) only gives fluctuations of order at most $2^{n/2}$ (Remark 5.1).

2 The hierarchical graphs and the recursion

A *two-terminal graph* is a finite multigraph H with oriented edges and two distinct vertices, its source and its sink. For two-terminal graphs H and K let $H[K]$ be the two-terminal graph obtained from H by replacing every edge e of H , oriented from u to w , by a copy $K^{(e)}$ of K whose source is identified with u and whose sink with w ; the inner vertices of different copies are distinct, and the terminals of $H[K]$ are those of H . Thus G_0 is a single edge, $G_n = G_{n-1}[G_1]$, and $G_0[K] = K$, $H[G_0] = H$. We write $S(A, B)$ and $P(A, B)$ for the series and the parallel composition. Every G_n is acyclic, with no edge entering its source or leaving its sink (by induction, from $G_n \cong S(A, P(B, C))$ in the proof below), so oriented walks are paths.

Lemma 2.1. *Let $n \geq 0$.*

- (a) *Every simple path from the source to the sink of G_n , in the unoriented multigraph, is oriented and has exactly 2^n edges, and there are exactly 2^{2^n-1} such paths. Consequently, for nonnegative weights, D_n is the minimum of the path weights over the oriented source–sink paths.*
- (b) *For $0 \leq k \leq n$, $G_n \cong G_{n-k}[G_k]$, and the oriented source–sink paths of $G_{n-k}[G_k]$ are obtained by choosing an oriented source–sink path π of G_{n-k} and an oriented source–sink path in each copy $G_k^{(e)}$, $e \in \pi$. Hence, with independent weights of law ξ on the edges of G_n ,*

$$D_n = \min_{\pi} \sum_{e \in \pi} D_k^{(e)}, \quad (4)$$

where π runs over the $2^{2^{n-k}-1}$ oriented source–sink paths of G_{n-k} and $D_k^{(e)}$ is the passage time of the copy $G_k^{(e)}$; the $D_k^{(e)}$, $e \in E(G_{n-k})$, are independent with the law of D_k .

- (c) *For $n \geq 1$, $D_n = D_{n-1}^{(e_1)} + \min(D_{n-1}^{(e_2)}, D_{n-1}^{(e_3)})$ with three independent copies of D_{n-1} ; so (1) holds for all $n \geq 1$, with $D_0 = \xi$.*

Proof. Substitution is associative: $(H[K])[L]$ and $H[K[L]]$ are both obtained from H by replacing every edge by a copy of K in which every edge is replaced by a copy of L . By induction on b this gives $G_{a+b} \cong G_a[G_b]$: the case $b = 0$ is trivial, and $G_{a+b+1} = G_{a+b}[G_1] \cong (G_a[G_b])[G_1] = G_a[G_b[G_1]] = G_a[G_{b+1}]$. This is the first assertion of (b), and for $a = 1$ it shows that $G_n \cong G_1[G_{n-1}] = S(A, P(B, C))$, where A, B, C are copies of G_{n-1} placed on e_1, e_2, e_3 .

Paths in compositions. In $S(A, B)$ the common vertex v separates the source from the sink, so a simple source–sink path visits v once and consists of a simple path of A followed by a simple path of B . In $P(A, B)$ a simple source–sink path stays in the part that contains its first edge, because it could leave that part only through the source, which it does not visit again, or through the sink, where it ends. Conversely, all these concatenations are simple paths. The same descriptions hold for oriented paths.

(a) By induction on n , the case $n = 0$ being trivial: by the two descriptions, a simple source–sink path of $S(A, P(B, C))$ is a simple path of A followed by a simple path of B or of C , so it is oriented and has $2^{n-1} + 2^{n-1}$ edges, and the number N_n of these paths satisfies $N_n = N_{n-1} \cdot 2N_{n-1}$, $N_0 = 1$, whence $N_n = 2^{2^n-1}$. For nonnegative weights every walk from the source to the sink contains a simple source–sink path whose weight is not larger.

(b) Substitution commutes with the compositions: $S(H, H')[K] = S(H[K], H'[K])$ and $P(H, H')[K] = P(H[K], H'[K])$. The description of the oriented paths of $G_{n-k}[G_k]$ follows by induction on $n - k$ from $G_{n-k} \cong S(A', P(B', C'))$, with copies A', B', C' of G_{n-k-1} , and from the description of oriented paths in compositions; the case $n = k$ is $G_0[G_k] = G_k$. Minimising first inside the copies gives (4); the copies have disjoint sets of edges, so the $D_k^{(e)}$ are independent, each with the law of D_k , and the number of paths is given by (a).

(c) is (b) for $k = n - 1$: the oriented paths of G_1 are e_1e_2 and e_1e_3 . $\square$

For weights that are only bounded below we *define* D_n as the minimum over oriented source–sink paths, as in convention (iii) of Section 1.1; then (b) and (c) hold verbatim, and $D_n(\xi + c) = D_n(\xi) + c2^n$. The program `check_note.py` (Section 7) confirms (a) and (c) by brute force on the graphs built literally from the definition, for $n \leq 4$.

3 The branching random walk constant

Let T_n and M_n be as in Section 1.1: T_n is a root edge followed by binary branching up to level n , with independent weights of law ξ , and M_n is the minimal weight of a path from the root to level n . Every such path has n edges, and there are 2^{n-1} of them.

Theorem 3.1 (Hammersley, Kingman, Biggins). *Let ξ be bounded below. Then, almost surely,*

$$\frac{M_n}{n} \longrightarrow \gamma_0 = \sup_{t>0} -\frac{\log(2 \mathbb{E} e^{-t\xi})}{t}.$$

If moreover $\mathbb{E}|\xi| < \infty$, the convergence also holds in L^1 , so $\mathbb{E}M_n/n \rightarrow \gamma_0$. We write $\gamma_{\text{brw}} := \gamma_0$.

