

The Subnetwork $S(1)$ of a Scale-Invariant Random Spatial Network Is Not a Tree

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Abstract

D. Aldous asked for a proof that in a scale-invariant random spatial network (SIRSN) the subnetwork $S(1)$, the union of the routes between the points of a Poisson process of rate one, cannot be a tree, even with Steiner points (Electron. J. Probab. 19 (2014), Open Problem 10; Open Problem 36 of the arXiv version). We prove this for every SIRSN and every weak SIRSN: the outcomes in which $S(1)$ contains no simple closed curve form a set of outer probability zero, so that $S(1)$ contains a circuit almost surely and, in particular, is not almost surely a tree. The same holds when “tree” means that the graph whose vertices are the Poisson points and the junctions of $S(1)$ has no finite cycle. More generally, for any process of compatible routes with the invariance properties of a SIRSN, if that set has outer probability $p > 0$, then $\mathbb{P}(D_1 \geq t) \geq cp/t$ for all large t , where D_1 is the route length between two points at distance one and $c > 7.7 \cdot 10^{-11}$ is explicit; the SIRSN axiom $\mathbb{E}D_1 < \infty$ excludes this. The core of the proof is Aldous’s argument for route lengths in invariant tree networks (Electron. Commun. Probab. 26 (2021)), whose percolation lemma is used as published. What the note adds is the transfer of this argument to the routes of a SIRSN, which need not consist of finitely many segments (a centroid bipartition and a route-length bound for finite unions of compatible routes), and the replacement of Palm probabilities by Campbell’s formula and scale invariance. This is an unrefereed note.

1 Introduction

1.1 The question

Aldous [Ald14] (see also [AG13]) introduced *scale-invariant random spatial networks* (SIRSNs): random routes $R(z, z')$ between all pairs of points of the plane, with joint distributions invariant under translations, rotations and scalings, and two finiteness conditions (Section 2). If Ξ is a Poisson process of rate one independent of the routes, the union $S(1)$ of the routes $R(\xi, \xi')$, $\xi, \xi' \in \Xi$, is the *subnetwork* on these points. One finiteness condition is that the route length D_1 between two points at distance one has finite mean.

In [Ald14, Section 8.7.2] Aldous recalls that the continuum limits of random spanning trees have $D_1 = \infty$, and he expects a simple proof of the next statement “because our definition of SIRSN requires $\mathbb{E}D_1 < \infty$ ”. The statement is [Ald14, Open Problem 10]: “In a SIRSN, the subnetwork $S(1)$ cannot be a tree (with Steiner points).” In arXiv:1204.0817v1, the only arXiv version of [Ald14], it is Open Problem 36 (all numbered statements share one counter there), and the preceding sentence reads $\mathbb{E}D_1 = \infty$. The problem is listed on Aldous’s page of open problems [Ald]; it is record AMR-096-0036 of the UnsolvedMath corpus. Steiner points are junctions that are not points of Ξ ; of tree

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networks with Steiner points Aldous writes “The tree property is that there are no circuits” [Ald21, Section 1]. We treat two readings of “ $S(1)$ is a tree”: $S(1)$ contains no simple closed curve; the graph whose vertices are the points of Ξ and the junctions has no finite cycle (Section 2.2).

1.2 Results

The *outer probability* of a set A of outcomes is $\mathbb{P}^*(A) = \inf\{\mathbb{P}(G) : G \supseteq A, G \text{ an event}\}$. We use it because we do not decide whether “ $S(1)$ contains no circuit” is an event for the σ -field outlined in [Ald14, Section 2.3].

Theorem 1.1. *Let $S(1)$ be the subnetwork of a SIRS or of a weak SIRS. The set of outcomes in which $S(1)$ contains no simple closed curve has outer probability zero, and so has the set of outcomes in which the junction graph of $S(1)$ has no finite cycle. Consequently, almost surely $S(1)$ contains a circuit, in the sense that the outcomes in which it contains none lie in an event of probability zero; every event on which $S(1)$ contains no circuit has probability zero (in particular that set of outcomes, if it is an event); and $S(1)$ is not almost surely a tree, in either reading.*

The theorem follows from a statement about general route processes. For $n \in \mathbb{N}$ write span_n for the union of the routes between the points of $\Xi \cap [0, n]^2$, and let A_∞ be the set of outcomes in which no span_n , $n \in \mathbb{N}$, contains a simple closed curve.

Theorem 1.2. *Let $(R(z, z'))$ be a process of feasible routes satisfying (2.4), (2.6)–(2.8) and (2.13) of [Ald14], and let Ξ be an independent Poisson process of rate one. Put $p := \mathbb{P}^*(A_\infty)$ and $c := 11/(32\sqrt{2}\pi \cdot 10^9) > 7.7 \cdot 10^{-11}$. There is a function $t_0 : (0, 1] \rightarrow (0, \infty)$, which does not depend on the route process, such that if $p > 0$, then*

$$\mathbb{P}(D_1 \geq t) \geq \frac{cp}{t} \quad \text{for all } t \geq t_0(p).$$

In particular $\mathbb{E}D_1 = \infty$ if $p > 0$.

Both readings of “tree” imply that the outcome lies in A_∞ (Section 2.2 and Proposition 7.3), and SIRSs and weak SIRSs satisfy $\mathbb{E}D_1 < \infty$; this gives Theorem 1.1. Theorem 1.2 does not use the conditions (2.14), (2.16), (2.20) of [Ald14] or the jaggedness of routes. The constant c is far from optimal.

1.3 The proof and its sources

Aldous [Ald21, Theorem 1.2] proved that there are constants $r_0 < \infty$ and $\beta > 0$ such that in every invariant tree network connecting the points of a Poisson process of rate one in the plane, with Steiner points allowed and edges that are line segments, $\mathbb{P}(D_r > d) \geq \beta r/d$ for $r_0 \leq r \leq d$, where D_r is the route length between Poisson points at distance at most r . In a SIRS the route length between points at distance s is distributed as sD_1 , so a tree $S(1)$ would contradict $\mathbb{E}D_1 < \infty$. This is the deduction suggested by the remark quoted above, but [Ald21, Theorem 1.2] does not apply to $S(1)$ as stated: the edges of its networks are line segments between vertices, while a route of a (weak) SIRS contains no segment starting at a Poisson point [Ald14, Proposition 5.1].

The proof in [Ald21] uses the network at two places only: (i) the subtree spanned by the N Poisson points in a square $[0, n]^2$ has a centroid v_* , which yields a bipartition $\{B, B^c\}$ of these points with $N/3 \leq |B| \leq N/2$; (ii) the route between a point of B and a point of B^c passes through v_* . We prove (i) and (ii) for finite unions of compatible routes (Sections 3 and 4), give the percolation part of [Ald21] with (i) and (ii) as its only network input, using its Lemma 2.2 as published (Section 5), replace its last step, which uses the Palm probability of two typical Poisson points, by Campbell’s

formula and scale invariance (Section 6), and treat the junction graph (Section 7). Section 8 explains why the inequality (2.16) and the constants of Corollary 2.3 of [Ald21] need repair; with Corollary 5.3 and Lemma 5.4, which apply verbatim to the networks of [Ald21], the proof of [Ald21, Theorem 1.2] goes through with other constants. The question and the core of the argument are Aldous's; the contributions of this note are listed under “Scope and priority” in Section 9.

