

On Conjecture 1 of Aldous for a Spatial Model of City Growth: A Proof of Part (a), and Counterexamples to Parts (b) and (c) for $2\alpha/\sqrt{3} < \beta < 2\alpha$

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Abstract

In rough notes of 2007 on a spatial model of city growth, D. J. Aldous stated a three-part conjecture about the unbalanced regime. Immigrants arrive one at a time at uniform random points of the unit square; a city of population n exerts the influence $c_0 n^\alpha r^{-\beta}$ at distance r ; an immigrant founds a new city with probability $1/(1+I)$, where I is the largest influence at its position, and otherwise joins the city whose influence there is largest. The conjecture asserts, almost surely: (a) if $\alpha > 1$, only finitely many cities are founded; (b) if $\beta < 2\alpha$, the share of the largest city tends to 1; (c) if $\alpha > 1$ and $\beta < 2\alpha$, only finitely many inhabitants ever live outside the largest city. We prove (a) for all $c_0 > 0$, $\alpha > 1$, $\beta > 0$. A packing bound for the cities founded where the influence is small, combined with Ville's maximal inequality, shows that the number of cities is $O(t^{2/(\beta+2)})$ almost surely for all parameters; for $\alpha > 1$ the exponent is then lowered in finitely many steps until the founding probabilities are summable. Parts (b) and (c) are false for every $c_0 > 0$, $\alpha > 1$ and $2\alpha/\sqrt{3} < \beta < 2\alpha$: with positive probability two cities near $(\frac{1}{2} \mp a, \frac{1}{2})$ share the population in nearly equal parts forever. The reason is that the two-city share map, the area of a city's sphere of influence as a function of its share of the population, has the slope $(2\alpha/\beta)(a + \frac{1}{12a})$ at the symmetric point, and this slope is less than 1 for a suitable spacing exactly when $2\alpha/\beta < \sqrt{3}$. Part (b) is also false for every $\alpha > 0$ in the same range of β , in particular in the range $0 < \alpha \leq 1$ of the companion paper of Aldous and Huang. An explicit instance ($\alpha = 2$, $\beta = 3$, any c_0) is given with all constants. Parts (b) and (c) remain open for $\beta \leq 2\alpha/\sqrt{3}$. The model and the conjecture are Aldous's, and the tools (Ville's and Doob's inequalities, circles of Apollonius) are standard. To our knowledge the results are new; no answer to the conjecture was found in the literature accessible to us, and a search that finds nothing is not a proof of novelty. This is an unrefereed note.

1 Introduction

1.1 The model

We use the model of Aldous's notes [Ald07, §1, (1)–(3)], which are linked as technical notes from his open problem page [Ald]; the companion paper of Aldous and Huang [AH12, §2] describes the same model for $0 < \alpha \leq 1$. At each time $t = 1, 2, 3, \dots$ there are $M(t)$ cities, indexed $i = 1, 2, \dots$ in the order of their founding, with positions x_i in the closed unit square $Q = [0, 1]^2$ and populations $N_i(t) \geq 1$ (integers); the total population is $\sum_i N_i(t) = t$. The model has three parameters

$$0 < c_0 < \infty, \quad 0 < \alpha < \infty, \quad \beta > 0.$$

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A city of population n exerts the *influence* $I_0(n, r) = c_0 n^\alpha r^{-\beta}$ at Euclidean distance r . The influence field and the *sphere of influence* of city i are

$$I(y, t) = \max_i I_0(N_i(t), |y - x_i|) = c_0 \max_i N_i(t)^\alpha |y - x_i|^{-\beta} \quad (y \in Q),$$

$$S(i, t) = \{y \in Q : I_0(N_i(t), |y - x_i|) = I(y, t)\}.$$

At time 1 there is one city, of population 1, at a uniform random point of Q . At time $t + 1$ an immigrant arrives at a uniform random point U of Q , independent of the past, and

- (i) with probability $1/(1 + I(U, t))$ it founds a new city at U , of population 1;
- (ii) otherwise, that is with probability $I(U, t)/(1 + I(U, t))$, it joins the city i with $U \in S(i, t)$, whose population becomes $N_i(t) + 1$.

The boundaries between spheres of influence have area zero (Lemma 2.1), so the convention used on them does not matter. Write $N_{(1)}(t) \geq N_{(2)}(t) \geq \dots$ for the ordered populations and $N_{(\geq 2)}(t) = t - N_{(1)}(t)$ for the population outside a largest city. Since $M(t)$ and $N_{(\geq 2)}(t)$ are nondecreasing in t (the largest population grows by at most one per step), the limits

$$M(\infty) = \lim_{t \rightarrow \infty} M(t) \in [1, \infty], \quad N_{(\geq 2)}(\infty) = \lim_{t \rightarrow \infty} N_{(\geq 2)}(t) \in [0, \infty]$$

exist [Ald07, §2]. Aldous's heuristics [Ald07, §1], [AH12, §3] predict *balanced growth*, with many cities of comparable size, for $0 < \alpha < 1$ and $\beta > 2\alpha$, and *unbalanced growth*, with most of the population in one city or in a few cities, for $\alpha > 1$ or $\beta < 2\alpha$.

1.2 The conjecture

Conjecture (Aldous [Ald07, §2, Conjecture 1]). (a) *If $\alpha > 1$, then $M(\infty) < \infty$ almost surely.* (b) *If $\beta < 2\alpha$, then $N_{(1)}(t)/t \rightarrow 1$ almost surely as $t \rightarrow \infty$.* (c) *If $\alpha > 1$ and $\beta < 2\alpha$, then $N_{(\geq 2)}(\infty) < \infty$ almost surely.*

Each part is claimed for every $c_0 > 0$ and every (α, β) in the stated range. A proof of a part must therefore cover all such parameters, while one parameter triple in the range, together with an event of positive probability on which the conclusion fails, refutes it. For $\alpha > 1$ and $\beta < 2\alpha$, part (c) implies parts (a) and (b), since $M(\infty) \leq 1 + N_{(\geq 2)}(\infty)$ and $N_{(\geq 2)}(\infty) < \infty$ forces $N_{(1)}(t)/t \rightarrow 1$. The conjecture is record AMR-096-0023 of the UnsolvedMath dataset [ula26], in the same form.

The notes prove weaker statements [Ald07, Theorem 2]: if $\alpha > 1$, then $\mathbf{P}(N_{(\geq 2)}(\infty) = 0) > 0$; if $\beta < 2\alpha$, then $\mathbf{P}(N_{(1)}(t) > t^{1-\varepsilon}) \rightarrow 1$ for every $\varepsilon > 0$; and if $\alpha < 1$ and $\beta < 2\alpha$, then $\mathbf{P}(N_{(1)}(t)/t \rightarrow 1) > 0$, a statement whose proof the notes write out only in part. In the notes the conjecture is followed by the words “What we can actually prove is somewhat weaker” [Ald07, p. 8], which introduce this theorem. Remark (ii) after their Theorem 2 observes that $M(\infty) < \infty$ almost surely if and only if $\mathbf{E} \text{area}\{y : \sum_t 1/I(y, t) < \infty\} = 1$, so that (a) would follow from $N_{(1)}(t)/t^{1-\varepsilon} \rightarrow \infty$ almost surely, and asks whether the convergence in probability can be improved to almost sure convergence. The web page [Ald] presents the notes as an unfinished attempt to prove some of the predictions rigorously.

1.3 Results

Throughout, $\kappa = 2^{\beta/2}/c_0$ (the constant of the proof of [Ald07, Theorem 2(a)]), $k = 2\alpha/\beta$, and p_t is the conditional probability that a city is founded at time $t + 1$, given the history up to time t .

Theorem A (all parameters). *Let $c_0, \alpha, \beta > 0$. Almost surely, $M(t) \leq 1 + \log_2 t + \left(\frac{16}{\pi} + \frac{2(e-1)}{c_0}\right)t^{2/(\beta+2)}$ for all large t ; in particular $\sup_t M(t) t^{-2/(\beta+2)} < \infty$.*

Part	Hypothesis	Outcome	Where
(a)	$\alpha > 1$	true for all $c_0 > 0$, $\alpha > 1$, $\beta > 0$	Theorems A and B
(b)	$\beta < 2\alpha$	false for all $c_0 > 0$, $\alpha > 0$, $2\alpha/\sqrt{3} < \beta < 2\alpha$ open for $\beta \leq 2\alpha/\sqrt{3}$	Theorem C ($\alpha > 1$), Theorem D Section 8
(c)	$\alpha > 1$, $\beta < 2\alpha$	false for all $c_0 > 0$, $\alpha > 1$, $2\alpha/\sqrt{3} < \beta < 2\alpha$ open for $\beta \leq 2\alpha/\sqrt{3}$	Theorem C Section 8

Table 1: The three parts of the conjecture: per-part outcome.

Theorem B (part (a) of the conjecture). *Let $c_0 > 0$, $\alpha > 1$ and $\beta > 0$. Then $\sum_t p_t < \infty$ and $M(\infty) < \infty$ almost surely. Consequently $N_{(1)}(t) \geq t/M(\infty)$ for all t , and $N_{(1)}(t)/t^{1-\varepsilon} \rightarrow \infty$ almost surely for every $\varepsilon > 0$.*

Theorem C (parts (b) and (c) fail for $\alpha > 1$). *Let $c_0 > 0$, $\alpha > 1$ and $2\alpha/\sqrt{3} < \beta < 2\alpha$. There are $\delta \in (0, \frac{1}{8})$ and an integer T such that, with positive probability, $M(t) = 2$ and $|N_1(t)/t - \frac{1}{2}| \leq \delta$ for all $t \geq T$. Consequently $\mathbf{P}(N_{(1)}(t)/t \rightarrow 1) < 1$ and $\mathbf{P}(N_{(\geq 2)}(\infty) = \infty) > 0$.*

Theorem D (part (b) fails for every $\alpha > 0$). *Let $c_0 > 0$, $\alpha > 0$ and $2\alpha/\sqrt{3} < \beta < 2\alpha$. There are $\delta \in (0, \frac{1}{8})$ and integers j_0, T such that, with positive probability, for all $t \geq T$: $N_1(t) + N_2(t) \geq \frac{3}{4}t$, $|N_1(t)/(N_1(t) + N_2(t)) - \frac{1}{2}| \leq \delta$, and every city other than cities 1 and 2 has fewer than j_0 inhabitants. Consequently $N_{(1)}(t)/t \leq \frac{1}{2} + \delta$ for all $t \geq T$ on this event, and $\mathbf{P}(N_{(1)}(t)/t \rightarrow 1) < 1$.*

Corollary 1.1. *Part (a) of the conjecture holds for every $c_0 > 0$, $\alpha > 1$, $\beta > 0$. Part (b) is false for every $c_0 > 0$, $\alpha > 0$ and $2\alpha/\sqrt{3} < \beta < 2\alpha$. Part (c) is false for every $c_0 > 0$, $\alpha > 1$ and $2\alpha/\sqrt{3} < \beta < 2\alpha$.*

Table 1 summarises the outcome. Section 6 gives an explicit counterexample to (b) and (c): $\alpha = 2$, $\beta = 3$, any $c_0 > 0$, cities near $(\frac{1}{4}, \frac{1}{2})$ and $(\frac{3}{4}, \frac{1}{2})$, $\delta = \frac{1}{20}$, and $T = \max(10^5, \lceil 100/c_0 \rceil)$, with all constants.