Proof. We use [Big10, Theorem 3.1], attributed there to Hammersley [Ham74, Theorem 2], Kingman [Kin75, Theorem 5] and, in the stated generality, Biggins [Big76, Theorem 4]. For a supercritical branching random walk in which every particle has at least one child, with children displaced according to a point process Z , it states: if $\kappa(\theta) = \log \int e^{\theta z} \mathbb{E}Z(dz)$ ($\theta \geq 0$; $\kappa = \infty$ on $(-\infty, 0)$) is finite for some $\theta > 0$, then the rightmost position $B(j)$ in generation j satisfies $B(j)/j \rightarrow \Gamma = \inf_{\theta} \kappa(\theta)/\theta$ almost surely. Let every particle have two children, displaced by independent copies of $-\xi$. Then $\kappa(\theta) = \log(2\mathbb{E}e^{-\theta\xi}) < \infty$ for $\theta \geq 0$, as ξ is bounded below, $\kappa(0) = \log 2 > 0$, and $\Gamma = -\gamma_0$. In T_n the root edge is followed by a binary tree of depth $n - 1$, whose path weights are the negatives of the positions, so $M_n = \xi_{\text{root}} - B(n - 1)$ and $M_n/n \rightarrow \gamma_0$ almost surely. If $\mathbb{E}|\xi| < \infty$, then $mn \leq M_n \leq S_n$, where S_n is the weight of a fixed root–leaf path, a sum of n independent copies of ξ ; S_n/n converges in L^1 by the law of large numbers, so (M_n/n) is uniformly integrable and converges in L^1 . $\square$

Remark 3.2. (a) Only the inequality $\gamma_{\text{brw}} \leq \gamma_0$ of Theorem 3.1 is used in the proof of Theorem A. The reverse inequality is the elementary first-moment bound; for D_n its analogue is Lemma 4.2 below with $k = 0$, which reproduces the bound $\gamma_{\text{rec}} \geq \gamma_{\text{brw}}$ of the source. (b) If $p := \mathbb{P}(\xi = m) \geq \frac{1}{2}$, then $2\mathbb{E}e^{-t(\xi-m)} \geq 2p \geq 1$, so $g_0(t) \leq m$ for all $t > 0$, while $g_0(t) \geq m - (\log 2)/t$; hence $\gamma_{\text{brw}} = m$, and the supremum is not attained. If $p < \frac{1}{2}$ it is attained (Lemma 4.4). (c) For the exponential law with mean 1, $g_0(t) = \log \frac{1+t}{2}/t$; at its maximiser $\log \frac{1+t}{2} = \frac{t}{1+t}$ and $g_0 = \frac{1}{1+t}$, which with $u = 1+t$ gives Theorem C(b). For the uniform law on $[0, 1]$, $\mathbb{E}e^{-t\xi} = (1 - e^{-t})/t$.

4 First-moment bounds and the proof of Theorem A

In this section and the next, ξ is bounded below with $\mathbb{E}|\xi| < \infty$, $m = \text{ess inf } \xi$, and g_k, γ_k, a_k, X_k are as in (2). Every oriented source–sink path of G_k has 2^k edges, so $2^k m \leq D_k \leq$ the weight of a fixed oriented path; hence $\mathbb{E}|D_k| < \infty$, $0 < \mathbb{E}e^{-tD_k} \leq e^{-t2^k m}$ for $t \geq 0$, $g_k(t)$ is finite, and $X_k \geq m$.

Lemma 4.1 (exact decrement). *For $n \geq 0$, $a_{n+1} = a_n - \frac{1}{4} \mathbb{E}|X'_n - X''_n|$, where X'_n, X''_n are independent copies of X_n . Consequently (a_n) is non-increasing and bounded below by m , the limit $\gamma_{\text{rec}} = \lim_n a_n$ exists (also for degenerate ξ), $\sum_{n \geq 0} \mathbb{E}|X'_n - X''_n| = 4(\mathbb{E}\xi - \gamma_{\text{rec}})$, and $X_n \rightarrow \gamma_{\text{rec}}$ in L^1 .*

Proof. By (1) and $\min(x, y) = \frac{x+y}{2} - \frac{|x-y|}{2}$, $\mathbb{E}D_{n+1} = \mathbb{E}D_n + \mathbb{E}\min(D'_n, D''_n) = 2\mathbb{E}D_n - \frac{1}{2}\mathbb{E}|D'_n - D''_n|$; divide by 2^{n+1} . Since $X_n \geq m$, also $a_n \geq m$, so the limit exists, and summing the identity over n gives the series. By the conditional Jensen inequality, $\mathbb{E}|X'_n - a_n| = \mathbb{E}|\mathbb{E}[X'_n - X''_n | X'_n]| \leq \mathbb{E}|X'_n - X''_n|$, which tends to 0; hence $\mathbb{E}|X_n - \gamma_{\text{rec}}| \leq \mathbb{E}|X_n - a_n| + (a_n - \gamma_{\text{rec}}) \rightarrow 0$. $\square$

In particular $a_1 = a_0 - \frac{1}{4}\mathbb{E}|\xi' - \xi''| < \mathbb{E}\xi$ for non-degenerate ξ , so $\gamma_{\text{rec}} < \mathbb{E}\xi$, in accordance with the bound $\gamma_{\text{rec}} < 1$ of the source for weights in $[0, 1]$.

