

Excursion Lengths in a Uniform Eulerian Circuit of the Complete Graph: Aldous's Conjecture Holds if and only if $i = o(n^{3/2})$

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September 30, 2026

Abstract

Let N_i be the number of excursions of length i from a fixed vertex in a uniformly random Eulerian circuit of the complete graph K_n in which every edge is replaced by two opposite arcs. Aldous and Yu (Open Problems in Mathematics, 2014) stated as the natural conjecture that $\mathbb{E}N_i$ is asymptotic to $e^{-i/n}$. We prove that for every fixed K , uniformly in $2 \leq i \leq Kn^{3/2}$,

$$\mathbb{E}N_i = \exp\left(-\frac{i}{n} - \frac{i^2}{2n^3}\right)(1 + O(n^{-1/2} \log n)),$$

and that $\mathbb{E}N_i e^{i/n} \rightarrow 0$ when $i/n^{3/2} \rightarrow \infty$. Consequently $\mathbb{E}N_i \sim e^{-i/n}$ holds if and only if $i = o(n^{3/2})$. This range contains fixed i , the scale $i \sim xn$ and essentially all excursions, and the length of a uniformly chosen excursion, divided by n , converges in distribution to the standard exponential law. At the scale $i \sim yn^{3/2}$ the ratio $\mathbb{E}N_i/e^{-i/n}$ tends to $e^{-y^2/2}$, so the conjecture read literally for all i is false. The proof is elementary. It combines the construction of a uniform Eulerian circuit from a uniform spanning tree (the BEST theorem, in the form used by Kandel, Matias, Unger and Winkler) with a hazard representation of the first excursion and a convexity bound. Exact computations for $n \leq 6$ and Monte Carlo simulations up to $n = 6400$ agree with the results. This is an unrefereed note.

1 Introduction

Let $n \geq 3$ and let G_n be the complete directed graph on a vertex set V with $|V| = n$: its arcs are all ordered pairs (a, b) of distinct vertices, so every edge of K_n is replaced by two opposite arcs. It has $n(n-1)$ arcs, and every vertex has in-degree and out-degree $n-1$. An *Eulerian circuit* of G_n is a cyclic sequence of arcs in which every arc occurs exactly once and the head of each arc is the tail of the next. Let $\mathcal{C}$ be a uniformly random Eulerian circuit. Fix a vertex v and put $U = V \setminus \{v\}$. The $n-1$ visits of $\mathcal{C}$ to v cut it into $n-1$ *excursions*, closed trails from v to v that do not visit v in between; the length of an excursion is its number of arcs. Let N_i be the number of excursions of length i . Then $\sum_i N_i = n-1$ and $\sum_i iN_i = n(n-1)$, so an excursion has mean length n , and $N_1 = 0$ because G_n has no loops.

In their note on random Eulerian circuits [AY14], Aldous and Yu ask mainly about the discrete torus, and at the end they briefly mention two other examples. Example (a) is the complete graph, for which the natural conjecture is that “the expected number of length- i excursions is asymptotic to $e^{-i/n}$ ” [AY14, p. 4]. (The same sentence says that the circuit has n excursions of mean length $n-1$. The correct numbers are $n-1$ excursions of mean length n ; this does not affect the conjecture.) The

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heuristic behind the conjecture is that a uniform Eulerian circuit should resemble a random walk: an excursion of simple random walk on K_n from v has a geometric length with mean n , and $n - 1$ such excursions include about $e^{-i/n}$ of length i . The question is recorded as AMR-096-0015 in the UnsolvedMath dataset [ula26], with status open, and Aldous’s open-problem page [Ald] still lists the topic “Random Eulerian circuits”. Aldous and Yu remark that they know of no nontrivial theory about properties of the uniform circuit on particular graphs, and Aldous’s 2023 talk [Ald23] describes the literature on uniform random Eulerian circuits as very small.

We prove the conjecture in the range where it is true, and we show that this range is sharp. Constants implied by $O(\cdot)$ are absolute unless a subscript indicates a dependence.