1.4 Related work

Kendall [Ken17] studied networks built from improper Poisson line processes, Kahn [Kah16] proved that they are SIRSs in every dimension, and [Ken20, Bla24, BCK24] study random flights on them, fractal properties and geodesics. None of these papers discusses whether $S(1)$ can be a tree. In [Ald14, Section 5.5] Aldous outlines an argument that the smallest edge-intensity of a SIRS exceeds the Steiner tree constant; it concerns networks of minimal length, not trees in general. The lattice analogue of [Ald21, Theorem 1.2] is [Ald21, Theorem 1.1], which goes back to [BLPS01]: for an invariant random spanning tree of $\mathbb{Z}^2$, the route length D between adjacent vertices satisfies $\mathbb{P}(D \geq i) \geq 1/(8i)$.

2 Setting

2.1 SIRSs

We follow [Ald14, Section 2.2] and keep its equation numbers (2.4)–(2.20); in the arXiv version they are (4)–(20).

Routes [Ald14, Aspect 1]. A feasible route $r(z, z')$, $z \neq z'$, is a non-self-intersecting chain of finitely or infinitely many line segments of finite total length, whose successive end points converge to z and to z' ; $r(z', z)$ is its reversal. With its endpoints, a route is a compact arc of finite length (a homeomorphic image of $[0, 1]$) with endpoints z, z' ; only this is used below. For points x, y of an arc r , $[x, y]_r$ is the sub-arc between them. Routes must be *compatible*: (2.4) if two routes meet at two points, then they coincide on the sub-route between the two meeting points.

Route processes [Ald14, Aspect 3]. For each finite set $\{z_1, \dots, z_k\}$ a law $\mu_{z_1, \dots, z_k}$ of a random feasible subnetwork $\text{span}(z_1, \dots, z_k)$ (a union of compatible routes $r(z_i, z_j)$) is given, such that (2.6) within $\text{span}(z_1, \dots, z_{k+1})$ the subnetwork on $z_1, \dots, z_k$ has the law $\mu_{z_1, \dots, z_k}$; (2.7) the family is invariant under translations and rotations; (2.8) it is invariant under the scalings $z \mapsto cz$; (2.13) $(z_1, \dots, z_k) \mapsto \mu_{z_1, \dots, z_k}$ is measurable. This defines a process of routes $R(z, z')$. With Ξ a Poisson process of rate one independent of it, $S(1)$ is the union of the routes $R(\xi, \xi')$ over pairs of distinct points of Ξ . The source omits the measure-theoretic details and imposes (2.13) so that $S(1)$ is measurable. We use the construction in this form: almost surely Ξ is locally finite, and for each $n \in \mathbb{N}$, conditionally on Ξ , the union span_n of the routes between the points of $\Xi \cap [0, n]^2$ has the law $\mu_{\Xi \cap [0, n]^2}$; so these routes are almost surely feasible and compatible, and by (2.6) the route between two points ξ, ξ' of Ξ has the conditional law $\mu_{\xi, \xi'}$. Counts of pairs of points of $\Xi \cap [0, n]^2$ with given route lengths are random variables.

Scaling. Put $D_1 := \text{len } R(0, \mathbf{1})$ with $\mathbf{1} = (1, 0)$. If $z \neq z'$, the similarity $x \mapsto z + |z' - z|\theta x$, with θ the rotation taking $\mathbf{1}$ to $(z' - z)/|z' - z|$, maps $0, \mathbf{1}$ to z, z' and multiplies lengths by $|z' - z|$. By (2.7) and (2.8),

$$\text{len } R(z, z') \text{ has the law of } |z - z'| D_1. \quad (1)$$

SIRS [Ald14, Aspect 4]. Besides (2.6)–(2.8) and (2.13), a SIRS satisfies (2.15) $1 < \mathbb{E} D_1 < \infty$ and (2.20) $p(1) < \infty$, where $p(1)$ is the edge-intensity of the long-range part of the network defined in (2.17)–(2.19). A *weak SIRS* satisfies (2.15) and (2.16) $\ell := \text{edge-intensity of } S(1) < \infty$, instead of (2.20). By [Ald14, Lemma 6.2], $\ell \leq 2p(1)$, so every SIRS is a weak SIRS. Under (2.16) the mean

length of $S(1)$ in a disc of radius ϱ is $\ell\pi\varrho^2$, so (2.14) holds: almost surely $S(1)$ has finite length in bounded sets, i.e. it is a feasible subnetwork on Ξ (2.9) and satisfies (2.5).

2.2 Circuits and trees

A *circuit* is a simple closed curve (a homeomorphic image of a circle). The set $S(1)$ is path-connected, since any two points of Ξ are joined by a route. We say that $S(1)$ is a tree (T) in the *topological reading* if $S(1)$ contains no circuit, junctions that are not points of Ξ playing the role of Steiner points; (J) in the *junction reading* if $S(1)$ is a feasible subnetwork and its junction graph $\Gamma(S(1))$ has no finite cycle, the vertices being the points of Ξ and the points where one route leaves another, and the edges the open sub-arcs of routes between consecutive vertices (Section 7). Let A_T and A_J be the corresponding sets of outcomes, and A_∞ the set of outcomes in which no span_n contains a circuit (Section 1). Since $\text{span}_n \subseteq S(1)$, we have $A_T \subseteq A_\infty$, and $A_J \subseteq A_\infty$ by Proposition 7.3.

Remark 2.1 (A degenerate reading). In [Ald14, Aspect 2] a subnetwork is described formally with the segments of its routes as edges. In that graph every point of Ξ is isolated, since almost surely no segment of $S(1)$ ends at a point of Ξ [Ald14, Proposition 5.1]; so $S(1)$ would never be a tree, for a trivial reason, while “no finite cycle of segments” would ignore all circuits through points of Ξ , which consist of infinitely many segments. Neither variant expresses the question, which concerns circuits [Ald21, Section 1]; we do not consider them.

3 Finite unions of compatible routes

In this section F is a finite set of points of the plane with $|F| \geq 2$, and for $z \neq z'$ in F , $r(z, z') = r(z', z)$ is a compact arc with endpoints z, z' ; these arcs are pairwise compatible in the sense of (2.4). We call them routes and put $G_F := \bigcup_{z \neq z' \in F} r(z, z')$. No other property of routes is used.

Lemma 3.1. *For two routes r, ρ the set $r \cap \rho$ is empty, a point, or a sub-arc of r which is also a sub-arc of ρ .*

Proof. If $a, b \in r \cap \rho$, then $[a, b]_r = [a, b]_\rho \subseteq r \cap \rho$ by (2.4). So $r \cap \rho$ is a compact connected subset of r : empty, a point, or a sub-arc $[a, b]_r = [a, b]_\rho$. $\square$

Let V be the set consisting of the points of F and of the endpoints of the sets $r \cap \rho$ for all pairs of distinct routes; V is finite. The points of V on a route r cut it into finitely many open sub-arcs, the *edges* of r .