Lemma 4.2 (first-moment bound at level k). *For $k \geq 0$ and $t > 0$, $\gamma_{\text{rec}} \geq g_k(t)$. Hence $\gamma_{\text{rec}} \geq \gamma_k$.*

Proof. Let $c < g_k(t)$, that is, $r := 2\mathbb{E}e^{-tD_k} e^{tc2^k} < 1$. For $n \geq k$ put $N := 2^{n-k}$. By (4), D_n is the minimum of 2^{N-1} sums S_π , each a sum of N independent copies of D_k . Markov's inequality for e^{-tS_π} gives $\mathbb{P}(S_\pi \leq c2^n) \leq e^{tc2^n} (\mathbb{E}e^{-tD_k})^N = (r/2)^N$, and the union bound gives

$$\mathbb{P}(D_n \leq c2^n) \leq 2^{N-1}(r/2)^N = r^N/2 \rightarrow 0 \quad (n \rightarrow \infty).$$

Since $X_n \geq m$, we have $a_n \geq \mathbb{E}[X_n \mathbf{1}\{X_n > c\}] + m\mathbb{P}(X_n \leq c) \geq c - |c - m|\mathbb{P}(X_n \leq c)$, which tends to c . So $\gamma_{\text{rec}} = \lim a_n \geq c$ for every $c < g_k(t)$. $\square$

For $k = 0$ this is literally the first-moment bound for M_{2^n} , and it gives the inequality $\gamma_{\text{rec}} \geq \gamma_{\text{brw}}$ of the source. The bound improves strictly from one level to the next:

Lemma 4.3 (strict increase). *For $k \geq 0$ and $t > 0$, $\mathbb{E}e^{-tD_{k+1}} < 2(\mathbb{E}e^{-tD_k})^2$. Hence $g_{k+1}(t) > g_k(t)$, and $\gamma_{k+1} \geq \gamma_k$.*

Proof. For real x, y we have $e^{-t\min(x,y)} = e^{-tx} + e^{-ty} - e^{-t\max(x,y)} < e^{-tx} + e^{-ty}$. With independent copies D', D'' of D_k this gives $\mathbb{E}e^{-t\min(D', D'')} < 2\mathbb{E}e^{-tD_k}$, and by (1) and independence $\mathbb{E}e^{-tD_{k+1}} = \mathbb{E}e^{-tD_k} \cdot \mathbb{E}e^{-t\min(D', D'')} < 2(\mathbb{E}e^{-tD_k})^2$. So $2\mathbb{E}e^{-tD_{k+1}} < (2\mathbb{E}e^{-tD_k})^2$, and it remains to take $-\log$ and to divide by $2^{k+1}t$. $\square$

The pointwise inequality $g_{k+1} > g_k$ gives $\gamma_{k+1} > \gamma_k$ as soon as γ_k is attained.

Lemma 4.4 (attainment). *Let Z be bounded below, $z_0 := \text{ess inf } Z$ and $q := \mathbb{P}(Z = z_0) < \frac{1}{2}$. Then $G(t) := -\log(2\mathbb{E}e^{-tZ})/t$ attains its supremum over $(0, \infty)$ at some $t^* \in (0, \infty)$, and $\sup G > z_0$.*

Proof. Put $\tilde{Z} := Z - z_0 \geq 0$ and $\phi(t) := \mathbb{E}e^{-t\tilde{Z}} \in (0, 1]$, so that $G(t) = z_0 + \tilde{G}(t)$ with $\tilde{G}(t) = -\log(2\phi(t))/t$. By dominated convergence ϕ is continuous, $\phi(t) \rightarrow 1$ as $t \downarrow 0$ and $\phi(t) \downarrow q$ as $t \rightarrow \infty$. Hence $\tilde{G}$ is continuous on $(0, \infty)$, $\tilde{G}(t) \rightarrow -\infty$ as $t \downarrow 0$, and, since $2q < 1$, $\tilde{G}(t_1) > 0$ for some t_1 . For every $\varepsilon > 0$, $\mathbb{P}(\tilde{Z} \leq \varepsilon) > 0$ and $\phi(t) \geq \mathbb{P}(\tilde{Z} \leq \varepsilon)e^{-t\varepsilon}$, so $\tilde{G}(t) \leq \varepsilon - \log(2\mathbb{P}(\tilde{Z} \leq \varepsilon))/t$; hence $\limsup_{t \rightarrow \infty} \tilde{G}(t) \leq 0$. The set $\{t > 0 : \tilde{G}(t) \geq \tilde{G}(t_1)\}$ is therefore non-empty, closed in $(0, \infty)$ and contained in a compact interval $[\delta, T] \subset (0, \infty)$; $\tilde{G}$ attains its maximum on it, and this is the supremum over $(0, \infty)$, at least $\tilde{G}(t_1) > 0$. $\square$

Lemma 4.5 (the atom at the minimum). *Let ξ be non-degenerate and $q_k := \mathbb{P}(D_k = 2^k m)$. Then $\text{essinf } D_k = 2^k m$, $q_0 = \mathbb{P}(\xi = m) < 1$, $q_{k+1} = q_k^2(2 - q_k)$, q_k decreases to 0, and every D_k is non-degenerate.*

Proof. By (1), $D_{k+1} \geq 2^{k+1}m$, with equality if and only if $D_k = 2^k m$ and $\min(D'_k, D''_k) = 2^k m$; this has probability $q_k(1 - (1 - q_k)^2) = q_k^2(2 - q_k)$. Since $D_{k+1} \leq D_k + D'_k$, for every $\varepsilon > 0$ we have $\mathbb{P}(D_{k+1} \leq 2^{k+1}m + \varepsilon) \geq \mathbb{P}(D_k \leq 2^k m + \varepsilon/2)^2 > 0$, which gives the essential infimum by induction. The map $f(q) = 2q^2 - q^3$ sends $[0, 1]$ to itself, $q - f(q) = q(1 - q)^2 > 0$ on $(0, 1)$, and its only fixed points in $[0, 1]$ are 0 and 1; as $q_0 < 1$ by non-degeneracy, q_k decreases to 0. If D_k were almost surely constant, the constant would be $2^k m$ and $q_k = 1$; but $q_k \leq q_0 < 1$. $\square$