Theorem 1.1. *For every $K \geq 1$ there is a constant C_K such that for all $n \geq 3$ and all integers i with $2 \leq i \leq Kn^{3/2}$,*

$$\left| \mathbb{E}N_i \cdot \exp\left(\frac{i}{n} + \frac{i^2}{2n^3}\right) - 1 \right| \leq C_K n^{-1/2} \log n.$$

Theorem 1.2. *For all $n \geq 23$ and all integers $i \geq 2$,*

$$\mathbb{E}N_i \leq e^{-i/n} \left(12 e^{-(i-2)^2/(2n^3)} + \varepsilon_n \right), \quad \varepsilon_n := n^2 e^{n/64} \left(\frac{e}{8} \right)^{n/4} + n e^{4.72-n/32768} + e^{3+e} n \frac{e^{M_n}}{M_n!},$$

where $M_n := \max\{\lfloor n/256 \rfloor - 1, 0\}$. In particular $\varepsilon_n \rightarrow 0$, and $\mathbb{E}N_i e^{i/n} \rightarrow 0$ whenever $i/n^{3/2} \rightarrow \infty$.

The constants in Theorem 1.2 are not optimized. A numerical evaluation shows $\varepsilon_n < 1$ only for n larger than about $5.9 \cdot 10^5$, so the theorem is informative only asymptotically.

Corollary 1.3. (a) *Let $i = i(n) \geq 2$ with $i/n^{3/2} \rightarrow y \in [0, \infty]$. Then $\mathbb{E}N_i e^{i/n} \rightarrow e^{-y^2/2}$, where $e^{-\infty} := 0$. Consequently $\mathbb{E}N_i \sim e^{-i/n}$ if and only if $i = o(n^{3/2})$. More precisely, $\mathbb{E}N_i = e^{-i/n} (1 + O(i^2/n^3 + n^{-1/2} \log n))$ uniformly in $2 \leq i \leq n^{3/2}$. In particular $\mathbb{E}N_i \rightarrow 1$ for every fixed $i \geq 2$, and $\mathbb{E}N_{\lfloor xn \rfloor} \rightarrow e^{-x}$ for every fixed $x > 0$.*

(b) $\sum_{i \geq 1} |\mathbb{E}N_i - e^{-i/n}| = O(n^{1/2} \log n)$.

(c) *Let Λ_n be the length of an excursion of $\mathcal{C}$ chosen uniformly at random among its $n-1$ excursions. Then $\mathbb{E}\Lambda_n = n$ and*

$$\sup_{x \geq 0} |\mathbb{P}(\Lambda_n \leq xn) - (1 - e^{-x})| = O(n^{-1/2} \log n).$$

In particular Λ_n/n converges in distribution to the exponential law with mean 1.

So the conjecture holds for fixed i , at the natural scale $i \sim xn$, and indeed for all $i = o(n^{3/2})$, which is where essentially all excursions lie: by Theorem 1.2, the expected number of excursions of length greater than $n^{3/2}$ is $O(ne^{-\sqrt{n}})$. Read literally as a statement about all $i \geq 2$, the conjecture is false, but only at a scale that is essentially never reached. Part (c) is the distributional form of the heuristic “excursion lengths are approximately exponential with mean n ”.

The correction factor $e^{-i^2/(2n^3)}$ has a simple explanation. In the tree-based construction of $\mathcal{C}$ recalled in Section 2, a walk that visits a vertex $w \neq v$ for the j -th time, before returning to v , steps to v with probability $1/(n-1-j)$ (or 0, if w is a child of v in the tree), instead of $1/(n-1)$ as for simple random walk. After s steps a typical vertex has been visited about s/n times, so the hazard of returning at time s is about $(1 + s/n^2)/n$, and the cumulative hazard up to time i is about $i/n + i^2/(2n^3)$.

The proof is elementary. Section 2 writes $\mathbb{E}N_i$ exactly through the first excursion of the tree-based walk of Kandel, Matias, Unger and Winkler [KMUW96], and a change of measure turns it

into an expectation of e^{-S_i} , where S_i is a cumulative hazard. Section 3 proves four simple estimates, Sections 4 to 6 prove the results, and Section 7 reports computations that are not used in the proofs.

Uniform random Eulerian circuits have been studied mostly from the enumerative and the algorithmic side. The number of Eulerian circuits of the complete graph and of other dense graphs was determined asymptotically by McKay and Robinson [MR98] and by Isaev [Isa11, Isa13], and that of random regular digraphs by Creed and Cryan [CC13]. Sampling algorithms were given by Kandel, Matias, Unger and Winkler [KMUW96], by Tetali and Vempala [TV01] and, recently, by Anari [Ana26]. None of these works concerns excursion lengths.

2 An exact representation

Throughout, $n \geq 3$, and V , v , U and G_n are as above. A *rooted circuit* is a sequence $(e_1, \dots, e_{n(n-1)})$ of arcs that forms an Eulerian circuit and whose first arc leaves v . Let $\mathcal{C}_v$ be a uniformly random rooted circuit, and let τ be the length of its first excursion, that is, the least t such that e_t enters v .