The mechanism. The threshold $\beta = 2\alpha$ governs a small city next to a large one: the sphere of influence of a city of population n near a city of population $N \gg n$ has area $O((n/N)^{2\alpha/\beta})$ [Ald07, Lemma 3], so for $2\alpha/\beta > 1$ the small city falls behind. It does not govern two large cities of comparable size. If only two cities are present, the share z of city 1 evolves like a generalized Pólya urn [HLS80, Pem07] whose urn function $f(z)$ is the area of the sphere of influence of city 1. For two cities at $(\frac{1}{2} \mp a, \frac{1}{2})$ we have $f(\frac{1}{2}) = \frac{1}{2}$ by symmetry and $f'(\frac{1}{2}) = k(a + \frac{1}{12a})$ (Lemma 4.1). The minimum over a of $a + \frac{1}{12a}$ is $1/\sqrt{3}$, so for $1 < k < \sqrt{3}$ there are spacings with $f'(\frac{1}{2}) < 1$; then the share is pushed back towards $\frac{1}{2}$ (Figure 1), and a martingale argument keeps it near $\frac{1}{2}$ forever with positive probability. For $k = \frac{4}{3}$ and $a = \frac{1}{4}$ the function $f(z) - z$ has, numerically, the zeros 0.3191, $\frac{1}{2}$ and 0.6809 in $(0, 1)$, the outer two unstable; so for the two-city dynamics domination by either city and coexistence are all locally stable.

1.4 Credits, novelty, and what remains open

The model, the heuristics and the conjecture are Aldous's [Ald07]; the companion paper [AH12] restricts to $0 < \alpha \leq 1$ and contains heuristics and simulations, the partial results being those of the notes. The tools are standard: Ville's maximal inequality [Vil39], Doob's maximal inequality for submartingales and optional stopping [Wil91], circles of Apollonius, and the trapping of an urn near a stable zero of its drift, for which we claim no novelty. The treatment of small cities in the proof of Theorem D follows the strategy of the notes' sketch of their Theorem 2(c) [Ald07, §2.1], including the comparison with the particle process of [Ald07, Lemma 4]. To our knowledge, and relative to the

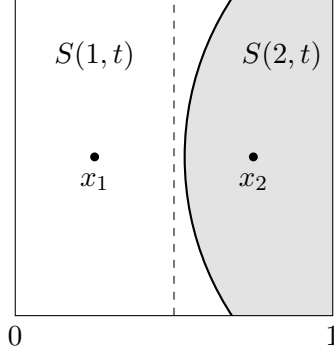


Figure 1: Two cities at $x_1 = (\frac{1}{4}, \frac{1}{2})$ and $x_2 = (\frac{3}{4}, \frac{1}{2})$, with $k = 2\alpha/\beta = \frac{4}{3}$. When city 1 holds the share $z = 0.6$ of their joint population, the boundary of its sphere of influence is the solid arc, part of a circle of Apollonius with centre $(1.447, \frac{1}{2})$ outside the square; $S(2, t)$ is shaded. For $z = \frac{1}{2}$ the boundary is the dashed line. The area of $S(1, t)$ is $f(0.6) \approx 0.581 < 0.6$, so city 1 grows more slowly than its share demands.

literature we found, Theorems A to D are new; no answer to [Ald07, Conjecture 1] was found in the literature accessible to us, and a search that finds nothing is not a proof of novelty (see “Scope and priority” at the end).

Two questions remain open: whether (b) holds for $0 < \beta \leq 2\alpha/\sqrt{3}$, and whether (c) holds for $\alpha > 1$ and $0 < \beta \leq 2\alpha/\sqrt{3}$. For $2\alpha/\beta > \sqrt{3}$ the symmetric two-city equilibrium is unstable for every spacing, so the method of this note says nothing there (Section 8).

Section 2 sets up the process and three lemmas, Section 3 proves Theorems A and B, Section 4 studies the two-city share map, Sections 5 and 7 prove Theorems C and D, Section 6 gives the explicit instance, Section 8 collects remarks, and Section 9 describes the computations, which test the statements but are not used in the proofs.

2 Preliminaries

Construction. Let $U_1, U_2, \dots$ be independent uniform points of Q and $V_2, V_3, \dots$ independent uniform variables on $[0, 1]$, all independent. City 1 is at U_1 . At time $t + 1$ ($t \geq 1$) the immigrant arrives at U_{t+1} ; it founds a city there if $V_{t+1} < 1/(1 + I(U_{t+1}, t))$, where $1/(1 + \infty) := 0$, and otherwise it joins the city i with $U_{t+1} \in S(i, t)$ (the smallest such i in case of a tie). Let $\mathcal{G}_t = \sigma(U_1, \dots, U_t, V_2, \dots, V_t)$. The state at time t (positions and populations) is $\mathcal{G}_t$ -measurable, and (U_{t+1}, V_{t+1}) is independent of $\mathcal{G}_t$, so this is a realization of the model. The conjecture concerns the law of the process, so we may work with this realization; the natural filtration of the process is contained in $(\mathcal{G}_t)$. “Found at $t + 1$ ” and “join city i at $t + 1$ ” refer to the immigrant arriving at time $t + 1$.

Lemma 2.1. (i) *Distinct cities have distinct positions. For two cities $i \neq j$ the set $\{y : N_i(t)^\alpha |y - x_i|^{-\beta} = N_j(t)^\alpha |y - x_j|^{-\beta}\}$ is a line or a circle; hence two spheres of influence meet in a set of area zero.*

(ii) *With $p_t := \mathbf{P}(\text{found at } t + 1 \mid \mathcal{G}_t)$,*

$$p_t = \int_Q \frac{dy}{1 + I(y, t)}, \quad \mathbf{P}(\text{join city } i \text{ at } t + 1 \mid \mathcal{G}_t) = \int_{S(i, t)} \frac{I(y, t)}{1 + I(y, t)} dy.$$

Proof. (i) A city is founded at U_{t+1} only if $I(U_{t+1}, t) < \infty$, that is, not at the position of an existing city. The set in question is $\{|y - x_i| = \sigma |y - x_j|\}$ with $\sigma = (N_i(t)/N_j(t))^{\alpha/\beta}$, a line if $\sigma = 1$ and a

circle of Apollonius otherwise. (ii) Integrate over (U_{t+1}, V_{t+1}) , which is independent of $\mathcal{G}_t$; by (i) the tie-breaking rule concerns a null set. $\square$

The display of the founding probability in [Ald07, §1.3] carries an extra indicator $1(y \in S(i, t))$; formula (ii) is the one implied by the dynamics of [Ald07, §1].

The following form of Ville's maximal inequality [Vil39] is used three times, in the proofs of Theorems A, B and D.

Lemma 2.2. *Let $Y_s \in \{0, 1\}$ be $\mathcal{G}_{s+1}$ -measurable and $b_s \geq 0$ be $\mathcal{G}_s$ -measurable with $\mathbf{P}(Y_s = 1 \mid \mathcal{G}_s) \leq b_s$ ($s \geq 1$). Then for all integers $t_0 \geq 1$ and all $x \geq 0$,*

$$\mathbf{P}\left(\exists t \geq t_0 : \sum_{s=t_0}^t Y_s \geq (e-1) \sum_{s=t_0}^t b_s + x \mid \mathcal{G}_{t_0}\right) \leq e^{-x} \quad \text{almost surely.}$$

In particular $\sup_{t \geq 1} (\sum_{s=1}^t Y_s - (e-1) \sum_{s=1}^t b_s) < \infty$ almost surely.

Proof. Let $q_s = \mathbf{P}(Y_s = 1 \mid \mathcal{G}_s) \leq b_s$. Then $\mathbf{E}[e^{Y_s} \mid \mathcal{G}_s] = 1 + (e-1)q_s \leq \exp((e-1)b_s)$. Put $Z_{t_0-1} = 1$ and $Z_t = Z_{t-1} \exp(Y_t - (e-1)b_t)$ for $t \geq t_0$. Then Z_t is $\mathcal{G}_{t+1}$ -measurable and $\mathbf{E}[Z_t \mid \mathcal{G}_t] \leq Z_{t-1}$, so $(Z_t)_{t \geq t_0-1}$ is a nonnegative supermartingale with respect to $(\mathcal{G}_{t+1})_{t \geq t_0-1}$, starting at 1. The event in question is $\{\exists t \geq t_0 : Z_t \geq e^x\}$. Let θ be the first $t \geq t_0$ with $Z_t \geq e^x$. By optional stopping at the bounded time $\theta \wedge n$ [Wil91], $e^x \mathbf{P}(\theta \leq n \mid \mathcal{G}_{t_0}) \leq \mathbf{E}[Z_{\theta \wedge n} \mid \mathcal{G}_{t_0}] \leq 1$; let $n \rightarrow \infty$. The last sentence follows with $t_0 = 1$ and $x \rightarrow \infty$. $\square$

Lemma 2.3 (packing). *Let $\lambda \geq c_0$ and $r = (c_0/\lambda)^{1/\beta} \leq 1$. Call a city $j \geq 2$, founded at time $s+1$, λ -low if $I(x_j, s) < \lambda$. Then on every path at most $\frac{16}{\pi}(\lambda/c_0)^{2/\beta}$ cities are λ -low.*

Proof. Every city has population at least 1, so $I(y, s) \geq c_0|y - x_i|^{-\beta}$ for every city i existing at time s . If city j is founded at time $s+1$ and $I(x_j, s) < \lambda$, then $|x_j - x_i| > r$ for every city i existing at time s . Cities are never removed, so any two λ -low cities are more than r apart. The open discs of radius $r/2$ around them are disjoint and lie in $[-\frac{r}{2}, 1 + \frac{r}{2}]^2$, whose area is $(1+r)^2 \leq 4$. Hence there are at most $4/(\pi r^2/4) = \frac{16}{\pi}(\lambda/c_0)^{2/\beta}$ of them. $\square$

Lemma 2.4. *For every t , $p_t \leq \kappa N_{(1)}(t)^{-\alpha}$ with $\kappa = 2^{\beta/2}/c_0$, and $N_{(1)}(t) \geq t/M(t)$.*

Proof. Every $y \in Q$ is within distance $\sqrt{2}$ of a largest city, so $I(y, t) \geq c_0 N_{(1)}(t)^{\alpha} 2^{-\beta/2}$ and $1/(1 + I(y, t)) \leq \kappa N_{(1)}(t)^{-\alpha}$. Integrate over Q (Lemma 2.1). The second inequality holds because the $M(t)$ populations sum to t . $\square$