Proof of Theorem A. By Lemma 4.5 there is a least $k \geq 0$ with $q_k < \frac{1}{2}$; $k = 0$ if and only if $\mathbb{P}(\xi = m) < \frac{1}{2}$. By Lemma 4.5, D_k is bounded below, $\text{essinf } D_k = 2^k m$, and its atom at $2^k m$ has mass $q_k < \frac{1}{2}$. Since $g_k = G/2^k$ with G as in Lemma 4.4 for $Z = D_k$, the supremum γ_k is attained at some $t_k^* \in (0, \infty)$. By Lemma 4.3 at $t = t_k^*$, $\gamma_k = g_k(t_k^*) < g_{k+1}(t_k^*) \leq \gamma_{k+1}$; by Lemma 4.2, $\gamma_{k+1} \leq \gamma_{\text{rec}}$; by Lemma 4.3 again, $\gamma_0 \leq \gamma_1 \leq \dots \leq \gamma_k$; and $\gamma_0 = \gamma_{\text{brw}}$ by Theorem 3.1. This proves (3). A positive law with finite mean is bounded below by 0, so it is covered. $\square$

Remark 4.6. (a) If $\xi \equiv c$, then $\gamma_{\text{brw}} = \gamma_k = \gamma_{\text{rec}} = c$ for every k ; no k as in Theorem A exists, the suprema γ_k are not attained, and the strict inequality fails. (b) If $\xi > 0$ and $\mathbb{E}\xi = \infty$, then $\gamma_{\text{rec}} = \infty$ by the definition of the source, while $\gamma_{\text{brw}} < \infty$. (c) If $\mathbb{P}(\xi = m) \geq \frac{1}{2}$, then $\gamma_{\text{brw}} = m$ (Remark 3.2(b)) and Theorem A says that $\gamma_{\text{rec}} > m$. Here q_k is the probability that G_k has a source-sink path of edges of weight m ; it tends to 0 for every $q_0 < 1$, while in the tree such paths exist with positive probability when $\mathbb{P}(\xi = m) > \frac{1}{2}$.

Remark 4.7. If $\mathbb{P}(\xi = m) < \frac{1}{2}$, then $k = 0$ in Theorem A, and with the maximiser t_0^* of g_0 and $\mathbb{E}e^{-t \min(\xi', \xi'')} = 2\mathbb{E}e^{-t\xi} - \mathbb{E}e^{-t \max(\xi', \xi'')}$ one obtains

$$\gamma_{\text{rec}} - \gamma_{\text{brw}} \geq g_1(t_0^*) - g_0(t_0^*) = -\frac{1}{2t_0^*} \log\left(1 - \frac{\mathbb{E}e^{-t_0^* \max(\xi', \xi'')}}{2\mathbb{E}e^{-t_0^* \xi}}\right) > 0.$$

For the uniform law on $[0, 1]$ this is at least 0.01941777, and for the exponential law with mean one at least 0.03150147 (Section 6). The true gaps are much larger (about 0.084 and 0.147 by Theorem C), because γ_k increases with k .

5 Characterisation and rate: the proof of Theorem B

Proof of Theorem B. (i) By Lemmas 4.2 and 4.3, (γ_k) is non-decreasing and bounded by γ_{rec} . Let $s > 0$. Taking $t = s/2^k$ in (2), $\gamma_k \geq g_k(s/2^k) = -\log(2\mathbb{E}e^{-sX_k})/s$. By Lemma 4.1, $X_k \rightarrow \gamma_{\text{rec}}$ in probability, and $x \mapsto e^{-sx}$ is bounded and continuous on $[m, \infty)$, which contains the values of X_k ; hence $\mathbb{E}e^{-sX_k} \rightarrow e^{-s\gamma_{\text{rec}}}$ and $\lim_k \gamma_k \geq \gamma_{\text{rec}} - (\log 2)/s$. Let $s \rightarrow \infty$.

(ii) This is Lemma 4.1 together with Lemma 4.2.

(iii) Fix n , number the $N = 3^n$ edges of G_n as $e_1, \dots, e_N$, and let $\xi_1, \dots, \xi_N$ be their weights and $\xi'_1, \dots, \xi'_N$ independent copies. Let $Z := D_n(\xi_1, \dots, \xi_N)$ and let Z_i be the same function of the weights $\xi^{(i)}$ in which ξ_i is replaced by ξ'_i . Since $2^n m \leq Z \leq$ the weight of a fixed path, $\mathbb{E}Z^2 < \infty$. The Efron–Stein inequality [ES81], in the form of Steele [Ste86], gives $\text{Var } Z \leq \frac{1}{2} \sum_i \mathbb{E}(Z - Z_i)^2$. The pair $(\xi, \xi^{(i)})$ is exchangeable, so $Z - Z_i$ is symmetric in law and $\mathbb{E}(Z - Z_i)^2 = 2\mathbb{E}(Z - Z_i)_+^2$; thus $\text{Var } Z \leq \sum_i \mathbb{E}(Z - Z_i)_+^2$.

Fix a measurable choice $\pi(w)$ of a minimal oriented path for the weights w , for instance the first minimal path in a fixed enumeration. The path $\pi(\xi^{(i)})$ has weight $Z_i + (\xi_i - \xi'_i)\mathbf{1}\{e_i \in \pi(\xi^{(i)})\}$ under ξ , so $(Z - Z_i)_+ \leq (\xi_i - \xi'_i)_+ \mathbf{1}\{e_i \in \pi(\xi^{(i)})\}$. Exchanging ξ_i and ξ'_i , which does not change the joint law,

$$\mathbb{E}(Z - Z_i)_+^2 \leq \mathbb{E}[(\xi'_i - \xi_i)_+^2 \mathbf{1}\{e_i \in \pi(\xi)\}] = \mathbb{E}[\mathbf{1}\{e_i \in \pi(\xi)\} \varphi(\xi_i)], \quad \varphi(x) := \mathbb{E}(\xi' - x)_+^2,$$

where we conditioned on ξ . For $x \geq m$, $\varphi(x) \leq \mathbb{E}(\xi - m)^2$, and $\xi_i \geq m$. Summing over i and using $|\pi(\xi)| = 2^n$, $\text{Var } D_n \leq 2^n \mathbb{E}(\xi - m)^2$. Then $\mathbb{E}|X'_j - X''_j| \leq (2 \text{Var } X_j)^{1/2} \leq \sqrt{2} \cdot 2^{-j/2} \|\xi - m\|_2$, and by Lemma 4.1

$$a_n - \gamma_{\text{rec}} = \frac{1}{4} \sum_{j \geq n} \mathbb{E}|X'_j - X''_j| \leq \frac{\sqrt{2}}{4} \|\xi - m\|_2 \frac{2^{-n/2}}{1 - 2^{-1/2}} = \frac{1 + \sqrt{2}}{2} \|\xi - m\|_2 2^{-n/2}. \quad \square$$