Lemma 2.1. $\mathbb{E}N_i = (n-1)\mathbb{P}(\tau = i)$ for every $i \geq 1$.

Proof. An Eulerian circuit has $n-1$ arcs leaving v . Cutting it just before one of them gives a rooted circuit, and every rooted circuit arises in this way from exactly one Eulerian circuit and one of its arcs leaving v . So if $\mathcal{C}$ is uniform and one of its $n-1$ arcs leaving v is then chosen uniformly, the resulting rooted circuit is uniform. Its first excursion is the excursion of $\mathcal{C}$ that begins with the chosen arc, which given $\mathcal{C}$ is a uniform choice among the $n-1$ excursions. Hence $\mathbb{P}(\tau = i) = \mathbb{E}[N_i/(n-1)]$. $\square$

Let T be a spanning tree of K_n , regarded as oriented towards v . For $w \in U$ write $p(w)$ for the parent of w , so that $(w, p(w))$ is the *tree arc* of w . Let $A := \{w \in U : p(w) = v\}$ be the set of children of v , and $C := |A| = \deg_T(v)$.

Lemma 2.2 ([KMUW96]; BEST theorem). *Let T be a uniformly random spanning tree of K_n . Given T , run the following walk $X_0 = v, X_1, X_2, \dots$ on V . If the current vertex is $w \neq v$ and w has an unused out-arc other than its tree arc, traverse one of these, chosen uniformly at random; if the tree arc is the only unused out-arc of w , traverse it. If the current vertex is v , traverse a uniformly chosen unused out-arc of v . Stop when the current vertex has no unused out-arc. Then the sequence of traversed arcs is a uniform rooted circuit.*

Proof. This is the algorithm of [KMUW96] that Aldous and Yu recall in [AY14]; we recall why it works. For a rooted circuit, the last out-arcs used by the vertices $w \neq v$ form a spanning arborescence oriented towards v . Conversely, for every such arborescence and every choice of orders of the out-arcs of all vertices in which the arborescence arc of each $w \neq v$ comes last, following these orders from v traverses every arc exactly once. These two maps are inverse to each other; this is the BEST theorem of van Aardenne-Ehrenfest and de Bruijn [vAEdB51] and Smith and Tutte [ST41] (see also [Sta99, Section 5.6]). Hence rooted circuits are in bijection with the tuples consisting of an arborescence, an order of the $n-2$ non-tree out-arcs of each $w \neq v$, and an order of the $n-1$ out-arcs of v . Arborescences of G_n oriented towards v correspond to spanning trees of K_n , so a uniform rooted circuit corresponds to a uniform tree together with independent uniform orders. The walk reveals each order one arc at a time, choosing uniformly among the arcs not yet revealed, and this produces independent uniform orders. $\square$

By Cayley's formula [Cay89] there are n^{n-2} spanning trees, so G_n has $n^{n-2}((n-2)!)^{n-1}(n-1)!$ rooted circuits and $n^{n-2}((n-2)!)^n$ Eulerian circuits.

From now on T is a uniform spanning tree and (X_s) is the walk of Lemma 2.2, so that $\tau = \min\{s \geq 1 : X_s = v\}$. For $u \in U$ and $s \geq 1$ let $k_u(s) := \#\{1 \leq t \leq s : X_t = u\}$, and let $j_s := k_{X_s}(s)$ when $X_s \in U$; thus the walk visits X_s for the j_s -th time at time s .

Lemma 2.3. X_1 is uniform on U . Let $1 \leq s < \tau$, $w := X_s$ and $j := j_s$.

(i) The only arc of v used so far is (v, X_1) , and no arc into v has been used. The vertex w has used exactly $j - 1$ out-arcs, none of them its tree arc.

(ii) If $j \leq n - 2$, then X_{s+1} is uniform on a set O_s of $n - 1 - j$ vertices, namely the heads of the unused non-tree out-arcs of w , and $v \in O_s$ if and only if $w \notin A$.

(iii) $j \leq n - 1$, and $j = n - 1$ implies $w = X_1 \in A$ and $X_{s+1} = v$.

Consequently the hazard $h_s := \mathbb{P}(X_{s+1} = v \mid T, X_1, \dots, X_s)$ satisfies

$$h_s = \frac{\mathbf{1}\{w \notin A\}}{n - 1 - j_s} \quad \text{if } j_s \leq n - 2, \quad h_s = 1 \quad \text{if } j_s = n - 1. \quad (1)$$

In particular $h_s = 1$ if $w \notin A$ and $j_s = n - 2$.