3 Part (a): finitely many cities when $\alpha > 1$

Proof of Theorem A. For integers $m \geq 1$ let $\lambda_m = c_0 2^{m\beta/(\beta+2)} \geq c_0$ and let $Y_s^{(m)}$ be the indicator that a city is founded at time $s+1$ at a point where $I(U_{s+1}, s) \geq \lambda_m$. By Lemma 2.1,

$$\mathbf{P}(Y_s^{(m)} = 1 \mid \mathcal{G}_s) = \int_Q \frac{\mathbf{1}\{I(y, s) \geq \lambda_m\}}{1 + I(y, s)} dy \leq \frac{1}{\lambda_m}.$$

Let $H_m(t) = \sum_{s=1}^{t-1} Y_s^{(m)}$ be the number of such foundings up to time t . Lemma 2.2 with $b_s = 1/\lambda_m$, $t_0 = 1$ and $x = m$ gives $\mathbf{P}(\exists t : H_m(t) \geq (e-1)t/\lambda_m + m) \leq e^{-m}$ (apply it at time $t-1$ and note $(t-1)/\lambda_m \leq t/\lambda_m$). Since $\sum_m e^{-m} < \infty$, the Borel–Cantelli lemma gives an almost surely finite m_0

such that $H_m(t) < (e-1)t/\lambda_m + m$ for all $m \geq m_0$ and all t . Every city other than city 1 is either λ_m -low or counted by H_m , so Lemma 2.3 gives, for $m \geq m_0$ and all t ,

$$M(t) \leq 1 + \frac{16}{\pi} 2^{2m/(\beta+2)} + \frac{(e-1)t}{c_0 2^{m\beta/(\beta+2)}} + m.$$

For $t \geq 2^{m_0}$ take $m = \lfloor \log_2 t \rfloor \geq m_0$, so that $2^m \leq t < 2^{m+1}$. Then $2^{2m/(\beta+2)} \leq t^{2/(\beta+2)}$ and $t 2^{-m\beta/(\beta+2)} < 2 \cdot 2^{m-m\beta/(\beta+2)} = 2 \cdot 2^{2m/(\beta+2)} \leq 2t^{2/(\beta+2)}$, hence $M(t) \leq 1 + \log_2 t + (\frac{16}{\pi} + \frac{2(e-1)}{c_0})t^{2/(\beta+2)}$. For $t < 2^{m_0}$ we have $M(t) \leq t < 2^{m_0}$. $\square$

Proof of Theorem B. Apply Lemma 2.2 to $Y_s = \mathbf{1}\{\text{found at } s+1\}$ and $b_s = p_s$. Since $M(t) - 1 = \sum_{s=1}^{t-1} Y_s$, the random variable $K = \max(0, \sup_t (M(t) - 1 - (e-1) \sum_{s=1}^{t-1} p_s))$ is finite almost surely, and

$$M(t) \leq 1 + K + (e-1) \sum_{s=1}^{t-1} p_s \quad \text{for all } t \geq 1. \quad (1)$$

Step. Suppose that, almost surely, $M(t) \leq C t^{1-g}$ for all $t \geq 1$, where $g \in (0, 1)$ is deterministic and C is an almost surely finite random variable. By Lemma 2.4, $N_{(1)}(t) \geq t^g/C$ and $p_t \leq \kappa C^\alpha t^{-\alpha g}$. If $\alpha g < 1$, then $\sum_{s=1}^{t-1} s^{-\alpha g} \leq 1 + \int_1^t x^{-\alpha g} dx \leq t^{1-\alpha g}/(1-\alpha g)$, and (1) gives $M(t) \leq (1 + K + (e-1)\kappa C^\alpha/(1-\alpha g))t^{1-\alpha g}$ for all $t \geq 1$: the hypothesis holds with αg in place of g . If $\alpha g > 1$, then $\sum_t p_t \leq \kappa C^\alpha \sum_t t^{-\alpha g} < \infty$, and (1) gives $M(\infty) < \infty$.

Iteration. Let $g_0 = \beta/(\beta+2) \in (0, 1)$, and choose $g \in (0, g_0]$ with $\alpha^n g \neq 1$ for all integers $n \geq 1$ (all but countably many g qualify). By Theorem A, $M(t) \leq C_0 t^{1-g_0} \leq C_0 t^{1-g}$ with $C_0 = \sup_t M(t) t^{-2/(\beta+2)} < \infty$. Since $\alpha > 1$, the integer $n^* = \min\{n \geq 0 : \alpha^{n+1} g > 1\}$ exists, and $\alpha^{n+1} g < 1$ for $n < n^*$. Applying the first case of the Step n^* times and then the second case once gives $\sum_t p_t < \infty$ and $M(\infty) < \infty$ almost surely (each application holds outside a null set, and there are finitely many). Finally $N_{(1)}(t) \geq t/M(t) \geq t/M(\infty)$ by Lemma 2.4. $\square$

Remark 3.1. (i) The hypothesis $\alpha > 1$ is used only to make $g \mapsto \alpha g$ increase. For $\alpha \leq 1$ the Step gives nothing beyond Theorem A, consistent with the heuristic expectation that $M(\infty) = \infty$ there.

(ii) By Lévy's extension of the Borel–Cantelli lemma [Wil91], $\{M(\infty) < \infty\} = \{\sum_t p_t < \infty\}$ up to a null set, so Theorem B is the integrated form of the criterion in Remark (ii) of [Ald07, Theorem 2]. It answers the question asked there, whether $N_{(1)}(t)/t^{1-\varepsilon} \rightarrow \infty$ almost surely, positively for $\alpha > 1$ and every $\beta > 0$; for $\alpha \leq 1$ the question stays open. The proof needs no lower bound on $N_{(1)}$ beyond $t/M(t)$: the packing bound gives a first polynomial gap, and $\alpha > 1$ amplifies it.

(iii) The exponent of Theorem A does not depend on α . The heuristic growth exponent of the notes for the balanced case, $\theta = (1-\alpha)/(1-\alpha+\beta/2)$ for $0 < \alpha < 1$ [Ald07, §1], satisfies $\theta < 2/(\beta+2)$ for every $0 < \alpha < 1$ (the inequality is equivalent to $(1-\alpha)\beta/2 < \beta/2$), and $\theta \rightarrow 2/(\beta+2)$ as $\alpha \rightarrow 0$. So Theorem A agrees with the heuristics, and in the limit $\alpha \rightarrow 0$ its exponent is the heuristic one.

4 The two-city share map

For distinct points $x = (x_1, x_2)$ of Q and $\sigma > 0$ let

$$A(x; \sigma) = \text{area}\{y \in Q : |y - x_1| \leq \sigma |y - x_2|\},$$

$$f_x(z) = A(x; s(z)), \quad s(z) = \left(\frac{z}{1-z}\right)^{\alpha/\beta} = \left(\frac{z}{1-z}\right)^{k/2},$$

for $0 < z < 1$, where $k = 2\alpha/\beta$. The function $A(x; \sigma)$ is nondecreasing in σ . If at time t the only cities are city 1 at x_1 and city 2 at x_2 , with N_1 and N_2 inhabitants, then city 1 has at least the influence of city 2 at y if and only if $N_1^\alpha |y - x_1|^{-\beta} \geq N_2^\alpha |y - x_2|^{-\beta}$, that is, $|y - x_1| \leq (N_1/N_2)^{\alpha/\beta} |y - x_2|$. So

$$\text{area } S(1, t) = f_x(z), \quad z = \frac{N_1}{N_1 + N_2}, \quad (2)$$

and $\text{area } S(2, t) = 1 - f_x(z)$ (Lemma 2.1).

4.1 The symmetric configuration

Fix $a \in (0, \frac{1}{2})$ and let $x^0 = ((\frac{1}{2} - a, \frac{1}{2}), (\frac{1}{2} + a, \frac{1}{2}))$. Put

$$q_a(h) = \frac{k(a + \frac{1}{12a})}{1 - \frac{(a^2 + 1/4)(2kh)^2}{2a^2}}, \quad h_a = \frac{a}{k(4a^2 + 1)}.$$

Lemma 4.1. (i) $f_{x^0}(\frac{1}{2} - h) = 1 - f_{x^0}(\frac{1}{2} + h)$ for $0 \leq h < \frac{1}{2}$.

(ii) Let $k \geq 1$. For $0 < h < h_a$ the denominator of $q_a(h)$ exceeds $\frac{1}{2}$, q_a is increasing on $(0, h_a)$, and $f_{x^0}(\frac{1}{2} + h) - \frac{1}{2} \leq h q_a(h)$.

(iii) For every $k > 0$, $(f_{x^0}(\frac{1}{2} + h) - \frac{1}{2})/h \rightarrow k(a + \frac{1}{12a})$ as $h \downarrow 0$. Hence f_{x^0} is differentiable at $\frac{1}{2}$ with $f'_{x^0}(\frac{1}{2}) = k(a + \frac{1}{12a})$.

Proof. Shift coordinates by $(\frac{1}{2}, \frac{1}{2})$: Q becomes $[-\frac{1}{2}, \frac{1}{2}]^2$, $x_1 = (-a, 0)$ and $x_2 = (a, 0)$.

(i) The reflection $R(y_1, y_2) = (-y_1, y_2)$ preserves Q and area and exchanges x_1 and x_2 , and $s(1 - z) = 1/s(z)$. Hence $f(1 - z) = \text{area}\{s(z)|y - x_1| \leq |y - x_2|\} = \text{area}\{s(z)|y - x_2| \leq |y - x_1|\} = 1 - f(z)$, the set $\{|y - x_1| = s(z)|y - x_2|\}$ being a null set.