Remark 5.1. Since $\|D_n - \mathbb{E}D_n\|_2 \leq 2^{n/2} \|\xi - m\|_2$, Theorem B(iii) gives $\|D_n - \gamma_{\text{rec}} 2^n\|_2 \leq \frac{3+\sqrt{2}}{2} \|\xi - m\|_2 2^{n/2}$: the fluctuations and the lower-order term of D_n are at most of the order of the square root of the height 2^n . Question 9.2 asks for the true order; we do not answer it. Non-certified floating-point computations suggest that $\mathbb{E}|X'_k - X''_k| = 4(a_k - a_{k+1})$ decreases by a factor of about 0.655 per level for both laws, so that the standard deviation of D_n would grow like $1.31^n \approx (2^n)^{0.39}$.

The following comparison is used for the exponential law, which is not bounded above.

Lemma 5.2 (truncation). *Let $L \in \mathbb{R}$ and $\eta := \min(\xi, L)$. Then for every n , $a_n(\eta) \leq a_n(\xi) \leq a_n(\eta) + \mathbb{E}(\xi - L)^+$, where $a_n(\cdot)$ denotes a_n for the indicated law of the weights.*

Proof. Couple the weights by $\eta_e = \min(\xi_e, L)$. The first inequality holds because D_n is non-decreasing in every weight. Let $\pi(\eta)$ be a measurable choice of a minimal path for the weights η . Then

$$D_n(\xi) \leq \sum_{e \in \pi(\eta)} \xi_e = D_n(\eta) + \sum_e (\xi_e - L)^+ \mathbf{1}\{e \in \pi(\eta)\}.$$

Fix an edge e and let η^e be η with η_e replaced by L . On $\{\xi_e > L\}$ we have $\eta = \eta^e$, so $(\xi_e - L)^+ \mathbf{1}\{e \in \pi(\eta)\} = (\xi_e - L)^+ \mathbf{1}\{e \in \pi(\eta^e)\}$, and η^e does not depend on ξ_e . Moreover $\{e \in \pi(\eta^e)\} \subset \{e \in \pi(\eta)\}$: if $\eta_e = L$ the two configurations coincide; if $\eta_e < L$ and the minimal path $\pi(\eta^e)$ contains e , then passing from η^e to η lowers the weight of every path through e by $L - \eta_e > 0$ and leaves the other paths unchanged, so every minimal path for η contains e . Hence $\mathbb{E}[(\xi_e - L)^+ \mathbf{1}\{e \in \pi(\eta)\}] = \mathbb{E}(\xi - L)^+ \mathbb{P}(e \in \pi(\eta^e)) \leq \mathbb{E}(\xi - L)^+ \mathbb{P}(e \in \pi(\eta))$, and summing over e , $\mathbb{E}D_n(\xi) \leq \mathbb{E}D_n(\eta) + \mathbb{E}(\xi - L)^+ \mathbb{E}|\pi(\eta)| = \mathbb{E}D_n(\eta) + 2^n \mathbb{E}(\xi - L)^+$. $\square$

6 Certified values: the proof of Theorem C

This section describes the computations behind Theorem C precisely enough to check the programs (Section 7) against it.

6.1 Reduction to enclosures of means

Let $L \in \mathbb{R}$, $\eta := \min(\xi, L)$ and $C := \frac{1+\sqrt{2}}{2}$. By Theorem B(ii),(iii) and Lemma 5.2, for every k ,

$$a_k(\eta) - C \|\xi - m\|_2 2^{-k/2} \leq \gamma_{\text{rec}} \leq a_k(\eta) + \mathbb{E}(\xi - L)^+. \quad (5)$$

For ξ uniform on $[0, 1]$ we take $L = 1$, so $\eta = \xi$, and $\|\xi - m\|_2 = 1/\sqrt{3}$; for ξ exponential with mean 1 we take $L = 32$, so $\mathbb{E}(\xi - L)^+ = e^{-32} < 1.3 \cdot 10^{-14}$, and $\|\xi - m\|_2 = \sqrt{2}$. It remains to enclose $a_k(\eta) = \mathbb{E}X_k(\eta)$, where $X_{k+1} \stackrel{d}{=} \frac{1}{2}(X_k + \min(X'_k, X''_k))$ and $X_0 = \eta$.

6.2 Grid variables

We write $Y \leq_{\text{st}} Y'$ if $\mathbb{P}(Y \leq x) \geq \mathbb{P}(Y' \leq x)$ for all x . If $Y_j \leq_{\text{st}} Y'_j$ for $j = 1, 2, 3$, with Y_1, Y_2, Y_3 independent and Y'_1, Y'_2, Y'_3 independent, then $Y_1 + \min(Y_2, Y_3) \leq_{\text{st}} Y'_1 + \min(Y'_2, Y'_3)$ (couple by quantile transforms). Let h be a power of 2. We propagate two variables with values in $h\mathbb{Z} \cap [0, T]$, where $T \geq \eta$ is the top of the grid, $V_k \leq_{\text{st}} X_k(\eta) \leq_{\text{st}} U_k$, each represented by its distribution function $F(i) = \mathbb{P}(\cdot \leq ih)$ on a window of grid points:

- $U_0 := h\lceil\eta/h\rceil$ and $V_0 := h\lfloor\eta/h\rfloor$;
- the minimum of two independent copies has survival function $(1 - F)^2$, which is rounded up for U (a stochastically larger variable) and down for V ;
- if Z is the sum of the variable and of this minimum (a convolution on the grid), the new variable has distribution function at most $\mathbb{P}(Z \leq 2ih)$ on the U side, which gives a variable $\geq_{\text{st}} Z/2$, and at least $\mathbb{P}(Z \leq (2i + 1)h)$ on the V side, which gives a variable $\leq_{\text{st}} Z/2$, since $Z \in h\mathbb{Z}$;
- pairs that involve the atom at T (U side) or at 0 (V side) are sent to that atom, tails of tiny mass are moved to the edge of the window in the safe direction, and the grid is refined ($h \rightarrow h/2$, which does not change the law) whenever the window has fewer than 2^{15} points.