Proof. The first arc of $\mathcal{C}_v$ is a uniform out-arc of v , so X_1 is uniform.

(i) Since $s < \tau$, the walk has not returned to v by time s , which gives the first sentence. Each of the $j - 1$ earlier visits of w was followed by an exit from w , so w has used $j - 1$ out-arcs. The j visits of w at times $1, \dots, s$ entered w through j distinct arcs, so $j \leq n - 1$ and $j - 1 < n - 1$; as the tree arc is used only when it is the last unused out-arc, it has not been used.

(ii) If $j \leq n - 2$, then w has $n - j \geq 2$ unused out-arcs, one of which is its tree arc, so the walk chooses uniformly among the $n - 1 - j$ unused non-tree out-arcs. By (i) the arc (w, v) is unused, and it is a non-tree arc if and only if $p(w) \neq v$, that is, $w \notin A$.

(iii) We saw $j \leq n - 1$. If $j = n - 1$, then all $n - 1$ arcs into w have been used, including (v, w) , so $w = X_1$ by (i). Suppose that $w = X_1 \notin A$, and let $s' < s$ be the time of the $(n - 2)$ -th visit of w . By (ii) at time s' , $O_{s'}$ has one element and contains v , so $X_{s'+1} = v$ and $\tau \leq s' + 1 \leq s$, a contradiction. Hence $w \in A$. At time s all $n - 2$ non-tree out-arcs of w are used, so the walk takes the tree arc, which leads to $p(w) = v$. Formula (1) follows from (ii) and (iii). $\square$

We now remove the event of returning to v by a change of measure. Fix T . The *auxiliary walk* $(\tilde{X}_s)_{s \geq 1}$ takes values in $U \cup \{\dagger\}$, where $\dagger$ is a cemetery state. Let $\tilde{X}_1$ be uniform on U . Suppose that $\tilde{X}_1, \dots, \tilde{X}_s \in U$ have been defined. Declare the arcs $(v, \tilde{X}_1), (\tilde{X}_1, \tilde{X}_2), \dots, (\tilde{X}_{s-1}, \tilde{X}_s)$ used, and define $k_u(s)$, j_s and O_s from this history exactly as above, and h_s by (1). (The history determines them, and a vertex is entered through distinct arcs at distinct visits, so $j_s \leq n - 1$.) If $h_s = 1$, put $\tilde{X}_t := \dagger$ for all $t > s$; we say that the walk is *absorbed*. If $h_s < 1$, let $\tilde{X}_{s+1}$ be uniform on $O_s \setminus \{v\}$. This set is not empty: if $w = \tilde{X}_s \in A$, then $v \notin O_s$ and $|O_s| = n - 1 - j_s \geq 1$; if $w \notin A$, then $h_s < 1$ forces $j_s \leq n - 3$ and $|O_s \setminus \{v\}| = n - 2 - j_s \geq 1$. The auxiliary walk never uses an arc twice and never visits v . After absorption we keep all k_u constant. We write $\tilde{\mathbb{P}}_T$ and $\tilde{\mathbb{E}}_T$ for its law and expectation given T , μ for the joint law of T and the auxiliary walk, and $\mu[\cdot]$ for the expectation under μ .

Lemma 2.4. For every tree T and every $i \geq 2$,

$$\mathbb{P}(\tau = i \mid T) = \tilde{\mathbb{E}}_T[W_i], \quad W_i := \mathbf{1}\{\tilde{X}_{i-1} \in U\} \prod_{s=1}^{i-2} (1 - h_s) \cdot h_{i-1}.$$

Proof. Let $x = (x_1, \dots, x_{i-1}) \in U^{i-1}$, and compute j_s , O_s and h_s from x as above. By Lemma 2.3, if $h_s < 1$ for all $s \leq i - 2$ then

$$\mathbb{P}(X_1 = x_1, \dots, X_{i-1} = x_{i-1}, X_i = v \mid T) = \frac{1}{n-1} \prod_{s=1}^{i-2} \frac{\mathbf{1}\{x_{s+1} \in O_s\}}{n-1-j_s} \cdot h_{i-1},$$

and otherwise the left side is 0, because the walk returns to v at the first time $s + 1$ with $h_s = 1$. On the other hand, if $h_s < 1$ for all $s \leq i - 2$, the auxiliary walk satisfies

