

Equilibria of Inverse-Square Repulsion on the Line Are Arithmetic Progressions

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Abstract

Benjamini asked whether every configuration of points on the real line that is in equilibrium under the inverse-square repulsive force must be an arithmetic progression. Georgakopoulos and Kolountzakis proved this when some gap between consecutive points has maximal or minimal length, and described the general (aperiodic) case as open. We show that the answer is yes. More generally, let $1 < s \leq 2$, and let $X \subset \mathbb{R}$ be a locally finite set with at least two points such that, for every $x \in X$, the total force $\sum_{y \in X \setminus \{x\}} |y - x|^{-s}$ is finite and the net force $\sum_{y \in X \setminus \{x\}} \text{sgn}(y - x) |y - x|^{-s}$ is zero. Then X is an arithmetic progression. No assumption on the gaps is needed. Subtracting the equilibrium equations of two consecutive points shows that the gaps g_n form a positive harmonic function for an explicit reversible random walk on $\mathbb{Z}$ with long-range jumps. Equilibrium also bounds the ratio of consecutive gaps, by $1.5386\dots$ when $s = 2$. With this bound, an energy estimate shows that the Doob transform of the walk by g is recurrent. Since $1/g$ is a positive harmonic function of the transformed walk, it is constant. This is an unrefereed note.

1 Introduction

In a one-page note of May 2015, I. Benjamini considered a locally finite configuration $(a_n)_{n \in \mathbb{Z}}$ of points on the real line, with a force such as $F(x, y) = |x - y|^{-2}$ between any two of them, and called the system in equilibrium if $\sum_{i \neq n} F(a_i, a_n) \text{sgn}(a_i - a_n) = 0$ for every n [Ben15]. Arithmetic progressions are in equilibrium, and Question 0.1 of [Ben15] asks whether, conversely, every equilibrium satisfies $a_n = \alpha n + \beta$ for all n . The same question is Question 5 in Section 4 of the problem list of the 2015 Warwick workshop “Random walks on graphs and potential theory” [Syl15]. With $F(x, y) = |x - y|^{-2}$ it is record AMR-099-0003 of the UnsolvedMath dataset [ula26].

Georgakopoulos and Kolountzakis [GK17] studied the question for a general strictly decreasing force law F . Their Problem 1 asks whether all distances between subsequent particles of an equilibrium must be equal. They proved that this is the case if some gap between consecutive particles has maximal or minimal length [GK17, Theorem 1]. In particular, periodic equilibria are arithmetic progressions, and finite equilibria on a circle are equally spaced. For $F(d) = d^{-2}$ and uniformly discrete configurations, they showed that if the particles on a half-line are in equilibrium, then their positions determine the positions of all other particles [GK17, Theorem 4]. They also noted that the ratio of consecutive gaps is bounded [GK17, Proposition 1]. Finally, for every continuous strictly decreasing F they constructed non-trivial configurations in which every particle except one is in equilibrium [GK17, Theorem 7]. For aperiodic configurations they describe the problem as open, even for $F(d) = d^{-2}$, and they state that it is open for every strictly decreasing F .

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We answer the question for $F(d) = d^{-2}$, and more generally for $F(d) = d^{-s}$ with $1 < s \leq 2$. The following definition makes the conventions of [GK17] explicit.

Definition 1.1. Let $s > 1$. An s -equilibrium is a set $X \subset \mathbb{R}$ with at least two points that is locally finite (every bounded interval contains only finitely many points of X) and satisfies, for every $x \in X$,

$$\sum_{y \in X \setminus \{x\}} |y - x|^{-s} < \infty, \quad (1)$$

$$\sum_{y \in X \setminus \{x\}} \operatorname{sgn}(y - x) |y - x|^{-s} = 0. \quad (2)$$

Condition (1) says that the total force exerted on each particle from each side is finite. It makes the series in (2) absolutely convergent, so that (2) does not depend on an order of summation. If (1) holds at one point $x_0 \in X$, then every interval $[x_0 - M, x_0 + M]$ contains at most $1 + M^s \sum_{y \neq x_0} |y - x_0|^{-s}$ points of X . So (1) at a single point already implies that X is locally finite, and then (1) holds at every point. We keep local finiteness in the definition because it is part of Benjamini's formulation.

Theorem 1.2. Let $1 < s \leq 2$, and let X be an s -equilibrium, that is, a locally finite set $X \subset \mathbb{R}$ with at least two points, such that for every $x \in X$ the total force (1) from both sides is finite and the net force (2) is zero. Then $X = \{\alpha n + \beta : n \in \mathbb{Z}\}$ for some $\alpha > 0$ and $\beta \in \mathbb{R}$. Equivalently, if $(a_n)_{n \in \mathbb{Z}}$ is the increasing enumeration of X (Lemma 2.1), then $a_n = \alpha n + \beta$ for all n . Conversely, every such set is an s -equilibrium.

For $s = 2$, Theorem 1.2 answers Benjamini's question, and Problem 1 of [GK17] for $F(d) = d^{-2}$, in the affirmative. No assumption is made on the gaps: they may be unbounded, and they may accumulate at 0.

Remark 1.3 (Conventions and scope). (a) The note [Ben15] does not say in which sense the series converges. We read $\sum_{i \neq n}$ as a sum over an unordered index set, that is, as an absolutely convergent series; this gives (1) and (2). The arXiv version of [GK17] defines equilibrium in the same way: the total force on each particle from each side is finite, and the net force is zero. The published version states the finiteness assumption for the force law instead (the one-sided forces are finite when the particles sit at the integers) and defines equilibrium as zero net force. Its proofs compare one-sided forces, which therefore have to be finite. A “principal value” reading, in which the one-sided forces are infinite and only symmetric partial sums cancel, is not covered by Theorem 1.2.