(ii) Let $z = \frac{1}{2} + h$, $\sigma = s(z) > 1$ and $c = (\sigma^2 + 1)/(\sigma^2 - 1) > 1$, so that $1/c = \tanh(\log \sigma)$. Since $z/(1 - z) = (1 + 2h)/(1 - 2h)$, we have $\log \sigma = \frac{k}{2} \log \frac{1+2h}{1-2h} = k \operatorname{artanh}(2h)$. The function $\tanh$ is concave on $[0, \infty)$ with $\tanh 0 = 0$, so $\tanh(kx) \leq k \tanh x$ for $x \geq 0$ and $k \geq 1$, and therefore

$$1/c = \tanh(k \operatorname{artanh}(2h)) \leq 2kh. \quad (3)$$

Expanding, $|y - x_1|^2 - \sigma^2 |y - x_2|^2 = (1 - \sigma^2)(|y|^2 + a^2) + 2a(1 + \sigma^2)y_1$, which is positive if and only if $|y|^2 + a^2 < 2acy_1$, that is, $(y_1 - ac)^2 + y_2^2 < a^2(c^2 - 1)$. So the complement of $S_1 := \{y \in Q : |y - x_1| \leq \sigma |y - x_2|\}$ in Q is the intersection of Q with the open disc of centre $(ac, 0)$ and radius $a\sqrt{c^2 - 1}$, a disc of Apollonius. For $h < h_a$, (3) gives $1/c < 2kh_a = 2a/(4a^2 + 1)$, that is,

$$c > \frac{4a^2 + 1}{2a} = \frac{2(a^2 + \frac{1}{4})}{a};$$

we call this inequality condition (V). Condition (V) implies $c^2 > (4a^2 + 1)^2/(4a^2) \geq 1 + 1/(4a^2)$ and $ac > 2a^2 + \frac{1}{2} > \frac{1}{2}$. For $|u| \leq \frac{1}{2}$ put $w(u) = \sqrt{a^2 c^2 - a^2 - u^2}$, which is real since $a^2(c^2 - 1) > \frac{1}{4}$, and

$$\psi(u) = ac - w(u) = \frac{a^2 + u^2}{ac + w(u)}.$$

The section of the disc at height u is the interval $(\psi(u), ac + w(u))$. Its right end is at least $ac > \frac{1}{2}$, and its left end satisfies $0 < \psi(u) \leq (a^2 + \frac{1}{4})/(ac) < \frac{1}{2}$ by (V). Hence the section of $Q \setminus S_1$ at height u is $(\psi(u), \frac{1}{2}]$, and

$$f_{x^0}(\frac{1}{2} + h) = 1 - \int_{-1/2}^{1/2} (\frac{1}{2} - \psi(u)) du = \frac{1}{2} + \int_{-1/2}^{1/2} \psi(u) du. \quad (4)$$

For $0 \leq B \leq A^2$ and $A > 0$ we have $\sqrt{A^2 - B} \geq A - B/A$ (square both sides when the right side is nonnegative). With $A = ac$ and $B = a^2 + u^2$ this gives $w(u) \geq ac - (a^2 + u^2)/(ac)$, hence

$$\psi(u) \leq \frac{a^2 + u^2}{2ac - (a^2 + \frac{1}{4})/(ac)} = \frac{a^2 + u^2}{2ac(1 - \frac{a^2 + 1/4}{2a^2}c^{-2})}.$$

Let $K_a = (a^2 + \frac{1}{4})/(2a^2)$. By (V), $K_a c^{-2} < K_a \cdot 4a^2/(4a^2 + 1)^2 = 1/(2(4a^2 + 1)) < \frac{1}{2}$, so the denominator is positive. Integrating, with $\int_{-1/2}^{1/2} (a^2 + u^2) du = a^2 + \frac{1}{12}$,

$$\int_{-1/2}^{1/2} \psi(u) du \leq \frac{a^2 + \frac{1}{12}}{2a} \cdot \frac{v}{1 - K_a v^2}, \quad v = 1/c.$$

For $0 < h < h_a$ we have $K_a(2kh)^2 < K_a(2kh_a)^2 = 1/(2(4a^2 + 1)) < \frac{1}{2}$; so the denominator of $q_a(h) = k(a + \frac{1}{12a})/(1 - K_a(2kh)^2)$ exceeds $\frac{1}{2}$, q_a is increasing on $(0, h_a)$, and the map $v \mapsto v/(1 - K_a v^2)$ is increasing on $[0, 2kh]$. By (3) we may replace v by $2kh$, which gives $\int \psi \leq h q_a(h)$. With (4) this proves (ii).

(iii) As $h \downarrow 0$ we have $c \rightarrow \infty$, so (V) holds for small h and the computation above applies for every $k > 0$ (only the step (3) used $k \geq 1$). From $\psi(u) \geq (a^2 + u^2)/(2ac)$ and the upper bound, $\int \psi = \frac{a^2 + 1/12}{2a} c^{-1} (1 + O(c^{-2}))$. Moreover $1/c = \tanh(k \operatorname{artanh}(2h)) = 2kh + O(h^3)$. Hence $(f_{x^0}(\frac{1}{2} + h) - \frac{1}{2})/h \rightarrow k(a^2 + \frac{1}{12})/a = k(a + \frac{1}{12a})$; by (i) the same limit holds from the left. $\square$

Remark 4.2 (the threshold $\sqrt{3}$). The minimum of $a + \frac{1}{12a}$ over $a > 0$ is $1/\sqrt{3}$, attained at $a = 1/(2\sqrt{3}) \in (0, \frac{1}{2})$. Hence some spacing gives $f'_{x^0}(\frac{1}{2}) < 1$ if and only if $k < \sqrt{3}$; for $k > \sqrt{3}$ the symmetric equilibrium is unstable for every a , and for $k = \sqrt{3}$ it is degenerate at $a = 1/(2\sqrt{3})$. For $a = \frac{1}{4}$ the slope is $7k/12$, less than 1 if and only if $k < 12/7$.

4.2 Nearby positions

Lemma 4.3. *Let $0 < \rho < r$ and $\omega = (1 + \rho/r)/(1 - \rho/r)$. If $|x_i - x_i^0| \leq \rho$ for $i = 1, 2$, then for every $\sigma > 0$*

$$A(x^0; \sigma/\omega) - 2\pi r^2 \leq A(x; \sigma) \leq A(x^0; \sigma\omega) + 2\pi r^2.$$

Proof. Let $B = D(x_1^0, r) \cup D(x_2^0, r)$, of area at most $2\pi r^2$. For $y \notin B$ and $i = 1, 2$ we have $(1 - \rho/r)|y - x_i^0| \leq |y - x_i| \leq (1 + \rho/r)|y - x_i^0|$. If $y \in Q \setminus B$ and $|y - x_1^0| \leq (\sigma/\omega)|y - x_2^0|$, then

$$|y - x_1| \leq (1 + \rho/r) \frac{\sigma}{\omega} |y - x_2^0| \leq (1 + \rho/r) \frac{\sigma}{\omega} \cdot \frac{|y - x_2|}{1 - \rho/r} = \sigma |y - x_2|.$$

This proves the left inequality; the right one is the same argument with the roles of x and x^0 exchanged. $\square$

Corollary 4.4 (strip estimates). *Let $1 \leq k < \sqrt{3}$ and fix $a \in (0, \frac{1}{2})$ with $q_0 := k(a + \frac{1}{12a}) < 1$. Choose $\delta \in (0, h_a/2)$ with $q_a(2\delta) < 1$ (possible since $q_a(h) \rightarrow q_0$ as $h \downarrow 0$), then $r > 0$ and $\rho \in (0, \min(r, \frac{1}{2} - a))$ so small that $\varepsilon := (\log \omega)/(2k) \leq \delta$ and*

$$\eta := \frac{\delta}{2} (1 - q_a(\delta + \varepsilon)) - \varepsilon - 2\pi r^2 > 0.$$

Let $G = D(x_1^0, \rho) \times D(x_2^0, \rho)$. Then for every $x \in G$

$$f_x(z) - z \leq -\eta \quad \text{for } z \in [\tfrac{1}{2} + \tfrac{\delta}{2}, \tfrac{1}{2} + \delta], \quad f_x(z) - z \geq \eta \quad \text{for } z \in [\tfrac{1}{2} - \delta, \tfrac{1}{2} - \tfrac{\delta}{2}].$$

Such a choice exists: take r with $2\pi r^2 < \frac{\delta}{4}(1 - q_a(2\delta))$, then ρ/r so small that $\varepsilon \leq \delta$ and $\varepsilon < \frac{\delta}{4}(1 - q_a(2\delta))$; since q_a is increasing and $\delta + \varepsilon \leq 2\delta < h_a$, this gives $\eta > 0$. Only $q_a(\delta + \varepsilon) < 1$ and $\eta > 0$ are used below; $q_a(2\delta) < 1$ is a convenient sufficient condition. The condition $\rho < \frac{1}{2} - a$ puts the discs $D(x_i^0, \rho)$ inside Q .

Proof. Let $z = \frac{1}{2} + h$ with $h \in [\frac{\delta}{2}, \delta]$, and define h' by $\operatorname{artanh}(2h') = \operatorname{artanh}(2h) + (\log \omega)/k$. Then $\log(s(z)\omega) = k \operatorname{artanh}(2h')$, that is, $s(z)\omega = s(\frac{1}{2} + h')$; and $h \leq h' \leq h + \varepsilon \leq 2\delta < h_a$, because $\tanh$ is 1-Lipschitz. Put $q = q_a(\delta + \varepsilon) < 1$. By Lemma 4.3, the monotonicity of A in σ , and Lemma 4.1(ii),

$$f_x(z) \leq f_{x^0}(\frac{1}{2} + h') + 2\pi r^2 \leq \frac{1}{2} + h'q + 2\pi r^2,$$

so $f_x(z) - z \leq (h + \varepsilon)q - h + 2\pi r^2 \leq -\frac{\delta}{2}(1 - q) + \varepsilon + 2\pi r^2 = -\eta$. For $z = \frac{1}{2} - h$ with the same h and h' we have $s(z)/\omega = s(\frac{1}{2} - h')$, and Lemma 4.3 with Lemma 4.1(i),(ii) gives $f_x(z) \geq f_{x^0}(\frac{1}{2} - h') - 2\pi r^2 = 1 - f_{x^0}(\frac{1}{2} + h') - 2\pi r^2 \geq \frac{1}{2} - h'q - 2\pi r^2$, so $f_x(z) - z \geq h(1 - q) - \varepsilon q - 2\pi r^2 \geq \eta$. $\square$

5 Proof of Theorem C

Let $c_0 > 0$, $\alpha > 1$ and $k = 2\alpha/\beta \in (1, \sqrt{3})$. Choose a with $k(a + \frac{1}{12a}) < 1$ (Remark 4.2) and δ, r, ρ, η, G as in Corollary 4.4. Since $h_a = a/(k(4a^2 + 1)) \leq 1/(4k) < \frac{1}{4}$, we have $\delta < h_a/2 < \frac{1}{8}$. Choose an integer $T \geq 2$ so large that

$$(T1) \quad \kappa(T/2)^{-\alpha} \leq \eta, \quad (T2) \quad \frac{64}{T\delta^2} + \kappa 2^\alpha \sum_{s \geq T} s^{-\alpha} \leq \frac{1}{2}, \quad (T3) \quad \frac{1}{T} \leq \frac{\delta}{4}; \quad (5)$$

this is possible because $\alpha > 1$.

The event E_T . Let E_T be the event that $U_1 \in D(x_1^0, \rho)$, that the immigrant of time 2 founds a city at a point of $D(x_2^0, \rho)$, and that the immigrants of times $3, \dots, T$ join city 2 and city 1 alternately, beginning with city 2. On E_T there are exactly two cities at time T , their positions $x = (x_1, x_2)$ lie in G , and $|N_1(T) - N_2(T)| \leq 1$, so that $|N_1(T)/T - \frac{1}{2}| \leq \frac{1}{2T} \leq \frac{\delta}{8}$. We have $\mathbf{P}(E_T) > 0$. Indeed, the first two requirements have probability $\int_{D(x_1^0, \rho)} \int_{D(x_2^0, \rho)} (1 + c_0|y - x|^{-\beta})^{-1} dy dx > 0$. Given $\mathcal{G}_t$, on the event that there are exactly two cities $i \neq j$, the conditional probability of joining city i at time $t + 1$ is $\int_{S(i, t)} I/(1 + I) dy > 0$, because $S(i, t)$ contains $Q \cap D(x_i, \sigma d/(1 + \sigma))$, where $d = |x_i - x_j|$ and $\sigma = (N_i(t)/N_j(t))^{\alpha/\beta}$: if $|y - x_i| \leq \sigma d/(1 + \sigma)$, then $|y - x_j| \geq d/(1 + \sigma) \geq |y - x_i|/\sigma$. By the tower property, $\mathbf{P}(E_T)$ is the expectation, over an event of positive probability, of a product of $T - 2$ positive conditional probabilities. Note that E_T is an event of the process started from its own initial condition.