$$\tilde{\mathbb{P}}_T(\tilde{X}_1 = x_1, \dots, \tilde{X}_{i-1} = x_{i-1}) = \frac{1}{n-1} \prod_{s=1}^{i-2} \frac{\mathbf{1}\{x_{s+1} \in O_s \setminus \{v\}\}}{|O_s \setminus \{v\}|},$$

and this probability is 0 otherwise. For each s with $h_s < 1$ and each $x_{s+1} \in U$, the s -th factors satisfy $\mathbf{1}\{x_{s+1} \in O_s\}/(n-1-j_s) = (1-h_s) \mathbf{1}\{x_{s+1} \in O_s \setminus \{v\}\}/|O_s \setminus \{v\}|$. Indeed, if $x_s \in A$ then $h_s = 0$ and $v \notin O_s$; if $x_s \notin A$ then $1-h_s = (n-2-j_s)/(n-1-j_s)$ and $|O_s \setminus \{v\}| = n-2-j_s$. Multiplying the factors and summing over x proves the lemma. $\square$

On the event $\{\tilde{X}_{i-1} \in U\}$ we have $h_s < 1$ for $s \leq i-2$, and we put

$$S_i := -\sum_{s=1}^{i-2} \log(1-h_s), \quad \text{so that} \quad W_i = \mathbf{1}\{\tilde{X}_{i-1} \in U\} e^{-S_i} h_{i-1}.$$

By Lemmas 2.1 and 2.4,

$$\mathbb{E}N_i = (n-1) \mathbb{E}_T \tilde{\mathbb{E}}_T[W_i] = (n-1) \mu[W_i] \quad (i \geq 2), \quad (2)$$

where $\mathbb{E}_T$ averages over the uniform tree T . Everything below rests on (2). Note that $0 \leq W_i \leq 1$.

Remark 2.5 (A second exact formula). Let $W = (v, x_1, \dots, x_{i-1}, v)$ be a closed trail with $x_1, \dots, x_{i-1} \in U$, let k_u be the number of indices t with $x_t = u$, and let H be G_n with the arcs of W removed. By the BEST theorem applied to H , the rooted circuits whose first excursion is W are counted by $t_v(H) (n-2)! \prod_{u \in U} (n-2-k_u)!$, where $t_v(H)$ is the number of spanning arborescences oriented towards v of the subgraph of H on its non-isolated vertices, and $(-1)! := 1$ (a vertex with $k_u = n-1$ is isolated in H). Dividing by the number of rooted circuits and using Lemma 2.1,

$$\mathbb{E}N_i = \sum_W \frac{t_v(H)}{n^{n-2}} \prod_{u \in U} \frac{(n-2-k_u)!}{(n-2)!},$$

the sum running over all such trails of length i . By the directed matrix-tree theorem [Tut48], $t_v(H)$ is a determinant. This formula does not use Lemmas 2.3 and 2.4; evaluated in exact integer arithmetic it gives one of the independent checks described at the end of the paper.

Remark 2.6 (Two closed forms). Formula (2) gives

$$\mathbb{E}N_2 = \frac{n-1}{n}, \quad \mathbb{E}N_3 = \frac{(n-1)(n^2-3n+3)}{n^2(n-2)}.$$

For $i = 2$, $W_2 = h_1 = \mathbf{1}\{\tilde{X}_1 \notin A\}/(n-2)$, so $\mathbb{E}N_2 = (n-1 - \mathbb{E}C)/(n-2)$, and $\mathbb{E}C = 2 - 2/n$ by Lemma 3.3 below. For $i = 3$, $W_3 = (1-h_1)h_2$ and $j_2 = 1$. If $\tilde{X}_1 \in A$, then $h_1 = 0$ and $\tilde{X}_2$ is uniform on $U \setminus \{\tilde{X}_1\}$; if $\tilde{X}_1 = w \notin A$, then $1-h_1 = (n-3)/(n-2)$ and $\tilde{X}_2$ is uniform on $U \setminus \{w, p(w)\}$. Averaging over $\tilde{X}_1$ gives $\mathbb{E}_T[W_3] = ((n-3)(n-1-C) + D_2)/((n-1)(n-2)^2)$, where D_2 is the number of vertices at distance 2 from v in T . Since exactly $3n^{n-4}$ labelled trees contain a given path with two edges (a special case of the classical count of trees containing a given forest; see [Moo70]), $\mathbb{E}D_2 = 3(n-1)(n-2)/n^2$, and the formula for $\mathbb{E}N_3$ follows. Both values agree with exact enumeration for $3 \leq n \leq 6$. Note that $\mathbb{E}N_2 e^{2/n}$ and $\mathbb{E}N_3 e^{3/n}$ are $1 + 1/n + O(n^{-2})$.

