

An Allocation-Invariant Diffuse Measure That Is Not Mass-Stationary

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Abstract

Invariance under every measurable equivariant measure-preserving allocation does not characterize mass-stationarity of a diffuse random measure without an independent stationary background. For every $d \geq 2$, we give a periodic measure supported on parallel hyperplanes with unequal weights. Its horizontal translation stabilizer prevents allocations from splitting layers, and a finite weighted quotient forces every preserving allocation to fix the rooted configuration. In contrast, sampling a new origin from the mass changes the root-type distribution. More generally, for m distinct positive layer weights, every law on the m rooted types is allocation-invariant, whereas exactly the weight-proportional law is mass-stationary. The family has a stationary ergodic probability parent of finite intensity. We distinguish this allocation-only obstruction from the Markovian-kernel problem and credit the related atomic, product and invariant-direction methods in the literature. No priority claim is made.

1 The question and the example

Thorisson [1, Problem 4.1] asks whether a diffuse measure and a random element are mass-stationary when their joint law is invariant under all preserving allocations, without extra stationary randomness. The corresponding question is Problem 7.5 of Last–Thorisson [2]. Neither formulation rules out translation stabilizers. We take the additional random element to be constant and give a negative answer on $\mathbb{R}^d$, $d \geq 2$.

Our shift convention is

$$\theta_z \mu(B) = \mu(B + z).$$

An allocation is a measurable map $\tau_\mu : \mathbb{R}^d \rightarrow \mathbb{R}^d$, defined on the translation orbit of the measure, satisfying

$$\tau_{\theta_z \mu}(s - z) = \tau_\mu(s) - z. \quad (1)$$

It is preserving when $(\tau_\mu)_* \mu = \mu$. Its induced shift is $\mu \mapsto \theta_{\tau_\mu(0)} \mu$. These are exactly the pointwise equivariant allocations obtained from the measurable shift functions in [1, Section 4].

Fix $m \geq 2$ and pairwise distinct positive numbers $w_0, \dots, w_{m-1}$. Extend the weights periodically to $\mathbb{Z}$, write $W = \sum_{i=0}^{m-1} w_i$, and set

$$\mu_0 = \lambda_{d-1} \otimes \sum_{k \in \mathbb{Z}} w_k \delta_k, \quad \mu_t = \theta_{(0,t)} \mu_0. \quad (2)$$

Here λ_{d-1} is Lebesgue measure on $\mathbb{R}^{d-1}$. The phase t is taken modulo m . The orbit

$$\Omega = \{\mu_t : t \in \mathbb{R}/m\mathbb{Z}\}$$

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carries the canonical action $\theta_{(x,y)}\mu_t = \mu_{t+y}$. The phase parametrization is continuous in the vague topology and injective: a period must preserve the integer support, and distinct weights force it to lie in $m\mathbb{Z}$. Thus it is a Borel isomorphism with the circle. All the measures are nonzero diffuse Radon measures. The condition $d \geq 2$ supplies a non-atomic horizontal factor.

Theorem 1. *For every probability vector $p = (p_0, \dots, p_{m-1})$, let*

$$Q_p = \sum_{i=0}^{m-1} p_i \delta_{\mu_i}.$$

Then Q_p is invariant under every measurable equivariant μ -preserving allocation. It is mass-stationary if and only if

$$p_i = \frac{w_i}{W}, \quad 0 \leq i < m. \quad (3)$$

The weight-proportional law is the normalized Palm law of the stationary ergodic uniform-phase probability on Ω , whose intensity is W/m .

The simplest counterexample uses $m = 2$, $(w_0, w_1) = (1, 2)$ and $Q = \delta_{\mu_0}$. In two dimensions, μ_0 is length measure with weight 1 on the even horizontal lines and weight 2 on the odd horizontal lines. Every preserving allocation leaves its rooted configuration unchanged. A mass-sampled origin instead has the odd-line rooted configuration with probability $2/3$.

2 The stabilizer forces a finite weighted permutation

Lemma 2. *Let τ be a preserving equivariant allocation for (2). There are vectors $c_i \in \mathbb{R}^{d-1}$ and integers $b_i \equiv i \pmod{m}$ such that*

$$\tau_{\mu_0}(x, i + m\ell) = (x + c_i, b_i + m\ell), \quad x \in \mathbb{R}^{d-1}, \quad \ell \in \mathbb{Z}. \quad (4)$$

Proof. Horizontal translations and translation by $(0, m)$ fix μ_0 . Applying (1) with these stabilizers gives (4), initially with $b_i \in \mathbb{R}$. Preservation forces $b_i \in \mathbb{Z}$: otherwise a whole positive-mass source hyperplane would be sent outside the support of μ_0 . This can be tested on a bounded horizontal box at the corresponding target height.

Put $\sigma(i) = b_i \bmod m$. Each residue- i family of source hyperplanes maps bijectively onto the residue- $\sigma(i)$ family, with a translation on every horizontal hyperplane. Comparing mass on a unit horizontal box at any residue- j height gives

$$\sum_{i: \sigma(i)=j} w_i = w_j, \quad 0 \leq j < m. \quad (5)$$

Every right-hand side is positive, so σ is onto. A map from an m -element set onto itself is a permutation. Consequently (5) gives $w_i = w_{\sigma(i)}$, and distinctness of the weights implies $\sigma(i) = i$. This proves the congruence asserted in the lemma. No irrationality assumption on the weights is needed. $\square$

At phase i , covariance and Lemma 2 yield

$$\tau_{\mu_i}(0) = \tau_{\mu_0}(0, i) - (0, i) = (c_i, b_i - i).$$

Its vertical coordinate belongs to $m\mathbb{Z}$, so

$$\theta_{\tau_{\mu_i}(0)}\mu_i = \mu_i. \quad (6)$$

This proves allocation invariance for every Q_p . If preservation is initially required only Q_p -almost surely, choose any phase with positive probability. Covariance transports preservation from that phase to the whole orbit, and the same argument applies.