(b) We take (a_n) to be the increasing enumeration of the configuration, as [GK17] do when they ask about the distances between subsequent particles. For an arbitrary bijection $\mathbb{Z} \rightarrow X$ the identity $a_n = \alpha n + \beta$ fails trivially, already for a re-enumerated arithmetic progression.

(c) For $s \leq 1$ an arithmetic progression violates (1). Problem 1 of [GK17] for a general strictly decreasing force law remains open, and we make no claim for exponents $s > 2$ (Remarks 5.3 and 5.4).

Idea of the proof. Let $g_n = a_{n+1} - a_n$. Subtract the equilibrium equation at a_{n+1} from the one at a_n , after shifting the summation index by one. This gives $\sum_{j \neq n} c(n, j)(g_j - g_n) = 0$ for explicit symmetric positive weights $c(n, j)$ (Lemma 2.3). For $s = 2$ they are $c(n, j) = (D + D')/(D^2 D'^2)$ with $D = |a_j - a_n|$ and $D' = |a_{j+1} - a_{n+1}|$. Thus g is a positive harmonic function of a reversible random walk on $\mathbb{Z}$ with long-range jumps. In this language, [GK17, Theorem 1] is the maximum principle (Remark 2.5). A Liouville theorem would finish the proof if the walk were known to be recurrent. We prove recurrence for the Doob transform of the walk by g , whose conductances are $c(n, j)g_n g_j$, by exhibiting test functions of small energy (Proposition 4.1). The estimate uses a bound on the ratio of consecutive gaps, which follows from equilibrium (Lemma 2.2), and, for $s = 2$, a quadratic bound on

the number of points in an interval, which follows from (1). Then $1/g$, a positive harmonic function of the recurrent transformed walk, is constant. Since g is harmonic, the transformed walk is recurrent if and only if the original one is. The transform does not create recurrence; it supplies the test functions (Remark 5.1).

2 The gaps form a harmonic function

Throughout this section $s > 1$, and X is an s -equilibrium.

Lemma 2.1. *The set X is unbounded above and below. Hence $X = \{a_n : n \in \mathbb{Z}\}$ for a strictly increasing sequence with $a_n \rightarrow \pm\infty$ as $n \rightarrow \pm\infty$, and this enumeration is unique up to a shift of the index.*

Proof. Suppose that X is bounded above. Since X is locally finite, it has a largest element x , and the sum in (2) at x equals $-\sum_{y \in X, y < x} (x - y)^{-s} < 0$, because X has at least two points. This contradicts (2), and the same argument applies below. A locally finite set that is unbounded in both directions has exactly one increasing enumeration by $\mathbb{Z}$ with a prescribed point at index 0. $\square$

From now on $(a_n)_{n \in \mathbb{Z}}$ is this enumeration, $g_n = a_{n+1} - a_n > 0$ are the gaps, and

$$F_n^- = \sum_{i < n} (a_n - a_i)^{-s}, \quad F_n^+ = \sum_{i > n} (a_i - a_n)^{-s}$$

are the forces on a_n from the left and from the right. By (1) and (2), $F_n^- = F_n^+ < \infty$ for every n .

Lemma 2.2 (Ratios of consecutive gaps). *Let r_s be the unique positive solution of $2r^{-s} + (1+r)^{-s} = 1$. Then $r_s > 1$, and*

$$r_s^{-1} g_{n-1} \leq g_n \leq r_s g_{n-1} \quad \text{for all } n \in \mathbb{Z}.$$

For $s = 2$, $r_2 = 1.538615 \dots$

Proof. Fix n , and write $y = a_n$, $z = a_{n+1}$, $u = g_{n-1}$ and $v = g_n$. For $i \geq n+2$ we have $a_i - y > a_i - z > 0$, hence

$$F_n^+ = v^{-s} + \sum_{i \geq n+2} (a_i - y)^{-s} \leq v^{-s} + F_{n+1}^+.$$

The terms $i = n$ and $i = n+1$ of F_{n+1}^- are v^{-s} and $(u+v)^{-s}$, and for $i \leq n-2$ we have $z - a_i > y - a_i > 0$. Hence

$$F_{n+1}^- \leq v^{-s} + (u+v)^{-s} + \sum_{i \leq n-2} (y - a_i)^{-s} = v^{-s} + (u+v)^{-s} + F_n^- - u^{-s}.$$

Using $F_n^- = F_n^+$ and $F_{n+1}^- = F_{n+1}^+$, we obtain

$$F_n^- = F_n^+ \leq v^{-s} + F_{n+1}^- \leq 2v^{-s} + (u+v)^{-s} + F_n^- - u^{-s}.$$

Since $F_n^- < \infty$, this gives $u^{-s} \leq 2v^{-s} + (u+v)^{-s}$, that is, $h_s(v/u) \geq 1$ with $h_s(r) = 2r^{-s} + (1+r)^{-s}$. The function h_s decreases strictly from $+\infty$ to 0 on $(0, \infty)$. So r_s is well defined, $v/u \leq r_s$, and $r_s > 1$ because $h_s(1) > 1$.

The reflected set $-X$ is again an s -equilibrium. Its increasing enumeration $b_m = -a_{-m}$ has the gaps $b_{m+1} - b_m = g_{-m-1}$. The bound just proved, applied to $-X$ at the index $m = -n$, gives $g_{n-1} \leq r_s g_n$. $\square$

Lemma 2.3 (The gaps are harmonic). *For integers $n \neq j$ put $D = |a_j - a_n|$ and $D' = |a_{j+1} - a_{n+1}|$, and let*

$$c(n, j) = \frac{D^{-s} - D'^{-s}}{D' - D} \quad \text{if } D \neq D', \quad c(n, j) = sD^{-s-1} \quad \text{if } D = D'.$$

Then $c(n, j) = c(j, n) > 0$, and for every $n \in \mathbb{Z}$

$$\sum_{j \neq n} c(n, j) (g_j - g_n) = 0, \quad (3)$$

where the series converges absolutely. For $s = 2$, $c(n, j) = (D + D')/(D^2 D'^2)$.