3 Four estimates

In this section T is fixed and all probabilities refer to the auxiliary walk. Let $\mathcal{F}_s := \sigma(\tilde{X}_1, \dots, \tilde{X}_s)$, $k_u(0) := 0$ and $M_s := \max_{u \in U} k_u(s)$, so that $M_0 = 0$ and M_s is nondecreasing in s . For $A' \subseteq U$ and $N \geq 0$ let $c_{A'}(N) := \#\{1 \leq s \leq N : \tilde{X}_s \in A'\}$.

Lemma 3.1 (Landing probabilities). $\tilde{\mathbb{P}}_T(\tilde{X}_1 = u) = 1/(n-1)$ for $u \in U$. If $s \geq 1$, $\tilde{X}_s \in U$ and $j_s \leq n-3$, then $\tilde{\mathbb{P}}_T(\tilde{X}_{s+1} = u \mid \mathcal{F}_s) \leq 1/(n-2-j_s)$ for every $u \in U$.

Proof. If $j_s \leq n-3$, then $h_s \leq 1/2$, and $\tilde{X}_{s+1}$ is uniform on $O_s \setminus \{v\}$, which has at least $n-2-j_s$ elements. $\square$

Lemma 3.2 (Visit counts). Let $1 \leq L \leq n-3$ and $q := 1/(n-2-L)$.

(a) Let $A' \subseteq U$ with $|A'| = a$ (it may depend on T), let $\lambda \geq 0$ and $N \geq 1$. Then

$$\tilde{\mathbb{E}}_T[e^{\lambda c_{A'}(N)}; M_{N-1} \leq L] \leq (1 + aq(e^\lambda - 1))^N \leq \exp(aqN(e^\lambda - 1)).$$

(b) For every $N \geq 1$, $\tilde{\mathbb{P}}_T(M_N > L) \leq (n-1)(eNq/(L+1))^{L+1}$.

Proof. (a) Put $Y_r := \mathbf{1}\{\tilde{X}_r \in A'\} \mathbf{1}\{M_{r-1} \leq L\}$ and $K_t := \sum_{r=1}^t Y_r$. We claim that $\tilde{\mathbb{P}}_T(Y_r = 1 \mid \mathcal{F}_{r-1}) \leq aq$ for every $r \geq 1$. For $r = 1$ this holds because $\tilde{\mathbb{P}}_T(\tilde{X}_1 \in A') = a/(n-1)$ and $q \geq 1/(n-1)$. Let $r \geq 2$. On $\{M_{r-1} > L\}$ we have $Y_r = 0$. On $\{\tilde{X}_{r-1} = \dagger\}$ the walk is absorbed and $Y_r = 0$. On the remaining event, $\tilde{X}_{r-1} \in U$ and $j_{r-1} \leq M_{r-1} \leq L \leq n-3$, so Lemma 3.1 gives $\tilde{\mathbb{P}}_T(\tilde{X}_r \in A' \mid \mathcal{F}_{r-1}) \leq a/(n-2-j_{r-1}) \leq aq$. This proves the claim. For a $\{0,1\}$ -valued Y with $\mathbb{P}(Y = 1 \mid \mathcal{F}) \leq p$ we have $\mathbb{E}[e^{\lambda Y} \mid \mathcal{F}] \leq 1 + p(e^\lambda - 1)$, so

$$Z_t := e^{\lambda K_t} (1 + aq(e^\lambda - 1))^{-t}$$

is a nonnegative supermartingale with $Z_0 = 1$, and $\tilde{\mathbb{E}}_T[e^{\lambda K_N}] \leq (1 + aq(e^\lambda - 1))^N$. Since M is nondecreasing, on $\{M_{N-1} \leq L\}$ all indicators $\mathbf{1}\{M_{r-1} \leq L\}$ with $r \leq N$ equal 1, so $K_N = c_{A'}(N)$ there. This proves (a); the last inequality is $1 + x \leq e^x$.