Lemma 3.2. *Two edges, of the same route or of different routes, are equal or disjoint. Hence G_F is a finite graph with vertex set V and edge set E , the set of all edges: each edge e is open in G_F , and its closure $\text{cl } e$ is e together with two distinct vertices, its ends.*

Proof. Edges of one route are its components of $r \setminus V$. Let e be an edge of r and f an edge of $\rho \neq r$ with a common point x . The set $I := r \cap \rho$ contains $x \notin V$, so it is not a point (a point would belong to V), and by Lemma 3.1 it is a sub-arc of r with endpoints in V . The open arc e meets I and contains no endpoint of I , hence $e \subseteq I \subseteq \rho$. The ends of e lie in $V \cap I \subseteq \rho$ and no point of V lies in e ; so e is a component of $\rho \setminus V$ containing x , that is, $e = f$. The complement of e in G_F is the union of V and of the closures of the other edges, a finite union of compact sets missing e ; so e is open in G_F . The ends of e are distinct because r is an arc. $\square$

Lemma 3.3. (i) *If a circuit $C \subseteq G_F$ meets an edge e , then $\text{cl } e \subseteq C$, and the edges contained in C form a cycle of length at least two of the multigraph (V, E) .* (ii) *If an arc $A \subseteq G_F$ has both endpoints in V and meets an edge e , then $\text{cl } e \subseteq A$, and the edges contained in A form a simple path of (V, E) between the endpoints of A .*

Proof. (i) Let $\varphi : [0, 1] \rightarrow \text{cl } e$ be a homeomorphism with $\varphi((0, 1)) = e$. Let K be a component of $C \cap e$. As e is open in G_F , K is open in C ; it is connected, and $K \neq C$ because e , an open arc, contains no circuit. So K is an open arc of C , and its closure is $K \cup \{p, q\}$ for the two ends p, q of K in C . Being a connected subset of e homeomorphic to an open interval, $K = \varphi(J)$ for an open interval $J = (a, b) \subseteq (0, 1)$, and its closure is the arc $\varphi([a, b])$, with the two distinct endpoints $\varphi(a), \varphi(b)$. Hence $\{p, q\} = \{\varphi(a), \varphi(b)\}$. If $a > 0$, then $\varphi(a) \in C \cap e$ and $K \cup \{\varphi(a)\}$ would be a connected subset of $C \cap e$ larger than K ; so $a = 0$, likewise $b = 1$, and $K = e$. Thus C is the union of the closures of the edges it meets and of finitely many vertices. At a vertex $v \in C$ the curve C is locally an arc through v ; each of its two sides at v lies in one edge with end v (it is connected and avoids V near v), the two sides lie in distinct edges (near v the closure of an edge with end v is a single arc ending at v , because the ends of an edge are distinct), and a third edge in C with end v would give a triod in C . So the edges in C form a finite connected 2-regular multigraph without loops, a cycle. (ii) is proved in the same way: the components of $A \cap e$ do not contain the endpoints of A , which lie in V , so they are open sub-arcs of A . $\square$

Lemma 3.4. *If G_F contains no circuit, then (V, E) is a tree, and any two points of G_F are joined by exactly one arc in G_F . In particular, for $z \neq z'$ in F the route $r(z, z')$ is the unique arc in G_F from z to z' , and it is the union of the closures of the edges of the path of the tree (V, E) from z to z' .*

Proof. Each route is an arc with endpoints in V , hence by Lemma 3.3(ii) a path of (V, E) through all vertices on it; any two routes from a common point of F meet, so (V, E) is connected. The closures of the edges of a cycle of length at least three, or of two edges with the same ends, form a circuit; so (V, E) is a simple graph without cycles, a tree. Given $x, y \in G_F$, add them to V , subdividing the edges containing them; the properties of Lemma 3.2 persist and the graph is still a tree. Arcs from x to y exist since G_F is a connected finite graph, and by Lemma 3.3(ii) two such arcs are unions of the edges of two paths of the tree between x and y , which coincide. $\square$

4 A centroid bipartition

The first statement is Jordan's weighted centroid [Jor69] with the merging step of [Ald21, Section 2.2].

Lemma 4.1. *Let T be a finite tree and W a set of $N \geq 3$ of its vertices. There are a vertex v_* and a set $B \subseteq W$ with $N/3 \leq |B| \leq N/2$ such that every path of T from a vertex of B to a vertex of $W \setminus B$ passes through v_* .*

Proof. The weight of a set of vertices is the number of its vertices in W . Start at any vertex v . If a component of $T - v$ has weight $> N/2$, move to the neighbour of v in it; the component of $T - v'$ containing the old vertex v then has weight $< N/2$, so the walk never turns back, and it stops at a vertex v_* all of whose branches (components of $T - v_*$) have weight at most $N/2$. The *pieces* are the sets $W \cap P$ for the branches P at v_* , and $\{v_*\}$ if $v_* \in W$; they partition W into sets of size at most $N/2$ (recall $N \geq 3$). Add pieces to B in order of decreasing size until $|B| \geq N/3$. If the first piece has size at least $N/3$, then B is that piece and $|B| \leq N/2$. Otherwise all pieces have size $< N/3$, and when the procedure stops, $N/3 \leq |B| < 2N/3$; if $|B| > N/2$, replace B by $W \setminus B$, which is also a union of pieces and has size in $(N/3, N/2)$. A vertex of B and a vertex of $W \setminus B$ lie in different pieces, hence in different branches at v_* or one of them is v_* ; so the path between them passes through v_* . $\square$

Corollary 4.2. *Let F and the routes be as in Section 3, with $N := |F| \geq 3$, and suppose that G_F contains no circuit and that all routes have finite length. Then there are a point $v_* \in \mathbb{R}^2$ and a set $B \subseteq F$ with*

(T1) $N/3 \leq |B| \leq N/2$, and

(T2) $\text{len } r(v, v') \geq |v - v_*| + |v_* - v'|$ for all $v \in B$ and $v' \in F \setminus B$.

Proof. By Lemma 3.4, (V, E) is a tree containing F among its vertices. Apply Lemma 4.1 with $W = F$. For $v \in B$ and $v' \in F \setminus B$ the route $r(v, v')$ is the union of the closures of the edges of the tree path from v to v' (Lemma 3.4), which passes through the vertex v_* . So $r(v, v')$ is the union of two sub-arcs, from v to v_* and from v_* to v' , meeting only at v_* , and their lengths are at least $|v - v_*|$ and $|v_* - v'|$. $\square$

The point v_* , a point of F or a junction, plays the role of the Steiner point in the proof of [Ald21, Theorem 1.2].

5 Many long routes

This section follows [Ald21, Section 2]; the network enters only through a point v_* and a bipartition with (T1). Fix integers $m, k \geq 1$ and put $n := mk$, $Q := [0, n]^2$. The *natural subsquares* are the closed squares $\sigma_{(i,j)} := [im, (i+1)m] \times [jm, (j+1)m]$, indexed by the grid $G_k := \{0, \dots, k-1\}^2$, in which (i, j) and (i', j') are adjacent if $|i - i'| + |j - j'| = 1$. The *strips* of a square $a + [0, m]^2$ are the ten rectangles $a + [\frac{(l-1)m}{5}, \frac{lm}{5}] \times [0, m]$ and $a + [0, m] \times [\frac{(l-1)m}{5}, \frac{lm}{5}]$, $1 \leq l \leq 5$. Consider finite configurations of points none of which lies on a line $x \in \frac{m}{5}\mathbb{Z}$ or $y \in \frac{m}{5}\mathbb{Z}$. Such a configuration is *balanced* in a square if each of its ten strips contains between $0.98m^2/5$ and $1.02m^2/5$ points [Ald21, p. 3] (where “[0, 1]” in the definition of the strips stands for $[0, m]$); the square then contains between $0.98m^2$ and $1.02m^2$ points.