Proof. For $m \in \mathbb{Z}$ let $\Phi_m = \sum_{i \neq m} \text{sgn}(a_i - a_m) |a_i - a_m|^{-s}$. By (1) and (2) this series converges absolutely, and $\Phi_m = 0$. Substituting $i = j + 1$ in Φ_{n+1} we get

$$0 = \Phi_n - \Phi_{n+1} = \sum_{j \neq n} \left(\text{sgn}(a_j - a_n) D^{-s} - \text{sgn}(a_{j+1} - a_{n+1}) D'^{-s} \right),$$

where D and D' depend on j as in the statement. The subtraction may be done termwise, because both series converge absolutely. If $j > n$, both signs are $+1$ and $D' - D = (a_{j+1} - a_j) - (a_{n+1} - a_n) = g_j - g_n$, so the j -th term is $D^{-s} - D'^{-s} = c(n, j)(g_j - g_n)$. If $j < n$, both signs are -1 and $D - D' = (a_n - a_j) - (a_{n+1} - a_{j+1}) = g_j - g_n$, so the j -th term is $D'^{-s} - D^{-s} = c(n, j)(g_j - g_n)$. (If $D = D'$, both expressions vanish.) This proves (3).

The numbers D and D' do not change when n and j are exchanged. Moreover $c(n, j)$ is the divided difference of the strictly decreasing function $t \mapsto t^{-s}$ at D and D' , with the sign reversed. Hence $c(n, j) = c(j, n) > 0$. For $s = 2$, $(D^{-2} - D'^{-2})/(D' - D) = (D + D')/(D^2 D'^2)$. $\square$

Lemma 2.4 (Comparison). *For $n \neq j$ let D, D' be as in Lemma 2.3. Then $\min\{D, D'\} \geq D/(1 + r_s)$ and*

$$c(n, j) \leq K_s |a_j - a_n|^{-s-1}, \quad K_s := s(1 + r_s)^{s+1}. \quad (4)$$

For $s = 2$ one also has $c(n, j) \leq (1 + r_2)(2 + r_2)|a_j - a_n|^{-3}$, with $(1 + r_2)(2 + r_2) = 8.983 \dots$. Consequently $\pi(n) := \sum_{j \neq n} c(n, j) < \infty$, and $\sum_{j \neq n} c(n, j) g_j = \pi(n) g_n$ for every n .

Proof. By symmetry we may assume $j > n$. If $j = n + 1$, then $D = g_n$ and $D' = g_{n+1} \geq g_n/r_s$ by Lemma 2.2. If $j \geq n + 2$, then $D \geq g_n + g_{n+1} \geq (1 + r_s^{-1})g_n$, so $g_n \leq r_s D/(1 + r_s)$ and $D' = D + g_j - g_n \geq D - g_n \geq D/(1 + r_s)$. In both cases $\min\{D, D'\} \geq D/(1 + r_s)$. By the mean value theorem, $c(n, j) = s\xi^{-s-1}$ for some ξ between D and D' , so $c(n, j) \leq s \min\{D, D'\}^{-s-1} \leq K_s D^{-s-1}$. For $s = 2$, $c(n, j) = D^{-1} D'^{-2} + D^{-2} D'^{-1} \leq ((1 + r_2)^2 + (1 + r_2)) D^{-3}$.

Every $j \neq n$ satisfies $|a_j - a_n| \geq \delta_n := \min\{g_{n-1}, g_n\} > 0$, so $\pi(n) \leq K_s \delta_n^{-1} (F_n^- + F_n^+) < \infty$. Finally, the series $\sum_j c(n, j)(g_j - g_n)$ converges absolutely with sum 0 by Lemma 2.3, and $\sum_j c(n, j) g_n = \pi(n) g_n$. Adding the two series gives the last identity. $\square$

Remark 2.5 (The maximum principle). Identity (3) contains [GK17, Theorem 1] for the forces d^{-s} . If g attains its maximum at n , all terms of (3) are ≤ 0 , so they all vanish, and $c > 0$ gives $g_j = g_n$ for all j ; the same holds for a minimum. The proof of Lemma 2.3 uses only that the force law is strictly decreasing and that the one-sided forces are finite (see Remark 5.4).

3 Reversible random walks with long-range jumps

Let V be a countably infinite set, and let $c: V \times V \rightarrow [0, \infty)$ be symmetric, with $c(x, x) = 0$ and

$$\pi(x) := \sum_{y \in V} c(x, y) \in (0, \infty) \quad \text{for all } x \in V.$$

We call c a *network*. A vertex may have infinitely many y with $c(x, y) > 0$. The random walk on c is the Markov chain $(X_t)_{t \geq 0}$ with transition probabilities $P(x, y) = c(x, y)/\pi(x)$; it is reversible with respect to π , and we always assume that it is irreducible. We write P_x for the law of the walk started at x . A function u on V is *harmonic* if $\sum_y P(x, y)|u(y)| < \infty$ and $Pu(x) := \sum_y P(x, y)u(y) = u(x)$ for every x . The walk, or the network, is *recurrent* if $P_x[X_t = x \text{ for some } t \geq 1] = 1$ for some, and then every, x . The *energy* of $f: V \rightarrow \mathbb{R}$ is $\mathcal{E}_c(f) = \frac{1}{2} \sum_{x, y} c(x, y)(f(x) - f(y))^2 \in [0, \infty]$.

For locally finite networks the three lemmas below are standard; see, for example, [LP16, Chapter 2], and [Doo57] for the transform in Lemma 3.3. We include short proofs, because our networks are not locally finite. The only property used is $\pi < \infty$.