(b) For $u \in U$ let K_t^u denote K_t for $A' = \{u\}$. Let $\sigma := \min\{s \geq 1 : M_s > L\}$. At each step at most one count k_u grows, by one, namely that of the vertex visited (and none after absorption). So on $\{M_N > L\}$ we have $\sigma \leq N$, $\tilde{X}_\sigma \in U$, the vertex $u^* := \tilde{X}_\sigma$ satisfies $k_{u^*}(\sigma) = L+1$, and $M_{r-1} \leq L$ for all $r \leq \sigma$; hence $K_N^{u^*} \geq K_\sigma^{u^*} = k_{u^*}(\sigma) = L+1$. By a union bound, Markov's inequality and the proof of (a), for every $\lambda \geq 0$,

$$\tilde{\mathbb{P}}_T(M_N > L) \leq \sum_{u \in U} \tilde{\mathbb{P}}_T(K_N^u \geq L+1) \leq (n-1) e^{-\lambda(L+1)} \exp(qN(e^\lambda - 1)).$$

If $qN < L+1$, the choice $e^\lambda = (L+1)/(qN)$ gives $(n-1)(eqN/(L+1))^{L+1} e^{-qN}$, which implies (b). If $qN \geq L+1$, the right side of (b) is at least 1. $\square$

Lemma 3.3 (Degree of the root). For a uniform spanning tree T of K_n , $C-1$ has the binomial distribution $\text{Bin}(n-2, 1/n)$. Hence $\mathbb{E}C = 2 - 2/n$, $\mathbb{E}e^{\theta C} \leq \exp(\theta + e^\theta - 1)$ for $\theta \geq 0$, and $\mathbb{P}(C = c) \leq 1/(c-1)!$ for $c \geq 1$.

Proof. Under the Prüfer bijection [Prüfer18], a uniform tree corresponds to a uniform word in V^{n-2} , and $\deg_T(v) - 1$ is the number of occurrences of v in the word. Then $\mathbb{E}e^{\theta C} = e^\theta (1 + (e^\theta - 1)/n)^{n-2} \leq \exp(\theta + e^\theta - 1)$ and $\mathbb{P}(C = c) \leq \binom{n-2}{c-1} n^{-(c-1)} \leq 1/(c-1)!$. $\square$

Lemma 3.4 (Convexity bound). *Let $i \geq 2$ and $B := A \cup \{\tilde{X}_1\}$. On the event $\{\tilde{X}_{i-1} \in U\}$ put $c' := c_B(i-2)$ and $m := i-2-c'$. Then, on this event,*

$$S_i \geq f(m), \quad f(m) := \frac{m}{n-2} + \frac{m^2/(n-2) - m}{2(n-2)^2}.$$

The function f is increasing on $[0, \infty)$, and for $0 \leq c' \leq i-2$,

$$f(i-2-c') \geq \frac{i}{n} + \frac{(i-2)^2}{2n^3} - \beta_1 - c'\beta_2, \quad \beta_1 := \frac{2}{n-2} + \frac{i-2}{2(n-2)^2}, \quad \beta_2 := \frac{1}{n-2} + \frac{i-2}{(n-2)^3}. \quad (3)$$

Proof. On $\{\tilde{X}_{i-1} \in U\}$ the walk is not absorbed before time $i-1$, so $h_s < 1$ for $s \leq i-2$. Since $-\log(1-h) \geq h \geq 0$, $S_i \geq \sum h_s$, the sum over those $s \leq i-2$ with $\tilde{X}_s \notin B$. For such s , $\tilde{X}_s \notin A$ and $\tilde{X}_s \neq \tilde{X}_1$. A vertex other than $\tilde{X}_1$ can be entered only through its $n-2$ in-arcs other than the one from v , so $j_s \leq n-2$ and $h_s = 1/(n-1-j_s)$. For $1 \leq j \leq n-2$,

$$\frac{1}{n-1-j} = \frac{1}{n-2} \left(1 - \frac{j-1}{n-2}\right)^{-1} \geq \frac{1}{n-2} + \frac{j-1}{(n-2)^2}.$$

A vertex $w \in U \setminus B$ is visited at times $\leq i-2$ with indices $1, \dots, k_w$, where $k_w := k_w(i-2)$, and $\sum_{w \in U \setminus B} k_w = m$. Therefore

$$S_i \geq \sum_{w \in U \setminus B} \sum_{r=1}^{k_w} \left(\frac{1}{n-2} + \frac{r-1}{(n-2)^2} \right) = \frac{m}{n-2} + \sum_{w \in U \setminus B} \frac{k_w^2 - k_w}{2(n-2)^2}.$$

As $\tilde{X}_1 \in B$, we have $|U \setminus B| \leq n-2$, so the Cauchy–Schwarz inequality gives $\sum_w k_w^2 \geq m^2/(n-2)$, and $S_i \geq f(m)$ follows.