Lemma 5.1 ([Ald21, Lemma 2.1]). *Let σ_1, σ_2 be adjacent natural subsquares in which the configuration is balanced, and colour its points in $\sigma_1 \cup \sigma_2$ blue and red. Unless both squares contain fewer than $0.1m^2$ blue points or both contain more than $0.88m^2$, there are at least $0.088m^4$ pairs of a blue and a red point in $\sigma_1 \cup \sigma_2$ at distance at most $\sqrt{2}m$.*

Proof. If a square with $0.98m^2 \leq (\text{number of points})$ has y blue points, $0.1m^2 \leq y \leq 0.88m^2$, it contains at least $y(0.98m^2 - y) \geq 0.1 \cdot 0.88m^4$ blue–red pairs, by concavity of $y \mapsto y(0.98m^2 - y)$, all at distance at most $\sqrt{2}m$. Otherwise one square, say σ_1 , has fewer than $0.1m^2$ blue points and σ_2 more than $0.88m^2$. If they are side by side horizontally, slide a square of side m from σ_1 to σ_2 in five steps of $m/5$. Each intermediate square is a union of five vertical strips of σ_1 and σ_2 , hence contains between $0.98m^2$ and $1.02m^2$ points, and each step changes the number of blue points by at most $1.02m^2/5 = 0.204m^2$. The first square with at least $0.1m^2$ blue points has fewer than $0.304m^2 \leq 0.88m^2$ of them, and the first case applies to it. For vertical neighbours use horizontal strips. $\square$

The next lemma is the percolation estimate of [Ald21]; we use it as published and do not prove it.

Lemma 5.2 ([Ald21, Lemma 2.2]). *For $\xi \subseteq G_k$ let $c(\xi)$ be the minimum, over all green–yellow colourings of G_k with at least $k^2/4$ vertices of each colour, of the number of adjacent green–yellow pairs with neither vertex in ξ . There is $q_0 > 0$ such that, if Ξ^k is the random subset of G_k containing each vertex independently with probability q_0 , then $\mathbb{P}(c(\Xi^k) < k/400) \rightarrow 0$ as $k \rightarrow \infty$.*

The proof in [Ald21, pp. 3–6] is a Peierls contour argument about the grid; no network enters, and the passage to the property (2.6) there is written as a sketch. As c is non-increasing under inclusion, a monotone coupling gives the conclusion for every $q \leq q_0$. As $c(\xi)$ is a minimum over colourings, it applies to colourings chosen after the configuration and the network are seen.

The next statement replaces [Ald21, Corollary 2.3]; see Section 8.2 for the reason. Put $\delta := 1/2000$. A *block* is a set $\{i_0, \dots, i_0 + s - 1\} \times \{j_0, \dots, j_0 + s - 1\} \subseteq G_k$ with $s = \lceil \delta k \rceil$.

Corollary 5.3. *Let $k \geq 16000$ and $\xi \subseteq G_k$ with $c(\xi) \geq k/400$. For every green–yellow colouring χ of G_k with at least $k^2/4 + 2\delta^2 k^2$ vertices of each colour and every block $\square$, there are at least $k/4000$ adjacent green–yellow pairs of χ with neither vertex in $\xi \cup \square$.*

Proof. Recolour the vertices of $\square$ green and call the result χ' . Since $\delta k \geq 8$, we have $[\delta k]^2 \leq (\delta k + 1)^2 \leq 2\delta^2 k^2$, so χ' has at least $k^2/4$ vertices of each colour, and $c(\xi) \geq k/400$ gives at least $k/400$ adjacent green–yellow pairs of χ' avoiding ξ . Such a pair with a vertex in $\square$ joins a vertex of $\square$ to one outside, and there are at most $4[\delta k] \leq k/500 + 4$ such adjacent pairs. The remaining at least $k/400 - k/500 - 4 = k/2000 - 4 \geq k/4000$ pairs avoid $\square$, where $\chi' = \chi$. $\square$

Lemma 5.4. *Let Ξ be a Poisson process of rate one and $\beta_0 := 11/2000000 = 5.5 \cdot 10^{-6}$. There is m_0 such that for all $m \geq m_0$ and $k \geq 16000$ there is an event $E_{m,k}$ determined by Ξ with $\lim_{k \rightarrow \infty} \mathbb{P}(E_{m,k}) = 0$ for each fixed $m \geq m_0$, such that the following holds outside $E_{m,k}$. The set $F := \Xi \cap Q$ has $N \geq 3$ points, and for every point $v_* \in \mathbb{R}^2$ and every $B \subseteq F$ with $N/3 \leq |B| \leq N/2$ there are at least $\beta_0 m^4 k$ pairs $(v, v') \in B \times (F \setminus B)$ with*

$$|v - v'| \leq \sqrt{2}m \quad \text{and} \quad |v - v_*| + |v' - v_*| \geq \delta n/2.$$

Proof. The exceptional event. Let E_1 be the event that $N < 0.99m^2 k^2$ or that a point of F lies on a line $x \in \frac{m}{5}\mathbb{Z}$ or $y \in \frac{m}{5}\mathbb{Z}$; $\mathbb{P}(E_1) \leq 10^4/(m^2 k^2)$ by Chebyshev's inequality. Let q_m be the probability that Ξ is not balanced in $[0, m]^2$; as each strip count is Poisson with mean $m^2/5$, $q_m \leq 125000/m^2$. The set $\Xi^k \subseteq G_k$ of natural subsquares in which Ξ is not balanced contains each vertex independently with probability q_m . Let N_σ be the number of points of Ξ in σ , $U := \sum_\sigma N_\sigma \mathbf{1}\{\sigma \in \Xi^k\}$ and $\psi(m) := 4q_m^{1/2}$. By Cauchy–Schwarz, $\mathbb{E}[N_\sigma \mathbf{1}\{\sigma \in \Xi^k\}] \leq (\mathbb{E}N_\sigma^2 q_m)^{1/2} \leq 2m^2 q_m^{1/2}$, since $\mathbb{E}N_\sigma^2 = m^4 + m^2 \leq 2m^4$; so $\mathbb{E}U \leq m^2 k^2 \psi(m)/2$, while $\text{Var } U \leq k^2 \mathbb{E}N_\sigma^2 \leq 2m^4 k^2$. With $E_2 := \{U > m^2 k^2 \psi(m)\}$, Chebyshev's inequality gives $\mathbb{P}(E_2) \leq 8/(k^2 \psi(m)^2)$. Let $E_3 := \{c(\Xi^k) < k/400\}$. Choose m_0 such that $q_m \leq q_0$ and $\psi(m) \leq 0.003$ for $m \geq m_0$; then $\mathbb{P}(E_3) \rightarrow 0$ as $k \rightarrow \infty$ by Lemma 5.2 and the remark after it. Put $E_{m,k} := E_1 \cup E_2 \cup E_3$. The rest of the proof is deterministic: fix $m \geq m_0$, $k \geq 16000$, an outcome outside $E_{m,k}$, a point v_* and a set B as in the statement. Then $N \geq 0.99m^2 k^2 \geq 3$.