Every step either is exact or replaces a law by a stochastically larger one (U side) or smaller one (V side), so by induction and the coupling remark $V_k \leq_{\text{st}} X_k(\eta) \leq_{\text{st}} U_k$, hence $\mathbb{E}V_k \leq a_k(\eta) \leq \mathbb{E}U_k$, for every k . The means are computed exactly as $h \sum_i (1 - F(i))$ in rational arithmetic.

6.3 Arithmetic

The U side (program `bracket.py`) works with exact integers: distribution functions in units of 2^{-31} , so that every product and every partial sum of a convolution is at most 2^{62} ; the total mass of every convolution is checked exactly, and random entries are recomputed with unbounded integers. On the V side (program `bracket_down.py`) one unit of rounding at the bottom of the window would be tripled at every level, so this side works in IEEE double precision with one-sided margins: every computed value of a distribution function is multiplied by $1 + 10^{-9}$, increased by 10^{-300} , capped at 1 and made non-decreasing.

Lemma 6.1. *On the V side every stored value of a distribution function is at least the exact value for the variables defined by the previously stored arrays.*

Proof. A stored value is a cumulative sum $\sum_{l \leq j} \sum_i p_i q_{l-i}$ (plus the mass of pairs sent to 0, itself rounded up), where the p_i, q_l are differences of consecutive entries of non-decreasing arrays of doubles. In round-to-nearest arithmetic with $u = 2^{-53}$, an addition, multiplication or subtraction of nonnegative doubles returns its exact result times a factor in $[1 - u, 1 + u]$ (a subtraction $a - b$ with $a \geq b \geq 0$ as

well), unless the result is subnormal, when the absolute error is at most 2^{-1075} [Hig02]. In any order of evaluation the computed value is therefore at least $(1 - u)^d$ times the exact one minus $P \cdot 2^{-1074}$, where d bounds the number of operations applied to a single term and P is the number of operations. The windows have at most $2^{20} + 1$ points, so $d < 2^{20} + 2^{21} + 10$ and $(1 - u)^d > 1 - 3.6 \cdot 10^{-10}$, while $P < 2^{43}$ and $P \cdot 2^{-1074} < 10^{-300}$. The factor $1 + 10^{-9}$, which as a double is at least $1 + 0.99 \cdot 10^{-9}$, and the addition of 10^{-300} (both computed with at most two further roundings) therefore produce an upper bound; capping at 1 and taking running maxima keep it. $\square$

The initial grid for the exponential law. The values $\lfloor 2^{31}(1 - e^{-ih}) \rfloor$ (U side) are computed with 60 digits and a check that their distance to the nearest integer exceeds 10^{-40} . On the V side the doubles $F(i) \geq 1 - e^{-(i+1)h}$ are obtained by rounding a 40-digit evaluation upwards; in the first version of the program this step had no error bound. The version in the package checks every one of the 2^{20} initial values on both sides against enclosures of the exact values computed in 200-bit interval arithmetic [The23]: all of them lie on the safe side, with smallest margin $9.3 \cdot 10^{-24}$ on the V side, and the outputs are unchanged. The second verification run had made the same check independently, with the same result.

6.4 Results for the uniform and the exponential law

The uniform law is started on the grid $h = 2^{-18}$ with $T = 1$, the exponential law on $h = 2^{-15}$ with $T = L = 32$, and both are run for 40 levels. Table 2 shows some of the enclosures of a_k ; all 41 levels are in the recorded outputs. By (5) the best upper bound is $\min_k \mathbb{E}U_k (+e^{-32})$, attained at $k = 40$ (uniform) and $k = 35$ (exponential; beyond it the rounding on the U side outweighs the decrease of a_k), and the best lower bound is $\max_k (\mathbb{E}V_k - C\|\xi - m\|_2 2^{-k/2})$, attained at $k = 40$, where $C\|\xi - m\|_2 2^{-20}$ is about $6.6 \cdot 10^{-7}$ and $1.6 \cdot 10^{-6}$. With interval arithmetic for $\sqrt{2}$, $\sqrt{3}$, e^{-32} and outward rounding to nine decimals this gives the enclosures of Theorem C(a),(b). As a test, the exact means a_1, a_2, a_3 , namely $\frac{3}{4}, \frac{29}{48}, \frac{52201}{100800}$ (Exp(1)) and $\frac{5}{12}, \frac{229}{630}, \frac{4747708908589}{14399277984000}$ ($U[0, 1]$), computed from the exact laws of D_k , lie in the recorded enclosures of a_1, a_2, a_3 .

	$U[0, 1]$	Exp(1)
$a_6 \in$	[0.2853880, 0.2854034]	[0.4142825, 0.4143976]
$a_8 \in$	[0.2758460, 0.2758652]	[0.3938533, 0.3939761]
$a_{40} \in$	[0.268786264, 0.268808710]	[0.379283856, 0.379413139]
$\gamma_{\text{rec}} \in$	[0.268785599, 0.268808710]	[0.379282228, 0.379412906]

Table 2: Certified enclosures $[\mathbb{E}V_k, \mathbb{E}U_k]$ of $a_k = 2^{-k}\mathbb{E}D_k$, rounded outwards (for Exp(1) including e^{-32}).