Notice that the argument classifies all measurable equivariant allocations on the support. It does not assume that an allocation is continuous, linear, invertible, or has bounded displacement. Arbitrary behavior off the support does not affect the conclusion. Pointwise equivariance also controls the image of the origin, despite singletons having zero mass.

3 The mass-sampled root law

We use the window definition of mass-stationarity in [2, Definition 6.1 and Remark 6.2]. For an admissible relatively compact Borel set C , let U be uniform on C , independently of μ , and conditionally choose V from the normalized restriction of μ to $C - U$. Mass-stationarity requires

$$(\theta_V \mu, U + V) \stackrel{d}{=} (\mu, U). \quad (7)$$

The admissibility conditions are positive finite ambient Haar measure and Haar-null boundary.

Take

$$C = [0, 1)^{d-1} \times [0, m).$$

For $\mu = \mu_i$, the translated window contains one hyperplane from each of the m weight residues, each with horizontal volume one. In particular, $\mu_i(C - U) = W$. A sampled point on the hyperplane at height $k - i$ reroots the measure to

$$\theta_{(x, k-i)} \mu_i = \mu_k.$$

Consequently, conditional on any starting phase i and almost every U , the new configuration has law

$$Q_w := \sum_{j=0}^{m-1} \frac{w_j}{W} \delta_{\mu_j}. \quad (8)$$

If (7) holds, its configuration marginal forces $Q_p = Q_w$. Since the phases μ_j are distinct, this proves the necessity of (3). For the binary example, the initial law assigns probability zero to μ_1 , whereas (8) assigns it $2/3$.

For sufficiency, define the uniform-phase probability

$$P = \frac{1}{m} \int_0^m \delta_{\mu_t} dt.$$

It is stationary. It is also ergodic, since the translation action on the phase circle is transitive. For any nonnegative measurable f on Ω and any Borel set B of finite volume, Tonelli's theorem gives

$$\begin{aligned} \mathbb{E}_P \int_B f(\theta_z \mu) \mu(dz) &= \frac{1}{m} \sum_{k \in \mathbb{Z}} w_k f(\mu_k) \int_0^m \int_{\mathbb{R}^{d-1}} \mathbf{1}_B(x, k - t) dx dt \\ &= \frac{|B|}{m} \sum_{i=0}^{m-1} w_i f(\mu_i). \end{aligned} \quad (9)$$

For the second equality, group $k = i + m\ell$ and observe that the vertical intervals swept out by $i + m\ell - t$, $0 \leq t < m$, partition $\mathbb{R}$ up to endpoints. Equation (9) identifies the intensity as W/m and the normalized Palm law as Q_w . The Palm equivalence [2, Theorem 6.3], applied to the stationary measure $(m/W)P$, now proves mass-stationarity of Q_w . This completes the proof of Theorem 1.

4 Randomized transports and the scope of the obstruction

The example is not invariant under every preserving Markov transport. Indeed, on the support of μ_0 , define

$$K_0((x, i + m\ell), \cdot) = \sum_{j=0}^{m-1} \frac{w_j}{W} \delta_{(x, j+m\ell)}, \quad 0 \leq i < m. \quad (10)$$

Use the identity kernel off the support. For μ_t , define

$$K_t(s, B) = K_0(s + (0, t), B + (0, t)).$$

The horizontal invariance and vertical m -periodicity of K_0 make this definition independent of the representative of t and equivariant. It is measurable. The incoming mass at a residue- j layer is

$$\sum_{i=0}^{m-1} w_i \frac{w_j}{W} = w_j,$$

so the kernel preserves the measure. Rerooting by it sends every Q_p to Q_w . Thus the separate Markovian-kernel question in [1, Problem 4.2] is not resolved by this counterexample.

The mechanism is not that non-atomic measures intrinsically resist splitting. Rather, equivariance forces a deterministic allocation to commute with the horizontal stabilizer, so it cannot split an individual layer between different layer types. Extra stationary randomness can break this symmetry.

There is substantial related prior work. Last–Thorisson [2, Example 7.1] give a weighted atomic obstruction. Their later work treats products with Lebesgue measure [3, Section 8]; its positive-density and independent-background results [3, Theorems 6–7] have hypotheses distinct from the present singular, background-free setting. Their parallel-line transport counterexample [4, Section 8 and Remark 8.1] also exhibits the importance of invariant directions, although it concerns balancing two different measures. These methods are not claimed as new here. The explicit periodic construction gives a negative answer to the unrestricted allocation-only question and a classification of its finite family of root laws. It makes no claim about measures without invariant directions, no claim of priority, and no claim of an exhaustive novelty search.

AI assistance was used to develop the argument, check the finite quotient calculations and prepare the text. This note is self-audited and unrefereed. The accompanying exact finite diagnostics are reproducibility aids, not formal verification or an independent review.

References

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