The block. Let H be the closed axis-parallel square of side $\delta n/2$ centred at v_* . A closed interval of length $\delta k m/2$ meets at most $\lceil \delta k/2 \rceil + 2 \leq \lceil \delta k \rceil$ of the intervals $[im, (i+1)m]$, because $\delta k \geq 8$. Hence there is a block $\square$ that contains all natural subsquares meeting H (if H misses Q , any block). Let $\square^+$ be the union of the subsquares in $\square$. A point of $F \setminus \square^+$ lies in a subsquare outside $\square$, which misses H ; so its distance from v_* exceeds $\delta n/4$, and $|v - v_*| + |v' - v_*| \geq \delta n/2$ for all $v, v' \in F \setminus \square^+$. It remains to find $\beta_0 m^4 k$ pairs $(v, v') \in B \times (F \setminus B)$ in $F \setminus \square^+$ with $|v - v'| \leq \sqrt{2}m$.

Counting. Colour the points of B red and those of $F \setminus B$ blue; there are at least $N/3 \geq 0.33m^2 k^2$ red and at least $N/2 \geq 0.495m^2 k^2$ blue points. Let b_σ be the number of blue points in σ . A balanced subsquare (one not in Ξ^k) contains between $0.98m^2$ and $1.02m^2$ points. Among the balanced subsquares let $S^<$, S and $S^>$ be the numbers of those with $b_\sigma < 0.09m^2$, with $0.09m^2 \leq b_\sigma \leq 0.93m^2$, and with $b_\sigma > 0.93m^2$. The unbalanced subsquares contain at most $0.003m^2 k^2$ points (outside E_2). Counting blue and red points square by square,

$$0.495 k^2 \leq 0.09 S^< + 0.93 S + 1.02 S^> + 0.003 k^2, \tag{2}$$

$$0.33 k^2 \leq 1.02 S^< + 0.93 S + 0.09 S^> + 0.003 k^2. \tag{3}$$

For (3): a subsquare counted by S has at most $1.02m^2 - 0.09m^2$ red points, one counted by $S^>$ fewer than $0.09m^2$.

Case A: $S > 0.005k^2$. A subsquare counted by S contains at least $b_\sigma(0.98m^2 - b_\sigma) \geq \min(0.09 \cdot 0.89, 0.93 \cdot 0.05) m^4 = 0.0465 m^4$ blue–red pairs, at distance at most $\sqrt{2}m$. At most $\lceil \delta k \rceil^2 \leq 0.001k^2$

subsquares lie in $\square$, so the subsquares counted by S outside $\square$ give at least $0.0465 \cdot 0.004 m^4 k^2 \geq \beta_0 m^4 k$ distinct pairs.

Case B: $S \leq 0.005 k^2$. Inserting $S^> \leq k^2 - S - S^<$ into (3) and $S^< \leq k^2 - S - S^>$ into (2) gives

$$0.93 S^< \geq 0.237 k^2 - 0.84 S, \quad 0.93 S^> \geq 0.402 k^2 - 0.84 S,$$

so $S^< \geq \frac{194}{775} k^2 \geq 0.2503 k^2$ and $S^> \geq \frac{663}{1550} k^2 \geq 0.4277 k^2$; both exceed $k^2/4 + 2\delta^2 k^2$. Colour a vertex σ of G_k yellow if $b_\sigma < 0.1 m^2$ and green otherwise. The subsquares counted by $S^<$ are yellow and those counted by $S^>$ are green, so Corollary 5.3 applies with $\xi = \Xi^k$ (outside E_3) and gives at least $k/4000$ adjacent pairs $\{\sigma, \sigma'\}$ of balanced subsquares outside $\square$, with σ yellow and σ' green. For each of them Lemma 5.1 applies, since $b_\sigma < 0.1 m^2 \leq b_{\sigma'}$ excludes both exceptions, and gives at least $0.088 m^4$ blue–red pairs of points of $(\sigma \cup \sigma') \setminus \square^+$ at distance at most $\sqrt{2} m$. A pair of points in one subsquare τ is counted for at most four such $\{\sigma, \sigma'\}$, those containing τ , and a pair in two different subsquares for at most one. So there are at least $0.088 \cdot (k/4000)/4 \cdot m^4 = \beta_0 m^4 k$ distinct pairs. $\square$

The bounds $194/775$ and $663/1550$ are the exact minima under (2), (3), $S \leq 0.005 k^2$ and $S^< + S + S^> \leq k^2$. On the differences from [Ald21, Lemma 2.4] see Section 8.1.

6 Campbell's formula and the proof of the theorems

Lemma 6.1. *Under the hypotheses of Theorem 1.2, let $n, r, d > 0$ and let X be the number of unordered pairs $\{\xi, \xi'\}$ of distinct points of $\Xi \cap [0, n]^2$ with $|\xi - \xi'| \leq r$ and $\text{len } R(\xi, \xi') \geq d$. Then*

$$\mathbb{E}X \leq \frac{\pi}{2} n^2 r^2 \mathbb{P}(D_1 \geq d/r).$$

Proof. Conditionally on Ξ , the route between two points ξ, ξ' of Ξ has the law $\mu_{\xi, \xi'}$ (Section 2.1), so by (1) its length exceeds or equals d with probability $g(|\xi - \xi'|)$, where $g(s) := \mathbb{P}(D_1 \geq d/s)$ is non-decreasing in s . Hence $\mathbb{E}[X \mid \Xi] \leq g(r) Y$, where Y is the number of unordered pairs of points of $\Xi \cap [0, n]^2$ at distance at most r . By Campbell's formula for the second factorial moment measure of a Poisson process (given $|\Xi \cap [0, n]^2| = j$ the points are independent and uniform, and $\mathbb{E}[j(j-1)] = n^4$ for a Poisson variable j of mean n^2),

$$\mathbb{E}Y = \frac{1}{2} \int_{[0, n]^2} \int_{[0, n]^2} \mathbf{1}\{|z - z'| \leq r\} dz dz' \leq \frac{1}{2} n^2 \pi r^2. \quad \square$$

Proof of Theorem 1.2. Let $m := m_0$ be the constant of Lemma 5.4; it does not depend on the route process. For $k \geq 16000$ put $n := mk$, $Q := [0, n]^2$, $r := \sqrt{2} m$, $d := \delta n/2$, and let X_k be the variable X of Lemma 6.1. Let Ω_0 be an event of probability one on which Ξ is locally finite and for every $n' \in \mathbb{N}$ the routes between the points of $\Xi \cap [0, n']^2$ are feasible and pairwise compatible (Section 2.1). We claim

$$A_\infty \cap \Omega_0 \setminus E_{m,k} \subseteq \{X_k \geq \beta_0 m^4 k\}. \quad (4)$$

Indeed, for an outcome on the left, $F := \Xi \cap Q$ has $N \geq 3$ points (Lemma 5.4), its routes are compact arcs of finite length satisfying (2.4), and their union span_n contains no circuit. Corollary 4.2 gives v_* and B with (T1) and (T2). By Lemma 5.4 there are at least $\beta_0 m^4 k$ pairs $(v, v') \in B \times (F \setminus B)$ with $|v - v'| \leq r$ and $|v - v_*| + |v' - v_*| \geq d$, hence, by (T2), with $\text{len } R(v, v') \geq d$. Distinct pairs in $B \times (F \setminus B)$ are distinct unordered pairs, so $X_k \geq \beta_0 m^4 k$.