No founding after T . Let $\tau = \inf\{t \geq T : \text{found at } t + 1\}$ ($\inf \emptyset = \infty$); then $\{\tau \geq t\} \in \mathcal{G}_t$. On $E_T \cap \{\tau \geq t\}$ there are exactly two cities at time t and $N_{(1)}(t) \geq t/2$, so $p_t \leq \kappa(t/2)^{-\alpha}$ by Lemma 2.4. Hence, on E_T ,

$$\mathbf{P}(\tau < \infty \mid \mathcal{G}_T) = \sum_{t \geq T} \mathbf{E}[\mathbf{1}\{\tau \geq t\} p_t \mid \mathcal{G}_T] \leq \kappa 2^\alpha \sum_{t \geq T} t^{-\alpha}. \quad (6)$$

The share. On $E_T \cap \{\tau \geq t\}$ let $z_t = N_1(t)/t$; there $N_1(t) + N_2(t) = t$. Let ξ_{t+1} be the indicator that the immigrant of time $t + 1$ joins city 1, and for $t \geq T$ let $D_t = \mathbf{1}_{E_T} \mathbf{1}\{\tau \geq t\} (\xi_{t+1} - z_t)/(t + 1)$. If $\tau \geq t + 1$, then $N_1(t + 1) = N_1(t) + \xi_{t+1}$ and $z_{t+1} = z_t + D_t$; so on $E_T \cap \{\tau = \infty\}$ we have $z_{t+1} = z_t + D_t$ for all $t \geq T$. Always $|D_t| \leq 1/(t + 1)$. By (2) and Lemma 2.1, on $E_T \cap \{\tau \geq t\}$

$$\mathbf{E}[D_t \mid \mathcal{G}_t] = \frac{f_x(z_t) - e_t - z_t}{t + 1}, \quad 0 \leq e_t := \int_{S(1, t)} \frac{dy}{1 + I(y, t)} \leq p_t \leq \kappa(t/2)^{-\alpha} \leq \eta, \quad (7)$$

the last inequality by (T1).

The martingale. Let $m_t = D_t - \mathbf{E}[D_t \mid \mathcal{G}_t]$ and $\Lambda_t = \sum_{s=T}^{t-1} m_s$ for $t \geq T$. Then $(\Lambda_t)_{t \geq T}$ is a martingale with respect to $(\mathcal{G}_t)_{t \geq T}$, $\Lambda_T = 0$, and $\mathbf{E}[m_t^2 \mid \mathcal{G}_t] \leq \text{Var}(\xi_{t+1} \mid \mathcal{G}_t)/(t+1)^2 \leq 1/(4(t+1)^2)$. Doob's maximal inequality [Wil91] for the nonnegative submartingale $(\Lambda_t^2)_{t \geq T}$ (Kolmogorov's inequality), $\mathbf{P}(\max_{T \leq t \leq n} |\Lambda_t| \geq c \mid \mathcal{G}_T) \leq c^{-2} \mathbf{E}[\Lambda_n^2 \mid \mathcal{G}_T] = c^{-2} \sum_{t=T}^{n-1} \mathbf{E}[m_t^2 \mid \mathcal{G}_T]$, applied with $c = \delta/16$ and then with $n \rightarrow \infty$, gives

$$\mathbf{P}\left(\exists t \geq T : |\Lambda_t| \geq \frac{\delta}{16} \mid \mathcal{G}_T\right) \leq \left(\frac{16}{\delta}\right)^2 \sum_{t \geq T} \frac{1}{4(t+1)^2} \leq \frac{64}{T\delta^2}. \quad (8)$$

No exit. Let $A = E_T \cap \{\tau = \infty\} \cap \{|\Lambda_t| < \delta/16 \text{ for all } t \geq T\}$. We claim that $|z_t - \frac{1}{2}| \leq \delta$ for all $t \geq T$ on A . If not, let $\nu = \min\{t \geq T : |z_t - \frac{1}{2}| > \delta\}$; then $\nu > T$. Suppose $z_\nu > \frac{1}{2} + \delta$, and let $\sigma' = \max\{s \in [T, \nu) : z_s \leq \frac{1}{2} + \frac{\delta}{2}\}$, which exists because $z_T \leq \frac{1}{2} + \frac{\delta}{8}$. For $\sigma' < s < \nu$ we have $z_s \in (\frac{1}{2} + \frac{\delta}{2}, \frac{1}{2} + \delta]$, so $\mathbf{E}[D_s \mid \mathcal{G}_s] \leq (f_x(z_s) - z_s)/(s+1) < 0$ by (7) and Corollary 4.4. Therefore

$$\frac{\delta}{2} < z_\nu - z_{\sigma'} = D_{\sigma'} + \sum_{s=\sigma'+1}^{\nu-1} (\mathbf{E}[D_s \mid \mathcal{G}_s] + m_s) \leq \frac{1}{\sigma'+1} + \Lambda_\nu - \Lambda_{\sigma'+1} < \frac{1}{T} + \frac{\delta}{8} \leq \frac{3\delta}{8},$$

a contradiction. If $z_\nu < \frac{1}{2} - \delta$, the same argument with $\sigma' = \max\{s \in [T, \nu) : z_s \geq \frac{1}{2} - \frac{\delta}{2}\}$ uses $\mathbf{E}[D_s \mid \mathcal{G}_s] \geq (\eta - e_s)/(s+1) \geq 0$ on the lower strip (Corollary 4.4 and (7)) and gives $z_{\sigma'} - z_\nu < \frac{3\delta}{8} < \frac{\delta}{2}$, again a contradiction.

Conclusion. By (6), (8) and (T2), $\mathbf{P}(A \mid \mathcal{G}_T) \geq \frac{1}{2}$ on E_T , so $\mathbf{P}(A) \geq \frac{1}{2} \mathbf{P}(E_T) > 0$. On A , for all $t \geq T$, $M(t) = 2$ and $|N_1(t)/t - \frac{1}{2}| \leq \delta$; hence $N_{(1)}(t)/t \leq \frac{1}{2} + \delta < 1$ and $N_{(\geq 2)}(t) \geq (\frac{1}{2} - \delta)t \rightarrow \infty$. The parameters satisfy $\alpha > 1$ and $\beta < 2\alpha$, the hypotheses of (b) and (c). $\square$

6 An explicit instance

Let $\alpha = 2$ and $\beta = 3$, so that $k = \frac{4}{3} \in (1, \sqrt{3})$, and let $c_0 > 0$ be arbitrary. Take $a = \frac{1}{4}$; the cities of Theorem C are then near $(\frac{1}{4}, \frac{1}{2})$ and $(\frac{3}{4}, \frac{1}{2})$. Table 2 lists the constants of Lemma 4.1, Corollary 4.4 and (5).

Quantity	Definition	Value
q_0	$k(a + 1/(12a))$	7/9
h_a	$a/(k(4a^2 + 1))$	3/20
$q_a(h)$	Lemma 4.1	$(7/9)/(1 - (160/9)h^2)$
δ	$\delta < h_a/2 = 3/40$	1/20; $q_a(2\delta) = 35/37$
r, ρ	$\rho < \min(r, 1/2 - a)$	1/100, 10^{-5}
ω	$(1 + \rho/r)/(1 - \rho/r)$	1001/999
ε	$(\log \omega)/(2k) = (3/4) \text{artanh}(10^{-3})$	$\in [0.00075000025, 0.00075000026]$
$q_a(\delta + \varepsilon)$		< 0.81510
η	$(\delta/2)(1 - q_a(\delta + \varepsilon)) - \varepsilon - 2\pi r^2$	> 0.0032441
κ	$2^{\beta/2}/c_0$	$2^{3/2}/c_0$
T	(T1)–(T3)	$\max(10^5, \lceil 100/c_0 \rceil)$

Table 2: Constants of the explicit instance $\alpha = 2$, $\beta = 3$, $a = \frac{1}{4}$.

Each entry is an elementary statement about explicit numbers. The value of ε follows from $x + x^3/3 \leq \text{artanh } x \leq x + x^3/(3(1 - x^2))$ for $0 < x < 1$, and the bound on η from it and $\pi < \frac{355}{113}$.

For (T1)–(T3) we use $\sum_{s \geq T} s^{-2} \leq 1/(T-1)$ and $2^{3/2} < \frac{99}{35}$. If $c_0 \geq 1$, then $T = 10^5$ and $\kappa < \frac{99}{35}$, so $\kappa(T/2)^{-2} < 1.2 \cdot 10^{-9}$ and $64/(T\delta^2) + 4\kappa/(T-1) < 0.2562$. If $0 < c_0 < 1$, then $T \geq 10^5$ and $c_0 T \geq 100$, so $\kappa(T/2)^{-2} = 4 \cdot 2^{3/2}/(c_0 T^2) \leq 4 \cdot 2^{3/2} c_0 / 10^4 < 0.0012$ and $64/(T\delta^2) + 4\kappa/(T-1) \leq 0.256 + 4 \cdot 2^{3/2}/(c_0 T - c_0) < 0.256 + 4 \cdot 2^{3/2}/99 < 0.371$. In both cases (T1) and (T2) hold, and (T3) holds because $T \geq 80$. The program `check_note.py` of Section 9 checks every entry of Table 2 and these inequalities in exact rational arithmetic.

Hence, for $\alpha = 2$, $\beta = 3$ and every $c_0 > 0$: with positive probability, from time $T = \max(10^5, \lceil 100/c_0 \rceil)$ on there are exactly two cities, near $(\frac{1}{4}, \frac{1}{2})$ and $(\frac{3}{4}, \frac{1}{2})$, and each has between $\frac{9}{20}t$ and $\frac{11}{20}t$ inhabitants at every time $t \geq T$. So $\mathbf{P}(N_{(1)}(t)/t \rightarrow 1) < 1$ and $\mathbf{P}(N_{(\geq 2)}(\infty) = \infty) > 0$, while $M(\infty) < \infty$ almost surely by Theorem B.

7 Proof of Theorem D

For $\alpha \leq 1$ new cities may keep being founded (the heuristics of the notes predict that they do), so the two-city picture must be protected against them. The argument for small cities follows the strategy of the notes' sketch of their Theorem 2(c) [Ald07, §2.1]: the bound below for a small city to reach j_0 inhabitants is a discrete form of [Ald07, Lemma 4], with the same condition on j_0 (their γ is our k). For a city i write $I_i(y) = N_i(t)^\alpha |y - x_i|^{-\beta}$.