First-moment bounds and γ_{brw} . The function $\varphi(t) = \ell(t) - t\ell'(t)$, $\ell(t) = \log(2\mathbb{E}e^{-t\xi})$, is strictly decreasing ($\varphi' = -t\ell'' < 0$ by strict convexity of ℓ), and $g'_0 = \varphi/t^2$; so g_0 increases and then decreases, and its maximiser t_0^* is the root of φ . Enclosing t_0^* by bisection in interval arithmetic and evaluating g_0 on the enclosure gives the enclosures of γ_{brw} . The bounds $\gamma_k \geq g_k(t)$ of Table 1 are evaluated in interval arithmetic: for $k = 1$ from $\mathbb{E}e^{-tD_1} = \mathbb{E}e^{-t\xi} \mathbb{E}e^{-t\min(\xi', \xi'')}$, where $\mathbb{E}e^{-t\min(\xi', \xi'')}$ is $\frac{2}{t} - \frac{2(1-e^{-t})}{t^2}$ (uniform) or $\frac{2}{2+t}$ (exponential); for the exponential law at $k = 2, 3$ from the exact rational values of $\mathbb{E}e^{-tD_k}$, e.g. $\mathbb{E}e^{-\frac{31}{20}D_2} = \frac{193280000}{2596876191}$. The gaps of Remark 4.7 are evaluated on the enclosures of t_0^* . A further lower bound of a different kind, Lemma 4.2 at level 40 with V_{40} in place of D_{40} (allowed since $V_{40} \leq_{\text{st}} X_{40}$), gives $\gamma_{\text{rec}} \geq 0.266475773$ (uniform, $s = 2^{40}t = 300$) and $\gamma_{\text{rec}} \geq 0.37581812$ (exponential, $s = 200$); it is weaker than the bound from Theorem B(iii) but does not use it.

6.5 Weights in $\{0, 1\}$

Let $\mathbb{P}(\xi = 0) = p \geq \frac{1}{2}$ and $\mathbb{P}(\xi = 1) = 1 - p$; by convention (iii) these stand for the positive weights $c, 1 + c$, with all constants increased by c . Here $\gamma_{\text{brw}} = 0$, the supremum defining γ_0 is not attained (Remark 3.2(b)), and Theorem A starts at a higher level. The laws of the integer-valued D_k are computed exactly, up to level 6 by the program `bernoulli_atom.py` and up to level 8 by `check_note.py` (Section 7). For $p = \frac{1}{2}$: $q_1 = \frac{3}{8} < \frac{1}{2}$, so $k = 1$ in Theorem A, and $\gamma_1 \geq g_1(\frac{57}{20}) \geq 0.0372545589$, $\gamma_2 \geq g_2(\frac{19}{10}) \geq 0.0666301146$. For $p = \frac{3}{4}$: $q_0, \dots, q_4 = 0.75, 0.703125, 0.6411\dots, 0.5585\dots, 0.4497\dots$, so $k = 4$, and $\gamma_4 \geq g_4(\frac{73}{20}) \geq 0.0014245853$, $\gamma_5 \geq g_5(\frac{21}{10}) \geq 0.0046486167$. At level 8 the lower bounds $\gamma_8 \geq g_8(\frac{31}{100}) \geq 0.12392609$ ($p = \frac{1}{2}$) and $\gamma_8 \geq g_8(\frac{19}{25}) \geq 0.00964880$ ($p = \frac{3}{4}$), together with the exact upper bounds a_8 , give Theorem C(c); these level-8 values are computed by `check_note.py`. For these laws Lemma 4.1 and the variance bound of Theorem B(iii) were also checked exactly for $k \leq 8$.

Proof of Theorem C. (a) and (b) follow from (5), Lemma 6.1 and the computations of Sections 6.2 to 6.4; (c) from Lemma 4.2, Theorem B(ii) and Section 6.5. The bound $\gamma_{\text{rec}} \geq g_1(\frac{11}{5})$ for the exponential law is Lemma 4.2: from $\mathbb{E}e^{-tD_1} = \frac{2}{(1+t)(2+t)}$ we get $g_1(t) = \log \frac{(1+t)(2+t)}{4} / (2t)$, and at $t = \frac{11}{5}$, $\frac{(16/5)(21/5)}{4} = 3.36$, so $g_1(\frac{11}{5}) = \frac{\log 3.36}{4.4} = 0.27544\dots > 0.23197 > \gamma_{\text{brw}}$. $\square$

In particular, the statement in the UnsolvedMath summary of the problem that γ_{rec} equals the branching random walk constant in the exponential case is incorrect: $\gamma_{\text{rec}} \geq 0.27544 > 0.23197 > \gamma_{\text{brw}}$ by the last computation alone, and $\gamma_{\text{rec}} - \gamma_{\text{brw}}$ is about 0.147 by Theorem C(b).

7 Computations and reproducibility

Theorems A and B are proved without computation; Theorem C is computer-assisted. The reproducibility package (folder `reproducibility/` of the source archive, Python [HMvdW⁺20, The23]) contains:

The programs of this note are in `programs/`, with their recorded outputs in `outputs/`: the programs `bracket.py`, `bracket_down.py` and `assemble.py` for Theorem C(a),(b); the program `certify_constants.py` for γ_{brw} and the first-moment bounds; `bernoulli_atom.py` for the weights in $\{0, 1\}$ up to level 6; two Monte Carlo programs on the literal graphs, whose estimates of a_6, a_8 lie within 2.5 standard errors of Table 2 (the larger run, for a_8 and the uniform law, within 0.3); and the exploration of Remark 5.1. The bracket programs run for one to two minutes (uniform law) and three and a half minutes (exponential law) each, the others for less than half a minute. At the writing of this note the interval check of Section 6.3 and a guard against overwriting outputs were added, and all programs were run again: every output is identical to the recorded one, up to running times and the line reporting the interval check.