The right-hand side of (4) is an event, and so are $E_{m,k}$ and Ω_0 . Hence $p = \mathbb{P}^*(A_\infty) \leq \mathbb{P}(X_k \geq \beta_0 m^4 k) + \mathbb{P}(E_{m,k})$, and by Markov's inequality and Lemma 6.1,

$$\beta_0 m^4 k (p - \mathbb{P}(E_{m,k})) \leq \mathbb{E}X_k \leq \pi m^4 k^2 \mathbb{P}(D_1 \geq \delta k/(2\sqrt{2})),$$

because $\frac{\pi}{2}n^2r^2 = \pi m^4k^2$ and $d/r = \delta k/(2\sqrt{2})$. The event $E_{m,k}$ is determined by the Poisson process, so its probability does not depend on the route process, and it tends to zero as $k \rightarrow \infty$. Let $k_0(p) \geq 16000$ be such that $\mathbb{P}(E_{m,k}) \leq p/2$ for all $k \geq k_0(p)$. Then

$$\mathbb{P}(D_1 \geq \delta k/(2\sqrt{2})) \geq \frac{\beta_0 p}{2\pi k} \quad (k \geq k_0(p)).$$

Put $t_0(p) := \delta k_0(p)/(2\sqrt{2})$. For $t \geq t_0(p)$ let $k := \lceil 2\sqrt{2}t/\delta \rceil \geq k_0(p)$. Then $t \leq \delta k/(2\sqrt{2})$ and $k \leq 2\sqrt{2}t/\delta + 1 \leq 4\sqrt{2}t/\delta$, so

$$\mathbb{P}(D_1 \geq t) \geq \mathbb{P}(D_1 \geq \delta k/(2\sqrt{2})) \geq \frac{\beta_0 p}{2\pi k} \geq \frac{\beta_0 \delta p}{8\sqrt{2}\pi t} = \frac{cp}{t},$$

with $c = \beta_0 \delta / (8\sqrt{2}\pi) = 11/(32\sqrt{2}\pi \cdot 10^9)$. Finally $\mathbb{E}D_1 \geq \int_{t_0(p)}^{\infty} \mathbb{P}(D_1 \geq t) dt = \infty$. $\square$

Proof of Theorem 1.1. A SIRS or weak SIRS satisfies the hypotheses of Theorem 1.2 and $\mathbb{E}D_1 < \infty$ (2.15), so $\mathbb{P}^*(A_\infty) = 0$, and $\mathbb{P}^*(A_T) = \mathbb{P}^*(A_J) = 0$ by Section 2.2 and Proposition 7.3. An event in A_T or A_J has probability zero; in particular no event of probability one lies in A_T or in A_J . $\square$

Scale invariance is what turns Lemma 5.4 into a statement about D_1 ; without it there is no contradiction: in the Euclidean minimal spanning tree the mean route length between Poisson points at a small distance is finite [Ald21, Section 3.1].

7 The junction graph

Let s be a feasible subnetwork on a locally finite set Z : for all $z \neq z'$ in Z a route $r(z, z')$ (a compact arc of finite length), the routes pairwise compatible (2.4), and (2.5) $\text{len}(s \cap D) < \infty$ for every disc D . For distinct routes r, ρ with $I := r \cap \rho \neq \emptyset$ (a point or a sub-arc, Lemma 3.1), an endpoint q of I is a *departure point on r* if q is not an endpoint of ρ . Then $\rho \setminus I$ has a component whose closure contains q ; its closure is a sub-arc φ of ρ from q to an endpoint e of ρ , a *free arc at q* . It satisfies $\varphi \cap r = \{q\}$ and $e \in Z \setminus r$.

Lemma 7.1. *Every route of s carries only finitely many departure points.*

Proof. Suppose that a route r carries infinitely many distinct departure points q_n , with routes ρ_n , sets $I_n := \rho_n \cap r$ and free arcs φ_n . Passing to a subsequence, $q_n \rightarrow q^* \in r$ with $q_n \neq q^*$. The set $Z \setminus r$ is locally finite and does not contain q^* , so some disc $D := \text{disc}(q^*, \varepsilon)$ misses it; discarding finitely many n , we may assume $|q_n - q^*| < \varepsilon/2$. Each φ_n joins q_n to a point of $Z \setminus r$ outside D , so $\text{len}(\varphi_n \cap D) \geq \varepsilon/4$. By (2.5), s contains no infinite family of pairwise disjoint arcs each of which has length at least $\varepsilon/4$ inside D .

Colour a pair $\{n, n'\}$ by (a) if $I_n \cap I_{n'} \neq \emptyset$; by (b1) if $I_n \cap I_{n'} = \emptyset$ and $\varphi_n \cap \varphi_{n'} = \emptyset$; by (b2) otherwise. By Ramsey's theorem there is an infinite set of indices all of whose pairs have the same colour.

Colour (a): we show $\varphi_n \cap \varphi_{n'} = \emptyset$. Let $t \in \varphi_n \cap \varphi_{n'}$ and $u \in I_n \cap I_{n'}$. By (2.4), $[t, u]_{\rho_n} = [t, u]_{\rho_{n'}}$. On ρ_n the arcs φ_n and I_n meet only at q_n , so $[t, u]_{\rho_n} \cap r = [q_n, u]_r$; likewise $[t, u]_{\rho_{n'}} \cap r = [q_{n'}, u]_r$. Two sub-arcs of r with the common end u coincide only if their other ends coincide, so $q_n = q_{n'}$, a contradiction. In colours (a) and (b1) the free arcs are pairwise disjoint, which is impossible.

Colour (b2): for $n \neq n'$, $I_n \cap I_{n'} = \emptyset$ and $\varphi_n \cap \varphi_{n'} \neq \emptyset$. The connected set $\rho_n \cap \rho_{n'}$ misses r (since $\rho_n \cap \rho_{n'} \cap r = I_n \cap I_{n'}$), so it lies in one component of $\rho_n \setminus I_n$; it meets φ_n , hence lies in $\varphi_n \setminus \{q_n\}$. Let q'_n be the other endpoint of I_n ($q'_n = q_n$ if I_n is a point) and φ'_n the sub-arc of ρ_n from q'_n to the endpoint of ρ_n beyond q'_n (on the side away from φ_n). Then $\varphi'_n \cap \rho_{n'} = \emptyset$, so the arcs φ'_n are

pairwise disjoint. The I_n are disjoint sub-arcs of r , so $\sum_n \text{len } I_n \leq \text{len } r$ and, discarding finitely many n , $|q'_n - q^*| < 3\varepsilon/4$. The arc φ'_n is a single point only if q'_n is an endpoint of ρ_n , so $q'_n \in Z \cap r$, a finite set whose points equal q'_n for at most one n each; discard these n . Every remaining φ'_n joins a point of $\text{disc}(q^*, 3\varepsilon/4)$ to a point of $Z \setminus r$ outside D , so $\text{len}(\varphi'_n \cap D) \geq \varepsilon/4$, and these arcs are pairwise disjoint, contradicting (2.5). $\square$

A related finiteness statement is [Ald14, Lemma 2.1(i)], proved there in outline.