We use the following form of Green's identity. If u is finitely supported and $\sum_y c(x, y)|w(y)| < \infty$ for every x , then

$$\frac{1}{2} \sum_{x, y} c(x, y)(u(x) - u(y))(w(x) - w(y)) = \sum_x u(x) \sum_y c(x, y)(w(x) - w(y)), \quad (5)$$

and all series converge absolutely. Indeed, by the symmetry of c ,

$$\sum_{x, y} c(x, y)|u(x) - u(y)||w(x) - w(y)| \leq 2 \sum_x |u(x)| \sum_y c(x, y)(|w(x)| + |w(y)|) < \infty.$$

Expanding the product and exchanging x and y in one of the two resulting sums gives (5).

Lemma 3.1 (Dirichlet principle). *Let $x_0 \in V$, and let $f: V \rightarrow \mathbb{R}$ be finitely supported with $f(x_0) = 1$. Then*

$$\pi(x_0) P_{x_0}[X_t \neq x_0 \text{ for all } t \geq 1] \leq \mathcal{E}_c(f).$$

Consequently, if $\inf \mathcal{E}_c(f) = 0$ over all such f , the walk is recurrent.

Proof. Let $B = \{x_0\} \cup \text{supp } f$ and $A = B \setminus \{x_0\}$, both finite. For $S \subset V$ let $\tau_S = \inf\{t \geq 0 : X_t \in S\}$ and $\tau_S^+ = \inf\{t \geq 1 : X_t \in S\}$. By irreducibility, from every $x \in A$ the walk leaves A within $|A|$ steps with positive probability. Since A is finite, the Markov property shows that, from any starting point, the walk stays in A forever with probability 0.

Let $h(x) = P_x[\tau_{\{x_0\}} < \tau_{V \setminus B}]$. Then $0 \leq h \leq 1$, $h(x_0) = 1$, $h = 0$ off B , and $Ph(x) = h(x)$ for $x \in A$ (first-step analysis). By the same analysis at x_0 ,

$$q := P_{x_0}[\tau_{V \setminus B}^+ < \tau_{\{x_0\}}^+] = 1 - Ph(x_0) = h(x_0) - Ph(x_0).$$

By (5) with $u = w = h$,

$$\mathcal{E}_c(h) = \sum_{x \in B} h(x) \pi(x)(h(x) - Ph(x)) = \pi(x_0) q.$$

The function $\eta = f - h$ vanishes at x_0 and off B , so it is supported in A . By (5) with $u = \eta$ and $w = h$,

$$\frac{1}{2} \sum_{x, y} c(x, y)(h(x) - h(y))(\eta(x) - \eta(y)) = \sum_{x \in A} \eta(x) \pi(x)(h(x) - Ph(x)) = 0.$$

All energies involved are finite, so $\mathcal{E}_c(f) = \mathcal{E}_c(h) + \mathcal{E}_c(\eta) \geq \pi(x_0)q$. Finally, if $X_t \neq x_0$ for all $t \geq 1$, then either the walk visits $V \setminus B$ before returning to x_0 , or it stays in A from time 1 on; the second event has probability 0. Hence the probability of no return is at most $q \leq \mathcal{E}_c(f)/\pi(x_0)$. If $\inf \mathcal{E}_c(f) = 0$, then x_0 is recurrent, and so is every vertex, by irreducibility. $\square$

Lemma 3.2 (Liouville property). *If the walk is recurrent, every positive harmonic function is constant.*

Proof. Let $u > 0$ be harmonic. Under P_x the process $u(X_t)$ is a nonnegative martingale, so it converges almost surely to a finite limit. By recurrence and irreducibility, the walk visits every vertex infinitely often, almost surely. Hence $u(y) = u(z)$ for all $y, z \in V$. $\square$

Lemma 3.3 (Doob transform). *Let $h: V \rightarrow (0, \infty)$ satisfy $\sum_y c(x, y)h(y) = \pi(x)h(x)$ for all x , that is, let h be a positive harmonic function. Put $c^h(x, y) = c(x, y)h(x)h(y)$. Then the following hold.*

- (a) $\pi^h(x) := \sum_y c^h(x, y) = \pi(x)h(x)^2 < \infty$, so c^h is a network. Its walk is irreducible and has transition probabilities $P^h(x, y) = P(x, y)h(y)/h(x)$.
- (b) The function $1/h$ is harmonic for P^h .
- (c) $(P^h)^t(x, y) = P^t(x, y)h(y)/h(x)$ for all $t \geq 0$. In particular, the walks P and P^h have the same return probabilities, and c^h is recurrent if and only if c is.
- (d) $\mathcal{E}_c(h\varphi) = \mathcal{E}_{c^h}(\varphi)$ for every finitely supported $\varphi: V \rightarrow \mathbb{R}$.

Proof. (a) We have $\pi^h(x) = h(x) \sum_y c(x, y)h(y) = \pi(x)h(x)^2$, and $c^h(x, y)/\pi^h(x) = P(x, y)h(y)/h(x)$. (b) Indeed, $\sum_y P^h(x, y)/h(y) = \sum_y P(x, y)/h(x) = 1/h(x)$. (c) The formula follows from (a) by induction on t . Along every path from x back to x the factors $h(y)/h(x)$ cancel, so the two walks return to x with the same probability. (d) Put $u = h\varphi^2$, which is finitely supported. For all x, y ,

$$(h(x)\varphi(x) - h(y)\varphi(y))^2 = h(x)h(y)(\varphi(x) - \varphi(y))^2 + (h(x) - h(y))(u(x) - u(y)).$$

Multiply by $c(x, y)/2$ and sum. The first terms give $\mathcal{E}_{c^h}(\varphi)$, which is finite because φ is finitely supported and $\pi^h < \infty$. By (5) with $w = h$, the second terms are absolutely summable, with sum $\sum_x u(x)(\pi(x)h(x) - \sum_y c(x, y)h(y)) = 0$. $\square$