The check program `check_note.py` (ten seconds, exact and interval arithmetic) checks Lemma 2.1 for $n \leq 4$, the exact laws and means of the low levels, all entries of Table 1, the enclosures of Theorem C(a),(b), re-assembled from the recorded exact means, and the constant of Theorem B(iii); for the weights in $\{0, 1\}$ it computes the laws of D_k up to level 8 and the enclosures of Theorem C(c).

The programs and outputs of the three verification runs described below, in `verification_run_A/`, `verification_run_B/` and `independent_run_2/`.

Verification. The results were first obtained, with proofs, with the programs in `programs/` in their first version. Two independent verification runs, both AI-assisted, followed; they did not see each other. Each re-derived every proof line by line, compared the statement with the source and wrote its own programs. The first recomputed γ_{brw} , the first-moment certificates, the exact low levels and the examples with weights in $\{0, 1\}$, and computed certified enclosures of γ_{rec} with its own program

(doubles with one-sided margins on both sides, exact halving of the grid, exact start at level 3 for the exponential law): $[0.2687907726, 0.2688051565]$ (uniform) and $[0.3793380918, 0.3793643414]$ (exponential), inside those of Theorem C. It reproduced every recorded mean of the bracket programs, ran a Monte Carlo simulation with the mirror pattern, and compared the formula for γ_{brw} with [Big10, Theorem 3.1]. It required the shift of convention (iii) for the weights in $\{0, 1\}$, and that the sentence after Question 9.1 be presented as an expectation of the authors, which Theorem B(ii) proves. The second computed the exact laws of D_k , $k \leq 3$, and certified enclosures that start from the exact law of D_3 and avoid rounding at every level: $[0.2687943832, 0.2688019634]$ and $[0.3793410791, 0.3793636293]$, inside those of the first run; its enclosures for the weights in $\{0, 1\}$ at level 8 agree with Theorem C(c) to four decimals. Its programs rejected deliberately wrong inputs and variants; it iterated the recursion for M_n up to $n = 32000$ to test the convention for the branching random walk, checked the 2^{20} initial values of both exponential grids in 200-bit interval arithmetic, reproduced the outputs of the programs of the note up to running times, and repeated the literature search. It required that the initial rounding in `bracket_down.py` be made rigorous, or its interval check recorded, and the DOIs and pages of [KKT16, HJ04, ABCD⁺20]. All corrections were made. Neither run found an error in a proof or in a stated number.

A third independent verification run, also AI-assisted, then examined the final text of the note, the package and the sources, as if nothing had been checked before. It read Section 9 of [Ben13] and Section 2 and Theorem 3.1 of [Big10] again, re-derived every proof of the note line by line, and wrote its own programs (folder `independent_run_2/`): a brute-force check of Lemma 2.1 on the graphs built literally from the definition (all simple paths of the unoriented multigraph for $n \leq 4$, Dijkstra’s algorithm against the recursion for $n \leq 7$); the exact laws of D_1 and D_2 for both laws, which reproduce the exact means a_1, a_2, a_3 and the exact values of $\mathbb{E}e^{-tD_k}$ of Section 6.4; γ_{brw} , all first-moment bounds of Table 1 and the gaps of Remark 4.7 in ball arithmetic (the FLINT library, through python-flint); the weights in $\{0, 1\}$ exactly up to level 8; and certified enclosures of γ_{rec} in exact integer arithmetic (distribution functions in units of 2^{-160} , exact convolutions of big integers with FLINT, the exponential law truncated at $L = 40$, start from the exact law of D_1): $[0.2687956521, 0.2688009027]$ (uniform) and $[0.3793364429, 0.3793661592]$ (exponential), inside those of Theorem C; its enclosures of a_6, a_8 and a_{40} lie inside those of Table 2. It ran all programs of the package again from the source archive, with outputs identical to the recorded ones up to running times and the line of the interval check, recomputed Table 2 and Theorem C(a),(b) from the recorded exact means, and repeated the literature search. It found no error in a proof or in a stated number. It required editorial corrections, which were made: the words “if true” of the question in the abstract, the reading of the source in convention (ii), the degenerate case in Remark 4.6(a), the name of the program that computes Theorem C(c), and the version of [Big10] that was read.

Scope and priority. The note answers the “in particular” part of Question 9.1 of [Ben13]: $\gamma_{\text{rec}} > \gamma_{\text{brw}}$ for every non-degenerate law of positive weights with finite mean (Theorem A). The “compute” part is answered only by the characterisation of Theorem B and the certified values of Theorem C; no closed formula is given, and Question 9.2 is not treated. The question and its setting are due to Benjamini [Ben13], whose Section 9 is based on a project with N. Curien; the formula for γ_{brw} is the theorem of Hammersley, Kingman and Biggins [Ham74, Kin75, Big76], used in the form of [Big10, Theorem 3.1]; the proof of Theorem A is a standard first-moment argument, and the Efron–Stein inequality [ES81, Ste86] is classical. No priority is claimed.

The source [Ben13] was read in the author-hosted version dated 16.07.12 (Section 9, Figure 1), from a copy saved earlier, the file being no longer available at the author’s address on 14 October 2026; the published chapter was not compared. Of [Big10], in the arXiv version (v1), whose numbering we use, Section 2 and Theorem 3.1 with the paragraph after it were read; of [KKT16, HJ04, ABCD⁺20,

BZ12, LM10] the abstracts, of [KKT16] also the introduction and Section 8.3, and of [DH91] a review (some of these only in the verification runs). The proofs of the cited theorems were not checked. The searches (14 October 2026) used the arXiv API, Crossref, OpenAlex, Semantic Scholar, zbMATH and a few web searches, and the works citing [Ben13] (15 in OpenAlex; 19 in Semantic Scholar, two of them through a record that merges [Ben13] with a different paper of 2026), of which only [KKT16] concerns hierarchical graphs; the third verification run repeated the searches in the arXiv API, Crossref, OpenAlex and zbMATH and one web search on the same day. No answer to Question 9.1 was found in the literature accessible to us; a search that finds nothing is not a proof of novelty.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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