Lemma 7.2. *Let $V(s)$ consist of the points of Z and the departure points on all routes. A point of $V(s)$ on a route r is a point of Z or a departure point on r ; hence every route contains only finitely many points of $V(s)$. The open sub-arcs of routes between consecutive points of $V(s)$ (edges) are pairwise equal or disjoint.*

Proof. Let $q \in r \setminus Z$ be a departure point on another route r' , an endpoint of $r' \cap \rho$ for a route ρ of which q is not an endpoint. If q were an inner point (relative to r) of both $r \cap \rho$ and $r \cap r'$, a sub-arc of r around q would lie in $\rho \cap r'$ and, since $q \notin Z$ is not an endpoint of r' , it would be a neighbourhood of q in r' , contradicting that q is an endpoint of $\rho \cap r'$. So q is an endpoint of $r \cap \rho$, and then a departure point on r since q is not an endpoint of ρ ; or q is an endpoint of $r \cap r'$, and then a departure point on r since $q \notin Z$. The set $Z \cap r$ is finite, and so, by Lemma 7.1, is $V(s) \cap r$. For the last assertion, let edges $e \subseteq r$ and $f \subseteq \rho \neq r$ meet. Every endpoint of $I := r \cap \rho$ is an endpoint of ρ (a point of Z) or a departure point on r , so it lies in $V(s)$, and the proof of Lemma 3.2 gives $e = f$. $\square$

So s is the union of the vertex set $V(s)$ and of the edge set $E(s)$; we call $\Gamma(s) := (V(s), E(s))$ the *junction graph* of s . It may have infinitely many vertices.

Proposition 7.3. *If $\Gamma(s)$ has no finite cycle, then no finite union of routes of s contains a circuit. In particular $A_J \subseteq A_\infty$.*

Proof. A finite union $U = r_1 \cup \dots \cup r_M$ of routes contains finitely many points of $V(s)$ and is the union of these points and of finitely many edges; an edge meeting U lies in U (Lemma 7.2). As in Lemma 3.2, every such edge is open in U , and its closure is the edge with two distinct ends. The proof of Lemma 3.3(i) used only these facts, so a circuit in U is the union of the closures of the edges of a cycle of this finite graph, which is a finite cycle of $\Gamma(s)$. For the last assertion apply this to $s = S(1)$ and $Z = \Xi$, and to the unions span_n . $\square$

8 Two steps of [Ald21] that need repair

This section explains why Corollary 5.3 and Lemma 5.4 differ from [Ald21, Corollary 2.3 and Lemma 2.4]. The changes concern a threshold and constants; the structure of the argument is unchanged.

8.1 The inequality (2.16)

In the proof of [Ald21, Lemma 2.4] the points of B are red, those of B^c blue, and $S, S^<, S^>$ count the balanced natural subsquares with $0.09m^2 \leq b_\sigma \leq 0.89m^2$, $b_\sigma < 0.09m^2$, $b_\sigma > 0.89m^2$. Outside events of small probability, by (2.13) and (2.14) of [Ald21], both colours have more than $0.33m^2k^2$ points and the unbalanced subsquares contain at most $m^2k^2\psi(m)$ points, $\psi(m) \rightarrow 0$. From the blue count, [Ald21, p. 8] derives $0.24k^2 \leq 0.8S + 0.93S^> + k^2\psi(m)$, and “with the corresponding inequality arising from counting red points” it states (2.16): for m sufficiently large, if $S \leq 0.005k^2$ then $\min(S^<, S^>) \geq k^2/4$.

With these thresholds the red count is $0.33k^2 \leq 1.02S^< + 0.93S + 0.13S^> + k^2\psi(m)$, since a subsquare counted by $S^>$ may contain up to $0.13m^2$ red points; eliminating $S^>$ gives only $S^< \geq (0.196 - \psi(m))k^2/0.89$ when $S \leq 0.005k^2$. This is the best the two counts give: with $\psi = 0$, the minimum of $S^</math> under the two inequalities, $S \leq 0.005k^2$ and $S^< + S + S^> \leq k^2$ is exactly $98/445 = 0.2202\dots$ (that of $S^>/k^2$ is $118/465 = 0.2537\dots$). In fact (2.16) is false, even for configurations that satisfy every other requirement of the proof.$

Example 8.1. Let $m \geq 130$ be a multiple of 10 and k a multiple of 100. Place m^2 points in every natural subsquare, $m^2/25$ in the interior of each of the 25 cells cut out by its strips, hence $m^2/5$ in each of its ten strips, so that every subsquare is balanced and $N = m^2k^2$. Colour blue no point in $0.2465k^2$ of the subsquares, $0.09m^2$ points in $0.005k^2$ of them, and $0.89m^2 + 1$ points in each of the remaining $0.7485k^2$; colour all other points red, and let B be the set of red points. Then $S^< = 0.2465k^2$, $S = 0.005k^2$, $S^> = 0.7485k^2$, no point lies in an unbalanced subsquare, and $|B| = 0.2465k^2m^2 + 0.005k^2 \cdot 0.91m^2 + 0.7485k^2(0.11m^2 - 1) = (0.333385m^2 - 0.7485)k^2$, which lies between $N/3$ and $N/2$ because $m^2 \geq 16900$. So B is admissible in [Ald21, Lemma 2.4], and both colours have more than $0.33m^2k^2$ points. Yet $\min(S^<, S^>) = 0.2465k^2 < k^2/4$.

The conclusion of [Ald21, Lemma 2.4] is not contradicted: with the threshold $0.93m^2$ of Lemma 5.4, the subsquares with $0.89m^2 + 1$ blue points are counted by S , and Case A applies. This is the repair: with the upper threshold $0.93m^2$ a subsquare counted by $S^>$ has fewer than $0.09m^2$ red points, as one counted by $S^<$ has fewer than $0.09m^2$ blue points, and the counts become symmetric. With $|B| \geq 0.33m^2k^2$, $|B^c| \geq 0.495m^2k^2$ and $\psi = 0.003$ the exact minima are $194/775$ for $S^</k^2$ and $663/1550$ for $S^>/k^2$ (with $0.33m^2k^2$ for both colours, $194/775$ for both), above $1/4 + 2\delta^2$, as Corollary 5.3 requires. The price is the constant 0.0465 instead of 0.08 in Case A, the analogue of (2.15) of [Ald21].

8.2 Corollary 2.3

[Ald21, Corollary 2.3] states that Lemma 5.2 remains true, with some αk in place of $k/400$, if only pairs outside a subsquare $\square$ of side asymptotic to $0.001k$, which may depend on Ξ^k , are counted; the proof is outlined: recolour $\square$ in one colour and apply Lemma 5.2. A block of side $0.001k$ has about $0.004k$ boundary edges, more than the $k/400$ pairs given by Lemma 5.2; after the recolouring all these pairs may be boundary edges of the block, and the outline gives no positive α . We do not know whether the statement holds for blocks of side $0.001k$; the outline does not prove it. The other points of the outline are sound: recolouring the block in the colour that has fewer vertices keeps $k^2/4$ vertices of each colour, and since the block may depend on Ξ^k , the statement applied to the worst block gives a bound valid for all blocks at once, as the proof of [Ald21, Lemma 2.4] requires (there the block is placed near the centroid, which depends on the network). Corollary 5.3 uses blocks of side $\lceil k/2000 \rceil$, whose at most $k/500 + 4$ boundary edges leave at least $k/4000$ of the $k/400$ pairs, and it holds for all colourings and blocks at once; it follows from Lemma 5.2 as published, because $c(\xi)$ is a minimum over colourings. (Its hypothesis of $k^2/4 + 2\delta^2k^2$ vertices of each colour could be weakened to $k^2/4$ by recolouring in the colour with fewer vertices; Case B of Lemma 5.4 provides it anyway.)