4 The energy estimate and the proof of the theorem

In this section $1 < s \leq 2$, X is an s -equilibrium, and we use the notation of Section 2. We write $r = r_s$ and $K = K_s$. By Lemma 2.4, c (with $c(n, n) := 0$) is a network on $\mathbb{Z}$ in the sense of Section 3; its walk is irreducible because $c(n, j) > 0$ for all $n \neq j$. By Lemma 2.4, g is a positive harmonic function for c . Let

$$c^*(n, j) = c(n, j)g_n g_j, \quad \pi^*(n) = \pi(n)g_n^2,$$

be the Doob transform of c by g (Lemma 3.3), and let P^* be its walk. For $R > 0$ let

$$\varphi_R(x) = \min\{1, \max\{0, 2 - |x|/R\}\} \quad (x \in \mathbb{R}), \quad f_R(n) = \varphi_R(a_n) \quad (n \in \mathbb{Z}).$$

Proposition 4.1. *Fix $R_1 \geq |a_0|$ such that $(-2R_1, 2R_1)$ contains at least two points of X . For $R \geq R_1$, the function f_R is finitely supported, $f_R(0) = 1$, and*

$$\begin{aligned} \mathcal{E}_{c^*}(f_R) &\leq K \left(\frac{7(1+r)}{2-s} + \frac{4(1+r)^{s+3}}{s} \right) R^{1-s} && \text{if } s < 2, \\ \mathcal{E}_{c^*}(f_R) &\leq \frac{K}{R} \left(7(1+r) \max\left\{1, \log \frac{(1+r)(1+16R^2(F_0^- + F_0^+))}{7}\right\} + 2(1+r)^5 \right) && \text{if } s = 2. \end{aligned}$$

In both cases $\mathcal{E}_{c^*}(f_R) \rightarrow 0$ as $R \rightarrow \infty$.

Proof. Since $|a_0| \leq R$, we have $f_R(0) = 1$, and f_R vanishes outside the finite set $\{n : |a_n| < 2R\}$. By (4),

$$\mathcal{E}_{c^*}(f_R) = \sum_{n < j} c(n, j) g_n g_j (f_R(n) - f_R(j))^2 \leq K(S_1 + S_2), \quad (6)$$

where S_1 is the sum of $g_n g_j (f_R(n) - f_R(j))^2 (a_j - a_n)^{-s-1}$ over the pairs $n < j$ with $a_j - a_n \leq R$, and S_2 is the same sum over the pairs $n < j$ with $a_j - a_n > R$.

Step 1: the cut-off. We have $0 \leq \varphi_R \leq 1$ and $|\varphi_R(x) - \varphi_R(y)| \leq |x - y|/R$. Moreover, φ_R is constant on each of $(-\infty, -2R]$, $[-R, R]$ and $[2R, \infty)$, and it vanishes on the first and the last. Hence, if $n < j$ and $f_R(n) \neq f_R(j)$, then $[a_n, a_j]$ meets $(-2R, -R) \cup (R, 2R)$, and at least one of a_n, a_j lies in $(-2R, 2R)$. If in addition $a_j - a_n \leq R$, then both a_n and a_j lie in $(-3R, 3R)$.

Step 2: comparison with integrals. Let $p > 0$ and $n < j$. Then

$$g_j (a_j - a_n)^{-p} \leq (1+r)^p \int_{a_j}^{a_{j+1}} (t - a_n)^{-p} dt, \quad (7)$$

$$g_n (a_j - a_n)^{-p} \leq r(1+r)^p \int_{a_{n-1}}^{a_n} (a_j - t)^{-p} dt. \quad (8)$$

For (7), note that $g_j \leq r g_{j-1} \leq r(a_j - a_n)$ by Lemma 2.2, because $j-1 \geq n$. So $t - a_n \leq a_{j+1} - a_n \leq (1+r)(a_j - a_n)$ for $t \in [a_j, a_{j+1}]$, and we integrate $(t - a_n)^{-p} \geq (1+r)^{-p}(a_j - a_n)^{-p}$ over this interval of length g_j . For (8), similarly, $g_{n-1} \leq r g_n \leq r(a_j - a_n)$, so $a_j - t \leq (1+r)(a_j - a_n)$ for $t \in [a_{n-1}, a_n]$. Integrating gives $g_{n-1}(a_j - a_n)^{-p} \leq (1+r)^p \int_{a_{n-1}}^{a_n} (a_j - t)^{-p} dt$, and $g_n \leq r g_{n-1}$.

Step 3: the near pairs. Let $I = \{n : |a_n| < 3R, g_n \leq R\}$, and for $n \in I$ let $J(n) = \{j > n : a_j - a_n \leq R\}$. If a pair $n < j$ with $a_j - a_n \leq R$ contributes to S_1 , then $|a_n| < 3R$ by Step 1, and $g_n \leq a_j - a_n \leq R$; so $n \in I$ and $j \in J(n)$. Using $(f_R(n) - f_R(j))^2 \leq (a_j - a_n)^2/R^2$ we obtain

$$S_1 \leq R^{-2} \sum_{n \in I} g_n \sum_{j \in J(n)} g_j (a_j - a_n)^{1-s}. \quad (9)$$

Let $n \in I$ with $J(n) \neq \emptyset$, and let $j^* = \max J(n)$. The intervals $[a_j, a_{j+1}]$, $j \in J(n)$, tile $[a_{n+1}, a_{j^*+1}]$, and $a_{j^*+1} - a_n \leq (1+r)(a_{j^*} - a_n) \leq (1+r)R$ by Step 2. By (7) with $p = s-1$ (and trivially if $J(n) = \emptyset$, since $g_n \leq R$),

$$\sum_{j \in J(n)} g_j (a_j - a_n)^{1-s} \leq (1+r)^{s-1} \int_{g_n}^{(1+r)R} \tau^{1-s} d\tau \leq \begin{cases} (1+r)R^{2-s}/(2-s), & s < 2, \\ (1+r) \log((1+r)R/g_n), & s = 2. \end{cases} \quad (10)$$