Lemma 7.1 (a small city next to a large one). *Let A and B be cities with $R := (N_A/N_B)^{\alpha/\beta} \geq 2$. Then $\text{area}\{y \in Q : I_B(y) \geq I_A(y)\} \leq 8\pi(N_B/N_A)^{2\alpha/\beta}$.*

Proof. Let $d = |x_A - x_B| \leq \sqrt{2}$. If $I_B(y) \geq I_A(y)$ and $s = |y - x_B|$, then $Rs \leq |y - x_A| \leq s + d$, so $s \leq d/(R-1) \leq 2d/R$. The set lies in a disc of area $4\pi d^2/R^2 \leq 8\pi R^{-2}$. $\square$

This is [Ald07, Lemma 3(b)] with a different constant and hypothesis.

Proof of Theorem D. Let $k = 2\alpha/\beta \in (1, \sqrt{3})$ and choose $a, \delta, r, \rho, \eta, G$ as in the proof of Theorem C; again $\delta < \frac{1}{8}$. Fix an integer $j_0 \geq 2$ with $\alpha + (k-1)(j_0-1) > 1$, put $C_1 = \prod_{i=1}^{j_0-1} 8\pi(6i)^k/(k-1)$ and $x_0 = \log 8$. For integers $2 \leq T \leq t$ let

$$B_T(t) = (e-1)\kappa 4^\alpha \sum_{s=T}^{t-1} s^{-\alpha} + x_0 \leq \bar{B}(t) := (e-1)\kappa 4^\alpha \sum_{s=1}^{t-1} s^{-\alpha} + x_0.$$

The majorant $\bar{B}(t)$ does not depend on T , and it is $O(t^{1-\alpha})$, $O(\log t)$ or $O(1)$ for $\alpha < 1$, $\alpha = 1$, $\alpha > 1$; since $k > 1$, $\bar{B}(t)/t \rightarrow 0$ and $\bar{B}(t)t^{-k} \rightarrow 0$. Choose T so large that, for all $t \geq T$,

$$(D1) \ j_0 \bar{B}(t) \leq t/4, \quad (D2) \ t \geq 6j_0 2^{\beta/\alpha}, \quad (D3) \ 8\pi(6j_0)^k \bar{B}(t) t^{-k} + \kappa(t/4)^{-\alpha} \leq \eta,$$

and moreover

$$(D4) \ \kappa 4^\alpha C_1 \sum_{s \geq T} s^{-\alpha-(k-1)(j_0-1)} \leq \frac{1}{8}, \quad (D5) \ \frac{512}{\delta^2(T-1)} \leq \frac{1}{8}, \quad (D6) \ \frac{4}{3T} \leq \frac{\delta}{4}.$$

Because $\bar{B}$ does not depend on T , these are conditions on T alone, and each holds for all large T ; so the choice of T is not circular. Write $B(t) = B_T(t)$.

Let E_T be the event of the proof of Theorem C; $\mathbf{P}(E_T) > 0$ by the same argument. Let $n_t = N_1(t) + N_2(t)$ and $z_t = N_1(t)/n_t$, and call the cities $j \geq 3$ *small*. Define the stopping times (all $\geq T$, $\inf \emptyset = \infty$)

$$\begin{aligned} \tau_z &= \inf\{t : |z_t - \tfrac{1}{2}| > \delta\}, & \tau_M &= \inf\{t : M(t) - 2 > B(t)\}, \\ \tau_j &= \inf\{t : \text{a small city has } j_0 \text{ inhabitants}\}, \end{aligned}$$

and $\tau^* = \min(\tau_z, \tau_j, \tau_M)$. On E_T we have $\tau^* > T$: at time T there is no small city, $|z_T - \frac{1}{2}| \leq \frac{\delta}{8}$, and $M(T) - 2 = 0 < x_0 = B(T)$. All events below are intersected with $E_T \in \mathcal{G}_T$, and all conditional probabilities given $\mathcal{G}_T$ are bounded on E_T .

Bounds before τ^ .* Let $T \leq t < \tau^*$, on E_T . There are at most $B(t)$ small cities, each with fewer than j_0 inhabitants, so $n_t \geq t - j_0 B(t) \geq \frac{3}{4}t$ by (D1). Since $\delta < \frac{1}{4}$, $N_i(t) \geq (\frac{1}{2} - \delta)n_t \geq t/6$ for $i = 1, 2$, and $N_{(1)}(t) \geq n_t/2 \geq t/4$, so $p_t \leq \kappa(t/4)^{-\alpha}$ by Lemma 2.4. For $i \in \{1, 2\}$ and a small city B , (D2) gives $N_i(t)/N_B(t) \geq t/(6j_0) \geq 2^{\beta/\alpha}$, so by Lemma 7.1

$$\text{area}\{y : I_B(y) \geq I_i(y)\} \leq 8\pi(6N_B(t)/t)^k. \quad (9)$$

Too many foundings. Apply Lemma 2.2 with $t_0 = T$, $b_s = \kappa(s/4)^{-\alpha}$, $Y_s = 0$ for $s < T$ and $Y_s = \mathbf{1}_{E_T} \mathbf{1}\{s < \tau^*, \text{ found at } s+1\}$ for $s \geq T$. Then Y_s is $\mathcal{G}_{s+1}$ -measurable, since $E_T \in \mathcal{G}_T$ and $\{s < \tau^*\} \in \mathcal{G}_s$, and $\mathbf{P}(Y_s = 1 \mid \mathcal{G}_s) \leq b_s$ for every $s \geq 1$, by the bound on p_s before τ^* . If $\tau^* = \tau_M < \infty$, then every founding at the times $T+1, \dots, \tau_M$ happens at a time $s+1$ with $s < \tau^*$, so $\sum_{s=T}^{\tau_M-1} Y_s = M(\tau_M) - 2 > B(\tau_M) = (e-1) \sum_{s=T}^{\tau_M-1} b_s + x_0$. Hence $\mathbf{P}(\tau^* = \tau_M < \infty \mid \mathcal{G}_T) \leq e^{-x_0} = \frac{1}{8}$.

A small city reaches j_0 . Consider a small city founded at time $t_1 + 1$ with $T \leq t_1 < \tau^*$, and let θ_i be the time at which it reaches i inhabitants ($\theta_1 = t_1 + 1$; $\theta_i = \infty$ if this never happens); these are stopping times. If $\theta_{i+1} \leq \tau^*$, the city grew from i to $i+1$ inhabitants at a time $s+1$ with $\theta_i \leq s < \tau^*$; given $\mathcal{G}_s$, on $\{s < \tau^*\}$, this has probability at most the area of its sphere of influence, which is at most $8\pi(6i/s)^k$ by (9), applied with city 1 and $N_B(s) = i$. Hence $\mathbf{P}(\theta_{i+1} \leq \tau^* \mid \mathcal{G}_{\theta_i}) \leq \sum_{s>t_1} 8\pi(6i/s)^k \leq 8\pi(6i)^k t_1^{1-k}/(k-1)$, and conditioning successively at $\theta_{j_0-1}, \dots, \theta_2$ (note $\{\theta_i \leq \tau^*\} \in \mathcal{G}_{\theta_i}$) gives

$$\mathbf{P}(\theta_{j_0} \leq \tau^* \mid \mathcal{G}_{t_1+1}) \leq C_1 t_1^{-(k-1)(j_0-1)}.$$

On E_T there is no small city at time T , so if $\tau^* = \tau_j < \infty$, the city that reaches j_0 inhabitants at time τ^* was founded at a time $t_1 + 1$ with $T \leq t_1 < \tau^*$. Since $\mathbf{P}(\text{found at } t_1 + 1, t_1 < \tau^* \mid \mathcal{G}_{t_1}) \leq \kappa(t_1/4)^{-\alpha}$, summing over t_1 and using (D4) gives $\mathbf{P}(\tau^* = \tau_j < \infty \mid \mathcal{G}_T) \leq \kappa 4^\alpha C_1 \sum_{t_1 \geq T} t_1^{-\alpha - (k-1)(j_0-1)} \leq \frac{1}{8}$.

The share leaves the strip. Let ζ_{t+1} and ζ'_{t+1} be the indicators that the immigrant of time $t+1$ joins city 1, city 2. For $t < \tau^*$ the share changes by $D_t = \mathbf{1}_{E_T} \mathbf{1}\{t < \tau^*\}(\zeta_{t+1}(1-z_t) - \zeta'_{t+1}z_t)/(n_t+1)$, that is, $z_{t+1} = z_t + D_t$, and $|D_t| \leq 1/(n_t+1) \leq 4/(3t)$. Let $S_i^{(2)}$ be the region of city i when only cities 1 and 2 are taken into account; by (2) its area is $f_x(z_t)$ for $i = 1$ and $1 - f_x(z_t)$ for $i = 2$. Up to null sets, $S(i, t) \subset S_i^{(2)}$, and $S_i^{(2)} \setminus S(i, t)$ lies in the union over small B of $\{I_B \geq I_i\}$, whose area is at most $\sigma_t := (M(t) - 2)8\pi(6j_0/t)^k \leq 8\pi(6j_0)^k B(t)t^{-k}$ by (9). With $P_i = \mathbf{P}(\text{join } i \text{ at } t+1 \mid \mathcal{G}_t) = \text{area } S(i, t) - \int_{S(i, t)} (1+I)^{-1}$ we get $f_x(z_t) - \sigma_t - p_t \leq P_1 \leq f_x(z_t)$ and $1 - f_x(z_t) - \sigma_t - p_t \leq P_2 \leq 1 - f_x(z_t)$. Since $(n_t + 1)\mathbf{E}[D_t \mid \mathcal{G}_t] = P_1(1 - z_t) - P_2 z_t$, it follows that

$$|(n_t + 1)\mathbf{E}[D_t \mid \mathcal{G}_t] - (f_x(z_t) - z_t)| \leq \sigma_t + p_t \leq \eta \quad (T \leq t < \tau^*)$$

by (D3). By Corollary 4.4, $\mathbf{E}[D_t \mid \mathcal{G}_t] \leq 0$ when $z_t \in [\frac{1}{2} + \frac{\delta}{2}, \frac{1}{2} + \delta]$ and $\mathbf{E}[D_t \mid \mathcal{G}_t] \geq 0$ when $z_t \in [\frac{1}{2} - \delta, \frac{1}{2} - \frac{\delta}{2}]$, for $T \leq t < \tau^*$. With $m_t = D_t - \mathbf{E}[D_t \mid \mathcal{G}_t]$ we have $\mathbf{E}[m_t^2 \mid \mathcal{G}_t] \leq (4/(3t))^2 \leq 2/t^2$, and Doob's maximal inequality, in the form used for (8), gives $\mathbf{P}(\exists t \geq T : |\sum_{s=T}^{t-1} m_s| \geq \frac{\delta}{16} \mid \mathcal{G}_T) \leq (\frac{16}{\delta})^2 \sum_{t \geq T} \frac{2}{t^2} \leq \frac{512}{\delta^2(T-1)} \leq \frac{1}{8}$ by (D5). Outside this event, $\tau^* = \tau_z < \infty$ is impossible: the no-exit argument of the proof of Theorem C applies on $[T, \tau^*]$, with the step bound $|D_t| \leq 4/(3T) \leq \delta/4$ of (D6) in place of $1/T$. So $\mathbf{P}(\tau^* = \tau_z < \infty \mid \mathcal{G}_T) \leq \frac{1}{8}$.