9 Computations and reproducibility

No theorem, lemma or corollary of this note depends on a computation. The program `check_note.py` of the reproducibility package (Python 3, standard library, about ten seconds, exact arithmetic except where marked) checks the finite and exact parts of the proofs: (A) the numerical facts of Lemma 5.1; (B) the inequalities of Corollary 5.3 for all $16000 \leq k \leq 200000$, and their analytic form; (C) the

constants of Lemma 5.4, with Case B as exact linear programs in $(S^<, S, S^>)$ solved by vertex enumeration (minima 194/775 and 663/1550); (D) the constant $c = 7.737 \dots \cdot 10^{-11}$ and the last chain of inequalities in the proof of Theorem 1.2 (floating point); (E) Lemma 4.1 on all trees with at most 10 vertices and all W with $|W| \geq 3$ (130,947 cases), including the path property, with two negative controls; (F) the linear programs of Section 8.1 (minima 98/445 and 118/465) and Example 8.1 for $m = 1000$, $k = 200$ and for $m \in \{130, \dots, 390\}$, $k \in \{100, 200, 300\}$. All 39 checks pass. The folder `independent_run_2/` of the package contains the programs of the last verification run (see the next paragraph) with their recorded outputs: `run2_constants.py` (39 checks), `run2_trees.py` (13 checks) and `run2_campbell.py` (a Monte Carlo sanity check of Lemma 6.1); they use Python 3 with NumPy, SciPy, SymPy, mpmath and NetworkX.

Verification. The result was first obtained with a written proof and a program checking its constants. Two independent verification runs, both AI-assisted, followed; each re-derived the proof and wrote its own programs, and neither read the other's report. The first run re-derived every step that involves the network and checked the statement and its numbering against both versions of [Ald14]; its program recomputed the constants exactly, solved linear relaxations of the counting step (also for the published thresholds), and tested Lemma 4.1 on 3,859 weighted trees and Lemma 5.1 on 191,021 colourings of strips. It found the outer-probability argument now used for Theorem 1.1, and it required the missing step in the proof of Lemma 7.2, an explicit statement of the dependence on [Ald21, Lemma 2.2], two corrected section references to [Ald14], Remark 2.1, and corrections in the account of the literature search. The second run compared the statements with the published texts of [Ald14, Ald21] (open copies on the EMIS mirror of the journal and in the repository of the University of California), whose numbering we use; its programs solved the linear programs exactly, with negative controls, tested Lemma 4.1 on all trees with at most 11 vertices (596,482 cases), searched for violations of Lemma 5.1 and found none (the bound used in its proof is attained), and tested Sections 3 and 4 on 1,000 random planar trees, with Delaunay networks as a negative control. It required the published numbering, the measurability proviso now expressed by outer probability, the missing step in Lemma 7.2, a block count valid for closed squares, and additions to the bibliography. Both runs re-ran the first program and obtained its recorded output (it has 13 checks, not 14 as first stated). Neither run found an error affecting the argument; both found that the steps of [Ald21] discussed in Section 8 need repair and that the repairs are correct. Both rest on [Ald21, Lemma 2.2] as published; its proof was read, not re-proved, and is a sketch at the step leading to (2.6) there. All corrections required by the two runs were made. Example 8.1, the program of this note and the remark that t_0 does not depend on the route process were added when the note was written.

A further independent verification run, also AI-assisted, then examined the final text, these additions first. It read the sources again, [Ald21] completely in its published version, re-derived every proof line by line, including Section 7, and confirmed Example 8.1 and the independence of t_0 from the route process. Its programs certified the four minima of the linear programs exactly by rational dual solutions, checked Example 8.1 in integer arithmetic for $m = 130, 140, \dots, 390$ with an explicit placement of the points, Corollary 5.3 for all $16000 \leq k \leq 2 \cdot 10^6$, the block count in the proof of Lemma 5.4 by brute force and the constant c to 50 digits, tested Lemma 4.1 on all trees with at most 12 vertices and all W with $|W| \geq 3$ (2,809,849 cases) and Sections 3 and 4 on 200 tree networks, with Delaunay networks as a negative control, and it re-ran `check_note.py` from the source archive and obtained the recorded output. It found no error in the theorems, lemmas and proofs. It found that two remarks of an earlier version of Section 8.2 on [Ald21, Corollary 2.3] were not correct, namely that the recolouring needs a margin over $k^2/4$ and that the corollary does not give a bound uniform in the block; they were removed. It also required the date, a sentence of Section 1.3 that read local finiteness into the hypotheses of [Ald21, Theorem 1.2], and the placement of the points in Example 8.1; these

corrections were made.

Scope and priority. The note answers [Ald14, Open Problem 10] (Open Problem 36 of the arXiv version) in the affirmative, for SIRSs and weak SIRSs, in both readings of Section 2.2, in the form of Theorem 1.1; whether the set of outcomes in which $S(1)$ has no circuit is an event is not decided. The question is Aldous’s, and the core of the argument is his proof of [Ald21, Theorem 1.2], whose Lemma 2.2 is used as published and is not re-proved here. What the note adds is the transfer of that proof to the routes of a SIRS (Sections 3, 4 and 7), the replacement of the Palm-probability step by Campbell’s formula and scale invariance, with the outer-probability formulation (Section 6), and the repairs of (2.16) and Corollary 2.3 of [Ald21] (Sections 5 and 8). The deduction is short once [Ald21] is available, and specialists may regard it as known; no priority is claimed. No printed answer to the question was found in the sources searched, listed below; a search that finds nothing is not a proof of novelty.

What was read: [Ald14], published and arXiv versions, Sections 1, 2, 5.1, 5.5, Lemma 6.2 and Section 8.7.2 (in the parts compared the versions agree in substance); [Ald21] completely, published and in arXiv:2103.00669v1, whose Theorem 2, Lemmas 1, 2, Corollary 1 and Lemma 3 are Theorem 1.2, Lemmas 2.1, 2.2, Corollary 2.3 and Lemma 2.4 of the published version. The arXiv sources of [Ken17, Kah16, Ken20, Bla24, BCK24] were searched for “tree”, “Steiner” and “open problem”; none treats the question. [AG13, BLPS01, Jor69] are cited for background and were not read for this note. The searches (16 and 17 October 2026) used arXiv (SIRSs, scale-invariant random spatial networks, invariant tree networks, route lengths, Steiner points with scale invariance, Aldous’s papers in probability since 2014), Crossref, zbMATH Open (no work citing [Ald21]; [Ken17, Kah16] citing [Ald14]), Semantic Scholar (no work citing [Ald21]; its 24 works citing [Ald14] belong to a record that merges [Ald14] with [AG13], because the arXiv record of 1204.0817 carries the DOI of [AG13]; none concerns the question), OpenAlex (citation counts 0 and 20; the lists were not available), Aldous’s web pages, including [Ald], which refers to the ten open problems of [Ald14, Section 8] without a note of progress, and web searches; the last verification run repeated the searches in arXiv, Crossref, zbMATH Open and OpenAlex and made one web search on 17 October 2026; it found nothing new. [Ald21] itself does not mention SIRSs, $S(1)$ or the open problem; it cites the SIRS programme only through [AG13], in its Section 3.5.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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