The intervals $[a_n, a_{n+1}]$, $n \in I$, are disjoint and lie in $(-3R, 4R)$, so $\sum_{n \in I} g_n \leq 7R$. For $s < 2$, (9) and (10) give

$$S_1 \leq \frac{7(1+r)}{2-s} R^{1-s}. \quad (11)$$

For $s = 2$, let $N = |I|$ and $\Sigma = \sum_{n \in I} g_n \leq 7R$. The function $\psi(t) = t \log((1+r)R/t)$ is concave on $(0, \infty)$, so Jensen's inequality gives $\sum_{n \in I} \psi(g_n) \leq N\psi(\Sigma/N) = \Sigma \log((1+r)RN/\Sigma)$. The function $\Sigma \mapsto \Sigma \log(\Lambda/\Sigma)$ increases on $(0, \Lambda/e]$ and is bounded by Λ/e . With $\Lambda = (1+r)RN$ we get $\sum_{n \in I} \psi(g_n) \leq 7R \max\{1, \log((1+r)N/7)\}$, and hence

$$S_1 \leq \frac{7(1+r)}{R} \max\left\{1, \log \frac{(1+r)N}{7}\right\} \quad (s = 2). \quad (12)$$

Finally, if $|a_n| < 3R$, then $|a_n - a_0| < 4R$ because $|a_0| \leq R$. Every $n \neq 0$ with $|a_n - a_0| \leq 4R$ contributes at least $(4R)^{-s}$ to $F_0^- + F_0^+$. Hence

$$N \leq \#\{n : |a_n| < 3R\} \leq 1 + (4R)^s (F_0^- + F_0^+), \quad (13)$$

which is $1 + 16R^2(F_0^- + F_0^+)$ for $s = 2$.

Step 4: the far pairs. By Step 1, every pair $n < j$ that contributes to S_2 has $a_n \in (-2R, 2R)$ or $a_j \in (-2R, 2R)$. We bound $(f_R(n) - f_R(j))^2 \leq 1$ and get

$$S_2 \leq \sum_{|a_n| < 2R} g_n \sum_{\substack{j > n \\ a_j - a_n > R}} \frac{g_j}{(a_j - a_n)^{s+1}} + \sum_{|a_j| < 2R} g_j \sum_{\substack{n < j \\ a_j - a_n > R}} \frac{g_n}{(a_j - a_n)^{s+1}}.$$

For fixed n , the intervals $[a_j, a_{j+1}]$ with $a_j - a_n > R$ are disjoint and lie in $[a_n + R, \infty)$. So by (7) with $p = s + 1$ the first inner sum is at most $(1 + r)^{s+1} \int_R^\infty \tau^{-s-1} d\tau = (1 + r)^{s+1} R^{-s}/s$. For fixed j , the intervals $[a_{n-1}, a_n]$ with $a_j - a_n > R$ are disjoint and lie in $(-\infty, a_j - R]$. So by (8) the second inner sum is at most $r(1 + r)^{s+1} R^{-s}/s$. Let $m < M$ be the smallest and the largest index with $a_m, a_M \in (-2R, 2R)$; there are at least two such indices because $R \geq R_1$. Then $\sum_{|a_n| < 2R} g_n = a_{M+1} - a_m = (a_M - a_m) + g_M$, where $a_M - a_m < 4R$ and $g_M \leq r g_{M-1} \leq r(a_M - a_m)$. So $\sum_{|a_n| < 2R} g_n \leq 4(1 + r)R$, and

$$S_2 \leq 4(1 + r)R \cdot \frac{(1 + r)^{s+1} + r(1 + r)^{s+1}}{s R^s} = \frac{4(1 + r)^{s+3}}{s} R^{1-s}. \quad (14)$$

For $s = 2$ this is $2(1 + r)^5/R$. Combining (6), (11), (12), (13) and (14) proves the proposition. $\square$

Proof of Theorem 1.2. Let X be an s -equilibrium with $1 < s \leq 2$. By Lemma 3.3, applied to the network c and its positive harmonic function g , the transform c^* is a network, its walk P^* is irreducible, and $1/g$ is a positive harmonic function for P^* . By Proposition 4.1 and Lemma 3.1, applied to c^* with $x_0 = 0$ and $f = f_R$,

$$\pi^*(0) P_0^* [X_t \neq 0 \text{ for all } t \geq 1] \leq \mathcal{E}_{c^*}(f_R) \xrightarrow{R \rightarrow \infty} 0,$$

so P^* is recurrent. By Lemma 3.2, $1/g$ is constant. Hence $g_n = g_0$ for all n , and $a_n = a_0 + g_0 n$.

Conversely, let $X = \{\alpha n + \beta : n \in \mathbb{Z}\}$ with $\alpha > 0$. For every $x \in X$, both one-sided sums equal $\sum_{k \geq 1} (\alpha k)^{-s} < \infty$, since $s > 1$. So (1) and (2) hold. $\square$

5 Remarks

Remark 5.1 (The role of the Doob transform). By Lemma 3.3(c) with $h = g$, the walk P^* has the same return probabilities as the walk of c . So, for an equilibrium, c is recurrent if and only if c^* is: the transform does not turn a transient walk into a recurrent one. Its role is to supply test functions. By Lemma 3.3(d), the finitely supported functions $n \mapsto g_n \varphi_R(a_n)/g_0$ take the value 1 at 0 and have c -energy $\mathcal{E}_{c^*}(f_R)/g_0^2 \rightarrow 0$. Thus the proof can also be phrased without P^* : the network c is recurrent by Lemma 3.1, and g , a positive harmonic function for c , is constant by Lemma 3.2. Heuristically, $c^*(n, j) \approx s g_n g_j |a_j - a_n|^{-1-s}$ is a Riemann sum, with mesh given by the gaps, of the translation-invariant kernel $s|x - y|^{-1-s} dx dy$ on $\mathbb{R}$. This kernel does not see the local density of the points; for $s = 2$ the density enters only through the logarithm in (12).