Conclusion. On E_T , $\mathbf{P}(\tau^* < \infty \mid \mathcal{G}_T) \leq \frac{3}{8}$, so $\mathbf{P}(E_T \cap \{\tau^* = \infty\}) \geq \frac{5}{8}\mathbf{P}(E_T) > 0$. On this event, for all $t \geq T$: $n_t \geq \frac{3}{4}t$, $|z_t - \frac{1}{2}| \leq \delta$, and every small city has fewer than $j_0 \leq t/6 \leq N_i(t)$ inhabitants by (D2); hence $N_{(1)}(t) = \max(N_1(t), N_2(t)) \leq (\frac{1}{2} + \delta)n_t \leq (\frac{1}{2} + \delta)t$. $\square$

8 Further remarks and open questions

Consistency with the notes. The results do not contradict [Ald07, Theorem 2]. For $\alpha > 1$ the event that the whole population stays in one city has positive probability (Theorem 2(a) there), and Theorem C exhibits a different event of positive probability; the notes already observe that no simple 0–1 law holds (Remark (i) after their Theorem 2). On the events of Theorems C and D we have $N_{(1)}(t) \geq t/2$ and $N_{(1)}(t) \geq \frac{3}{8}t$, in agreement with Theorem 2(b) there. For $\alpha < 1$ and $2\alpha/\sqrt{3} < \beta < 2\alpha$, the probability of $\{N_{(1)}(t)/t \rightarrow 1\}$, positive by Theorem 2(c) there (whose proof the notes write out only in part), is less than 1 by Theorem D. The small-city bound [Ald07, Lemma 3] (our Lemma 7.1) shows that for $k > 1$ the shares 0 and 1 are also stable for two cities; the counterexample is a third, interior stable state. For $k = \frac{4}{3}$ and $a = \frac{1}{4}$, high-precision quadrature in the two verification runs puts the two unstable zeros of $f_{x^0}(z) - z$ at 0.31907 and 0.68093; this is a numerical observation, not used in any proof.

The associated dynamic system. For n cities at fixed positions and no foundings, the notes introduce the deterministic system $dz_i/dt = -z_i + \text{area } S_i(z_1, \dots, z_n)$, where S_i is defined with the weights $z_i^\alpha |y - x_i|^{-\beta}$, and conjecture [Ald07, Conjecture 32(a)] that some $z_i(t) \rightarrow 1$ if $\alpha > 1$ or $0 < \beta < 2\alpha$, for positions and initial values in general position. For $n = 2$ the system is $dz/dt = f_x(z) - z$. If $1 \leq k < \sqrt{3}$, choose a, δ, ρ, η, G as in Corollary 4.4; for $x \in G$ the field $f_x(z) - z$ is at most $-\eta$ on $[\frac{1}{2} + \frac{\delta}{2}, \frac{1}{2} + \delta]$ and at least η on $[\frac{1}{2} - \delta, \frac{1}{2} - \frac{\delta}{2}]$, so no solution starting in $[\frac{1}{2} - \delta, \frac{1}{2} + \delta]$ leaves this interval (f_x is continuous and nondecreasing, so solutions exist; at a first exit time the solution would have crossed a strip in which its derivative has the opposite sign). The positions in the open set G and the initial values in an interval form an open set of data, which a general-position assumption cannot exclude. Hence Conjecture 32(a) of the notes fails for every (α, β) with $2\alpha/\sqrt{3} < \beta < 2\alpha$, and for $\beta = 2\alpha$ with $\alpha > 1$. For $0 < k < 1$ and $\alpha > 1$ the same conclusion should follow by replacing (3) with $1/c \leq k \operatorname{artanh}(2h) \leq 2kh/(1 - 4h^2)$, which changes $q_a(h)$ only by factors $1 + O(h^2)$, so that the strip estimates hold whenever $k(a + \frac{1}{12a}) < 1$; we have not written this out, and this part is only a sketch. Independently of it, the system depends on (α, β) only through α/β , since S_i is unchanged when the weights are raised to the power $1/\beta$; so for $\alpha > 1$ and $\beta > 2\alpha$ the two clauses (a) and (b) of Conjecture 32, the latter applied to $(\alpha/m, \beta/m)$ with $m > \alpha$, make incompatible predictions for the same system. The dynamic system is not part of Conjecture 1.

The open range. For $k = 2\alpha/\beta > \sqrt{3}$ the symmetric two-city equilibrium is unstable for every spacing (Remark 4.2), and for $k = \sqrt{3}$ it is degenerate. Whether some other configuration, such as two cities placed asymmetrically or near the boundary, or three or more cities, has a stable interior equilibrium for some $k \geq \sqrt{3}$ is not settled here. So it remains open whether (b) holds for $0 < \beta \leq 2\alpha/\sqrt{3}$ and whether (c) holds for $\alpha > 1$ and $0 < \beta \leq 2\alpha/\sqrt{3}$. An exploratory grid scan of two-city configurations (Section 9) is consistent with $\sqrt{3}$ being the two-city threshold but proves nothing. The question of Remark (ii) of [Ald07, Theorem 2], whether $N_{(1)}(t)/t^{1-\varepsilon} \rightarrow \infty$ almost surely, also remains open for $\alpha \leq 1$ and $\beta < 2\alpha$.

9 Computations and reproducibility

Proved, and checkable by reading: Theorems A to D, all lemmas and corollaries, and the inequalities of Table 2; no proof depends on a computation. The inequalities of Table 2 are derived in Section 6 with rational bounds and are also checked by programs in exact or interval arithmetic. *Only tested:* the location of the unstable zeros 0.319 and 0.681 of $f_{x^0}(z) - z$, the size (about 0.0055) of the margin in Corollary 4.4 for the explicit instance, the classifications made by the grid programs, and every statement about simulations; simulations cannot prove almost sure statements, and grid or quadrature computations of areas are not certified. The reproducibility package (Python 3.13.5 with numpy 2.5.2

and mpmath 1.3.0, scipy 1.18.1 for one comparison, sympy 1.14.0 for one symbolic check, and one C program) has five parts; all programs use fixed seeds.

Written with this note (writing_stage/). The program `check_note.py` uses only the Python standard library and exact rational arithmetic. It checks the hypotheses $k = \frac{4}{3} \in (1, \sqrt{3})$, every entry of Table 2, the denominator bound of Lemma 4.1(ii) at h_a , and (T1)–(T3) for $T = \max(10^5, \lceil 100/c_0 \rceil)$, both for five values of c_0 from 10^{-6} to 10^3 and for all $c_0 > 0$ at once through the two cases of Section 6; 41 checks, all passed, in well under a second; it also proves $\pi < \frac{355}{113}$ from Machin’s formula.

Programs with which the results were first obtained (original/). `explicit_constants.py` certifies the constants of Table 2 for $c_0 = 1$, $T = 10^5$ in interval arithmetic; all checks passed. `f_sym_check.py` compares (4), evaluated by quadrature, with a count on a 6000×6000 grid (agreement to about 10^{-5} , the grid resolution), and evaluates the bound of Lemma 4.1(ii) and the difference quotients at $\frac{1}{2}$, which approach 0.7778, 0.7698, 1.1667 and 0.875 for $(k, a) = (\frac{4}{3}, \frac{1}{4})$, $(\frac{4}{3}, \frac{1}{2\sqrt{3}})$, $(2, \frac{1}{4})$ and $(\frac{3}{2}, \frac{1}{4})$, as Lemma 4.1(iii) predicts. The simulator `sim_city.py` implements the model literally. With it, two cities of 500 inhabitants at $(\frac{1}{4}, \frac{1}{2})$ and $(\frac{3}{4}, \frac{1}{2})$, $\alpha = 2$, $c_0 = 1$, four runs each to $t = 10^7$, gave: for $\beta = 3$ (slope $\frac{7}{9}$) the share of the larger city was within 0.013 of $\frac{1}{2}$ at $t = 117,549$ and within 0.006 at $t = 10^7$; for $\beta = 2.4$ (slope $\frac{35}{36}$) it stayed between 0.50 and 0.53; for $\beta = 2.2$ and $\beta = 2$ (slopes 1.06 and $\frac{7}{6}$) it drifted slowly away from $\frac{1}{2}$, up to 0.62; for $\beta = 1$ one city held at least 99.96% at $t = 10^7$; no city was founded in any of these runs. Started at $t = 10^5$ with share z_0 of city 1 ($\alpha = 2$, $\beta = 3$, two runs each), the share at $t = 10^7$ was 0.513–0.563 for $z_0 \in \{0.54, 0.60, 0.64\}$ and 0.92–0.98 for $z_0 \in \{0.72, 0.78\}$, in agreement with the zeros 0.319, $\frac{1}{2}$, 0.681 of $f_{x^0}(z) - z$. For $(\alpha, \beta) = (1, 1.5)$ and $(0.8, 1.2)$ (so $k = \frac{4}{3}$), two cities of 500, three runs each to $2 \cdot 10^6$: shares within 0.003 of $\frac{1}{2}$, and 1 to 40 further cities with at most 2 inhabitants each, as in Theorem D. From the model’s own initial condition with $\alpha = 2$, $\beta = 3$ and $c_0 = 1$, no run of 40 had a second city with more than 20% of the population at $t = 10^6$; with $c_0 = 0.01$ and $c_0 = 0.001$, 7 runs of 40 each did, and in these runs the share of the second city changed by at most 0.07 between $t = 117,549$ and $t = 10^6$. For $\alpha \in \{1.2, 2\}$ and $\beta \in \{1, 3\}$ (16 runs to 10^6), $M(10^6) \leq 4$ and the last founding was at $t \leq 573,382$. Two exploratory programs estimate $f_x(z) - z$ on grids: `f_general.py` for given positions, and `scan_two_city.py` on 2463 two-city configurations per value of k ; the scan found configurations with a robustly stable interior zero for $k = 1.5$ (8 of them) and none for $k = 1.7, 1.75, 1.8, 2, 2.5$. At $k = 1.7$ stable symmetric configurations exist (slope 0.98 for $a = 1/(2\sqrt{3})$) but are too weak for the scan’s robustness margin.