Proposition 4.1 uses equilibrium only through Lemma 2.2 and, for $s = 2$, through $F_0^- + F_0^+ < \infty$. Its proof applies to every increasing sequence whose consecutive gap ratios lie in $[r_s^{-1}, r_s]$ and which, when $s = 2$, satisfies the point count (13). For such a sequence that is not an equilibrium, g is in general

not harmonic for the weights c of Lemma 2.3, c^* is not a Doob transform of c , and Proposition 4.1 says nothing about the recurrence of c . The informal idea that c may be transient when the gaps shrink quickly concerns only such non-equilibrium sequences. We neither need nor prove it.

Remark 5.2 (What the proof uses). (a) Equilibrium of every particle is used in Lemma 2.2 and Lemma 2.3. If a single particle a_k is not in equilibrium, then (3) may fail at $n = k - 1$ and $n = k$, and the argument breaks down. This is consistent with [GK17, Theorem 7], which provides non-trivial configurations in which all particles but one are in equilibrium.

(b) The finiteness condition (1) is used in Lemma 2.3 and, for $s = 2$, in the point count (13). For $s = 2$, any bound with $\log \#\{n : |a_n| < 3R\} = o(R)$ would suffice in place of (13).

(c) Local finiteness, which follows from (1), is used in Lemma 2.1 and to make f_R finitely supported.

Remark 5.3 (Exponents $s > 2$; not claimed). Lemmas 2.1 to 2.4 and the bound (14) hold for every $s > 1$, with the same proofs. For $s > 2$, however, the integral in (10) is of order g_n^{2-s} . Then S_1 is controlled by $R^{-2} \sum_{n \in I} g_n^{3-s}$, which is large when many gaps are small. A sketch suggests that Hölder's inequality, together with the bound $\#\{n : |a_n| < 3R\} = o(R^s)$ that follows from (1), gives $S_1 = o(R^{s^2-3s+1})$ for $2 < s < 3$, and hence the conclusion of Theorem 1.2 for $2 < s \leq (3 + \sqrt{5})/2$. We have not written out these details with the same care, and we make no claim for $s > 2$.

Remark 5.4 (Other force laws). Let $F : (0, \infty) \rightarrow (0, \infty)$ be strictly decreasing, and assume that the one-sided forces $\sum_{y \in X, y > x} F(y - x)$ and $\sum_{y \in X, y < x} F(x - y)$ are finite and equal for every $x \in X$. The proof of Lemma 2.3 then gives (3) with $c(n, j) = (F(D) - F(D'))/(D' - D) > 0$ when $D \neq D'$ (when $D = D'$ the term vanishes, and any positive value of $c(n, j)$ will do). So Remark 2.5 recovers [GK17, Theorem 1] for such configurations. What is missing for general F is recurrence: our estimates use the homogeneity of d^{-s} in Lemmas 2.2 and 2.4. Problem 1 of [GK17] for general F remains open, and so does the principal-value reading mentioned in Remark 1.3(a).

Related work. The question is a force-based variant of the crystallization problem; see [BL15] for a review. In one dimension, Duneau and Katz [DK84] showed that the property of a two-body potential to give rise to a one-dimensional lattice, defined through sequences of finite stable equilibria, is stable under perturbations of the potential; they do not classify infinite equilibria. Since the pair potential $V(d) = d^{1-s}/(s - 1)$ is convex, a routine argument shows that an s -equilibrium minimizes the formal energy $\sum_{i < j} V(a_j - a_i)$ with respect to compactly supported perturbations that preserve the order of the points. This relates the problem to Aubry–Mather theory for long-range interactions in statistical mechanics [CdIL98, dILV10], where minimizers are typically shown to be ordered with respect to their translates. In the present translation-invariant setting, a configuration is ordered with respect to all of its translates $(a_{n+k} + t)_{n \in \mathbb{Z}}$, $k \in \mathbb{Z}$, $t \in \mathbb{R}$ (meaning that $a_{n+k} + t - a_n$ has the same sign for all n), if and only if it is an arithmetic progression. We have not checked whether the results of [CdIL98, dILV10] apply to the singular interaction d^{1-s} and to configurations that are not uniformly discrete, and [GK17] do not mention such a reduction.

Verification. The claims were checked as follows.

1. A program by the author, `check_general_s.py`, does the following.
 - It computes r_s for $s \in \{1.05, 1.25, 1.5, 1.75, 2\}$ to 50 digits.
 - It checks the finite-range form of the identity behind Lemma 2.3 in 60-digit arithmetic for $s \in \{\frac{5}{4}, \frac{3}{2}, \frac{7}{4}, 2\}$: 777 identities on random configurations with gaps between 10^{-6} and 10^6 , and relative errors below $2 \cdot 10^{-59}$.
 - It checks the inequality behind Lemma 2.2, which holds for every finite configuration, in 1006 cases.

- It checks Lemma 2.4 on adversarial gap patterns. For $s = 2$, the largest value of $c(n, j) |a_j - a_n|^3$ that it finds is 3.906, well below the constant 8.983... of Lemma 2.4.
 - It checks the inequalities $\mathcal{E}_{c^*}(f_R) \leq K(S_1 + S_2)$, (11) and (14) in floating point for $s = \frac{5}{4}$ and $s = \frac{3}{2}$, on five gap profiles and for $R = 10, 30, 100$.
2. Earlier programs, written while the proof was being found, treat $s = 2$.
- In exact rational arithmetic they check the identity of Lemma 2.3 term by term (6240 terms), the comparison bound of Lemma 2.4 (25,200 pairs), and the algebra of Lemma 3.3(b).
 - Numerically, they solve 900 two-particle equilibrium problems inside fixed external configurations. The largest gap ratio found is 1.4870, below r_2 .
 - They compute $\mathcal{E}_{c^*}(f_R)$ for five gap profiles and $R \in \{10, 30, 100, 300\}$; the values of $\mathcal{E}_{c^*}(f_R) R / \log R$ lie between 2.0 and 6.6.

These numerical runs illustrate the statements; the proofs do not use them.

3. Two independent verification runs, both AI-assisted, re-derived all proofs line by line and wrote their own code. Neither found a mathematical error.
- In exact rational arithmetic, the code of the first run checks the finite-range identity behind Lemma 2.3 (530 cases), the inequality behind Lemma 2.2 (470 cases), the identities of Lemma 3.3(b) and (d) (40 random networks), and Lemma 2.4 on adversarial patterns. It tests every inequality in the proof of Proposition 4.1 for $s = 2$ on six further gap profiles. It also solves numerically for 200 particles in equilibrium between two fixed arithmetic tails. Identity (3), divided by $\pi(n)$, holds at all 199 interior indices up to an error of $5 \cdot 10^{-8}$, which comes from truncating the tails, and it fails, as it should, next to the fixed particles. The first run asked for corrections to the explanation of the role of the Doob transform, to the statement of the conventions, to the numbering of the results cited from [GK17], and to the range of exponents claimed. These have been made.
 - The second run read the sources again and wrote its code before it read the programs above; it also re-ran all of them. Its code checks the finite-range identity behind Lemma 2.3 (1689 identities) and the two comparisons in the proof of Lemma 2.2 (1891 cases), exactly for $s = 2$ and in 100-digit arithmetic for several $s \in (1, 2)$. It checks Lemma 2.4 on 11,536 pairs from adversarial gap patterns. In exact arithmetic it checks Green's identity (5), Lemma 3.1 and Lemma 3.3(a)–(d) on random finite networks, most of them complete graphs. It tests every inequality in the proof of Proposition 4.1, and its final bound, for $s \in \{1.1, 1.25, 1.5, 1.75, 1.9, 2\}$ on eight gap profiles and for $R \leq 300$ (186 cases). It asked for an update of the record of the literature search below and of the verification report. These have been made.

All programs and outputs are in the source files.

Scope and priority. Theorem 1.2 answers Question 0.1 of [Ben15], which is Question 5 in Section 4 of [Syl15], under the conventions of Remark 1.3. It also answers Problem 1 of [GK17] for the force laws d^{-s} with $1 < s \leq 2$. It does not cover general force laws, exponents $s > 2$, or a principal-value reading of the equilibrium condition. We read [Ben15] and [Syl15] in their PDF versions, and both the arXiv version and the published version of [GK17]. Of [DK84] we read the abstract and the introduction; of [CdIL98, dILV10] we know only the bibliographic data. We searched arXiv, Crossref, OpenAlex,

zbMATH and Semantic Scholar, including the works citing [GK17], most recently in October 2026. Apart from duplicate entries and one evidently misattributed entry, OpenAlex and Semantic Scholar list the same two works citing [GK17], and neither addresses the question. The publication lists of the two authors of [GK17] contain no follow-up. Three general web searches found only [GK17] and copies of it. We found no resolution of the question and no partial result beyond [GK17]. MathSciNet and Google Scholar were not searched. This negative search is not a proof of priority.

AI-assisted tools were used to search the problem corpus and the literature, to find the proof, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

References

- [Ben15] Itai Benjamini. Points in equilibrium. One-page note dated May 2015, formerly at <https://www.wisdom.weizmann.ac.il/~itai/equilibrium.pdf>, 2015. Question 0.1. Archived copy: https://arquivo.pt/wayback/20201231041612id_/https://www.wisdom.weizmann.ac.il/~itai/equilibrium.pdf; accessed October 2026.
- [BL15] Xavier Blanc and Mathieu Lewin. The crystallization conjecture: a review. *EMS Surveys in Mathematical Sciences*, 2(2):255–306, 2015. doi:10.4171/EMSS/13.
- [CdL98] A. Candel and R. de la Llave. On the Aubry–Mather theory in statistical mechanics. *Communications in Mathematical Physics*, 192(3):649–669, 1998. doi:10.1007/s002200050313.
- [DK84] M. Duneau and A. Katz. Structural stability of classical lattices in one dimension. *Annales de l’Institut Henri Poincaré. Physique Théorique*, 41(3):269–290, 1984. Zbl 0557.58017. URL: http://www.numdam.org/item?id=AIHPA_1984__41_3_269_0.
- [dlLV10] Rafael de la Llave and Enrico Valdinoci. Ground states and critical points for Aubry–Mather theory in statistical mechanics. *Journal of Nonlinear Science*, 20(2):153–218, 2010. doi:10.1007/s00332-009-9055-0.
- [Doo57] J. L. Doob. Conditional Brownian motion and the boundary limits of harmonic functions. *Bulletin de la Société Mathématique de France*, 85:431–458, 1957. doi:10.24033/bsmf.1494.
- [GK17] Agelos Georgakopoulos and Mihail N. Kolountzakis. On particles in equilibrium on the real line. *Proceedings of the American Mathematical Society*, 145(8):3501–3511, 2017. arXiv: 1604.01649, doi:10.1090/proc/13492.
- [LP16] Russell Lyons and Yuval Peres. *Probability on Trees and Networks*, volume 42 of *Cambridge Series in Statistical and Probabilistic Mathematics*. Cambridge University Press, Cambridge, 2016. doi:10.1017/9781316672815.
- [Syl15] John Sylvester. Open problems from “Random walks on graphs and potential theory”. Workshop at the University of Warwick, 18–22 May 2015, <https://warwick.ac.uk/fac/sci/maths/research/events/2014-15/nonsymposium/random/Openprobs.pdf>, 2015. Edited by J. Sylvester; Section 4 (I. Benjamini), Question 5; accessed October 2026.
- [ula26] ulamai. UnsolvedMath. Hugging Face dataset, <https://huggingface.co/datasets/ulamai/UnsolvedMath>, 2026. Version 1.6.0; record AMR-099-0003.