Verification run A (verification_run_A/). `va_constants.py` re-checks the instance in exact rational arithmetic (15 checks passed; $\eta \geq 0.0032439$; its search for a T satisfying (T1)–(T3) stops at 51,229 for $c_0 = 1$ and at 73,856 for $c_0 = 10^{-3}$; the smallest such T , with the tail bound $1/(T - 1)$, is 51,223 and 73,828, by independent run 2 below). `va_area.py` computes the disc of Apollonius from scratch and integrates its area in Q : the bound of Lemma 4.1(ii) holds at 12 values of $h \in [10^{-6}, 0.149]$ (smallest margin $1.1 \cdot 10^{-17}$ at $h = 10^{-6}$, as expected since both sides are $\frac{7}{9}h + O(h^3)$); the zeros of $f_{x^0}(z) - z$ are 0.319066, 0.5, 0.680934; for 40 random positions in G the largest value of $f_x(z) - z$ on the upper strip is -0.005498 and the smallest on the lower strip 0.005498 (for $\rho = 0.01$ the strip estimate fails, as expected); and the smallest sampled slope at $\frac{1}{2}$ is 0.9815 for $k = 1.7$ and 1.0392 for $k = 1.8$. Its own simulator `va_sim.py` gave, for $\alpha = 2$, $\beta = 3$, $c_0 = 1$: from two cities of 500, largest deviation of the share from $\frac{1}{2}$ after $t = 10^5$ at most 0.0088 in 3 runs to $3 \cdot 10^6$ and at most 0.0145 in 20 runs to 10^6 ; from two cities of 50,000 at the explicit time $T = 10^5$, at most 0.0018 in 10 runs to 10^6 ; no founding in any of these runs. It also gave monopoly for $\beta = 1$ ($N_{(1)}/t \geq 0.996$ at 10^6), the picture of Theorem D for $\alpha = 0.8$, $\beta = 1.2$, and a second city above 20% in 11 of 30 fresh runs with $c_0 = 10^{-3}$ at $t = 2 \cdot 10^5$.

Verification run B (verification_run_B/). `B1_share_map.py` computes f_x for arbitrary positions by an independent exact method (vertical chords of the disc, closed-form antiderivatives, 50

digits): the slope at $\frac{1}{2}$ agrees with $k(a + \frac{1}{12a})$ to within $3 \cdot 10^{-15}$ for six pairs (k, a) , and a deliberately wrong formula is detected; Lemma 4.1(i) holds to 10^{-47} at 299 values of h ; the bound of Lemma 4.1(ii) holds at these 299 values and at 975 points for 25 pairs (k, a) with $1 \leq k \leq 1.73$. `B2_constants_exact.py` re-checks the instance in exact rational arithmetic for $c_0 = 1$ ($T = 10^5$) and $c_0 = 10^{-2}, 10^{-3}$ ($T = 10^6$), with five negative controls ($\delta = 0.07$, $r = 0.03$, $T = 1000$, $\beta = 2$, $a = \frac{1}{20}$), each of which violates a condition. `B5_strip_check.py` finds the margin of Corollary 4.4 to be about 0.0055 against the certified $\eta > 0.0032$ (for 40 positions in G), still above 0.0046 for $\rho = 10^{-3}$, and violated for $\rho = 10^{-2}$. Its C simulator `B3_simcity.c` was tested against the exact transition probabilities of Lemma 2.1 ($|z| \leq 2.14$ in 12 comparisons; a kernel with β doubled is clearly rejected, with $|z|$ up to 1116). Its campaign `B4_sim_campaign.py` (runs to 10^7) agrees with the runs above, including the basins and the picture of Theorem D for $\alpha \in \{1, 0.8, 0.5\}$, and found from the model's own initial condition ($\alpha = 2$, $\beta = 3$, 100 runs to 10^6) a second city above 20% in 1, 21 and 37 runs for $c_0 = 1$, 0.01 and 0.001; with more runs (`B6_*`) the frequency for $c_0 = 0.001$ is between 34% and 39% (34 of 100 with `sim_city.py` on new seeds, 77 of 200 with the C simulator), so the 7 of 40 reported above is a low draw. `B7_packing_thmA.py` records the influence at every founding and confirms Lemma 2.3 on every tested path (12 runs, 16 values of λ).

Independent run 2 (`independent_run_2/`). Its three programs were written for the final text of this note. The program `r2_constants.py` checks Table 2 and (T1)–(T3) in exact rational arithmetic with enclosures different from those above (artanh by its series with a tail bound, π by the Bailey–Borwein–Plouffe series, $\sqrt{2}$ by bisection): 28 checks passed, $\eta \geq 0.003244197$; (T1)–(T3) hold for every $c_0 > 0$ because $c_0 T \geq 100$, $T \geq 10^5$ and $c_0(T - 1) \geq 99$, and were also checked at 67 values of c_0 from 10^{-9} to 10^6 ; the smallest T satisfying (T1)–(T3), with the tail bound $1/(T - 1)$, is 51,223 for $c_0 = 1$ and 73,828 for $c_0 = 10^{-3}$. `r2_sharemap.py` computes f_x by Green's theorem on the boundary of the intersection of the disc of Apollonius with the square (40 to 60 digits): (4) to 10^{-40} ; Lemma 4.1(i) to 10^{-38} ; the bound of Lemma 4.1(ii) at 590 points for ten pairs (k, a) with $k \geq 1$; the slope $k(a + \frac{1}{12a})$ at six pairs, two of them with $k < 1$; exactly three zeros 0.31907, $\frac{1}{2}$, 0.68093 of $f_{x^0}(z) - z$ for $k = \frac{4}{3}$, $a = \frac{1}{4}$; the centre, radius and area of Figure 1; and the margin of Corollary 4.4, 0.0055 at x^0 and over 29 position pairs in G (violated for $\rho = 10^{-2}$). `r2_symbolic.py` checks 44 algebraic steps of the proofs with sympy. Each of the three programs takes a few seconds.

Re-runs. On 14 October 2026 every program of the package was run again with the command lines of the README, first from a copy of the package when this note was written and again, by independent run 2, from an extracted copy of the archive; both times all 33 outputs were identical to the recorded files. (One recorded file of run B, a negative control for Lemma 2.3, was produced by a temporary variant of the C simulator that is not part of the package.) One recorded file of `original/`, `f_general_simE.out`, had been saved without the table lines beginning with `z=`; the complete output was regenerated and is included as `f_general_simE_complete.out`, and it agrees with the recorded file after removing those lines. The longest single program takes about six minutes. Details and command lines are in `reproducibility/README.md`.

Verification. The results were first obtained, with proofs, together with the programs in `original/`. Two independent verification runs, both AI-assisted, followed; each re-derived the proofs step by step and wrote its own programs, without access to the other's report. Run A compared the statement with the notes and the companion paper, re-derived every lemma and theorem, tried to break Theorem B in three ways, certified the explicit instance in exact rational arithmetic, and wrote an independent area code and an independent simulator. Run B re-derived the proofs, computed the share map by an independent exact method, re-checked the constants with negative controls, tested its own simulator against the exact transition probabilities, re-ran all programs of `original/` (with identical outputs, except the filtered file mentioned above), checked the bibliography against Crossref and the arXiv, and

repeated the literature search. Both runs confirmed the outcome stated in Corollary 1.1 and found no mathematical gap. They required corrections of wording, which were made: the outcome is stated part by part everywhere; the comparison with the heuristic exponent θ is restricted to $0 < \alpha < 1$ (Remark 3.1(iii)); the case $0 < k < 1$ of the remark on Conjecture 32 is marked as a sketch; the dependence of $B_T(t)$ on T in the proof of Theorem D is resolved by the T -free majorant $\bar{B}(t)$; the description of the companion paper does not attribute theorems to it; a range of simulated shares was corrected to 0.50–0.53; and the filtered output file was regenerated. The runs examined the first written version of the results. This note follows that version with these corrections, with the choice $T = \max(10^5, \lceil 100/c_0 \rceil)$ for every c_0 in Section 6, and with the proofs rearranged.

A second round of verification, one further independent AI-assisted run (independent run 2), then examined the final text of this note. It compared every statement about [Ald07, AH12, Ald] with the sources, re-derived every lemma, theorem and corollary line by line, wrote three programs of its own (`independent_run_2/`), re-ran every program of the package from an extracted copy of the archive with identical outputs, and repeated the literature search. It found no mathematical error and required ten corrections, which were made: the inequality used for (8) is named as Doob’s maximal inequality for the submartingale Λ_t^2 , in the form used; in the proof of Theorem D, Lemma 2.2 is applied with $Y_s = 0$ for $s < T$, so that its hypotheses hold; the opening sentence of Section 7 no longer asserts that new cities keep being founded for $\alpha \leq 1$, which is not proved; the bound for small cities is credited to [Ald07, Lemma 4]; the position of the sentence quoted from the notes, the status of their Theorem 2(c) in Section 8, the number of uses of Lemma 2.2, an inequality sign in Section 6, the definition of $N_{(\geq 2)}(t)$ and the description of a search of run A for admissible T were corrected; and this record of the run was added.

Scope and priority. The note answers Conjecture 1 of [Ald07] part by part: part (a) is proved for all parameters in its range (Theorem B); parts (b) and (c) are disproved for every $c_0 > 0$ and $2\alpha/\sqrt{3} < \beta < 2\alpha$ (with $\alpha > 0$ for (b) and $\alpha > 1$ for (c); Theorems C and D); whether (b) and (c) hold for $\beta \leq 2\alpha/\sqrt{3}$ remains open. The model, its heuristics and the conjecture are due to Aldous [Ald07, Ald]; the companion paper of Aldous and Huang [AH12] restricts the model to $0 < \alpha \leq 1$ and contains heuristics and simulations, while the partial results are those of [Ald07, Theorem 2]. The constant κ and the small-city bound (Lemma 7.1) are taken from the notes, and the treatment of small cities in the proof of Theorem D follows the notes’ sketch of their Theorem 2(c), with the comparison of their Lemma 4. The tools are standard: Ville’s inequality [Vil39], Doob’s maximal inequality, optional stopping and Lévy’s extension of the Borel–Cantelli lemma [Wil91], and circles of Apollonius; the trapping of an urn near a stable zero of its drift is of the kind studied for generalized Pólya urns [HLS80, Pem07], and we claim no novelty for it. New, to our knowledge and relative to the literature we found, are Theorems A to D, that is, the proof of (a) and the counterexamples to (b) and (c) in the stated range.

What was read: the notes [Ald07], in the author’s file, pages 1–14 and 38–41 in full and pages 15–37 (on the balanced and critical cases, not used here) for their statements, while verification run A read all 41 pages and independent run 2 read pages 1–15 and 37–41 again; the companion paper [AH12] in full (version 1, the only version on the arXiv); and the web page [Ald]. The papers [HLS80, Pem07] and the books [Vil39, Wil91] were not read for this note; none of their results is used beyond the standard facts named above, and Lemma 2.2 is proved in the form used.

The searches were made on 14 October 2026, when the results were first obtained, again by verification runs A and B, and once more by independent run 2: queries to the arXiv (for the title and authors of [AH12], for city growth, spheres of influence, weighted Voronoi tessellations combined with urns or reinforcement, and spatial Chinese-restaurant processes, and for all later arXiv papers of Aldous), to Crossref, to zbMATH Open (no document on the model), to OpenAlex (no work citing

[AH12], checked again at the writing and by run 2) and to Semantic Scholar (an empty list of citing works), and seven web searches in all. They found the sources above, two undergraduate simulation reports linked from Aldous’s pages, and unrelated work; neither the reports nor any other item found addresses Conjecture 1. No answer to Conjecture 1 was found in the literature accessible to us; a search that finds nothing is not a proof of novelty, and we make no priority claim.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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