For $m \geq 0$ and $n \geq 3$, $f'(m) = 1/(n-2) - 1/(2(n-2)^2) + m/(n-2)^3 > 0$. For (3) write $a := i-2$, so $m = a - c'$. Using $(a - c')^2 \geq a^2 - 2ac'$ and $m \leq a$,

$$f(m) \geq \frac{a}{n-2} + \frac{a^2}{2(n-2)^3} - \frac{a}{2(n-2)^2} - c'\beta_2.$$

Finally $a/(n-2) \geq i/n - 2/(n-2)$, because $i/n - (i-2)/(n-2) = 2(n-i)/(n(n-2)) \leq 2/(n-2)$, and $a^2/(2(n-2)^3) \geq (i-2)^2/(2n^3)$. $\square$

4 Proof of Theorem 1.1

Fix $K \geq 1$. For n below any fixed threshold $n_0(K)$ the left side of Theorem 1.1 is bounded, since $\mathbb{E}N_i \leq n-1$ and $\exp(i/n + i^2/(2n^3)) \leq \exp(K\sqrt{n} + K^2/2)$. So it suffices to prove the bound for $n \geq n_0(K)$, where $n_0(K)$ is chosen so large that all the inequalities marked “for large n ” below hold. Let

$$L := \lceil \sqrt{n} \log n \rceil, \quad q := \frac{1}{n-2-L}, \quad 2 \leq i \leq Kn^{3/2}, \quad G := \{M_{i-1} \leq L\}.$$

For large n we have $L \leq n/4$, hence $q \leq 2/n$ and $1/(n-1-L) \leq 2/n$.

The event G . By Lemma 3.2(b) with $N = i-1$,

$$\mu(G^c) \leq (n-1) \left(\frac{e(i-1)q}{L+1} \right)^{L+1} \leq n \left(\frac{2eK}{\log n} \right)^{L+1} \leq n e^{-2\sqrt{n} \log n}$$

for large n , since $e(i-1)q/(L+1) \leq eKn^{3/2}(2/n)/(\sqrt{n}\log n)$. As $e^{-i/n-i^2/(2n^3)} \geq e^{-K\sqrt{n}-K^2/2}$, this gives, for large n ,

$$(n-1)\mu(G^c) \leq n^{-1/2} e^{-i/n-i^2/(2n^3)}. \quad (4)$$

On G , every $s \leq i-1$ has $j_s \leq M_{i-1} \leq L \leq n-3$. Hence on G the walk is not absorbed before time $i-1$, and for $s \leq i-1$ we have $h_s = 0$ if $\tilde{X}_s \in A$ and $h_s = 1/(n-1-j_s) \leq 2/n$ if $\tilde{X}_s \notin A$.

Lower bound. Let $E := G \cap \{\tilde{X}_{i-1} \notin A\}$; on E , $h_{i-1} \geq 1/(n-2)$. For $1 \leq j \leq L$,

$$\frac{1}{n-1-j} \leq \frac{1}{n-2} + \alpha(j-1), \quad \alpha := \frac{1}{(n-2)(n-1-L)}.$$

Using $-\log(1-h) \leq h + h^2$ for $0 \leq h \leq 1/2$, we get on G

$$S_i \leq \sum_{s=1}^{i-2} (h_s + h_s^2) \leq \frac{i-2}{n-2} + \frac{4i}{n^2} + \alpha P, \quad P := \sum_{s=1}^{i-2} (j_s - 1).$$

(P is the number of pairs $t < s \leq i-2$ with $\tilde{X}_t = \tilde{X}_s$; on G it is well defined.) We have $j_1 = 1$. For $2 \leq s \leq i-2$, on $\{M_{s-1} \leq L, \tilde{X}_{s-1} \in U\}$, Lemma 3.1 gives

$$\tilde{\mathbb{E}}_T[k_{\tilde{X}_s}(s-1) \mid \mathcal{F}_{s-1}] = \sum_{u \in U} \tilde{\mathbb{P}}_T(\tilde{X}_s = u \mid \mathcal{F}_{s-1}) k_u(s-1) \leq q \sum_{u \in U} k_u(s-1) = q(s-1),$$

and $j_s - 1 = k_{\tilde{X}_s}(s-1)$ on $\{\tilde{X}_s \in U\}$. Since $G \subseteq \{M_{s-1} \leq L\} \cap \{\tilde{X}_s \in U\}$ and all terms are nonnegative, $\mu[P \mathbf{1}_G] \leq q \sum_{s=2}^{i-2} (s-1) \leq qi^2/2$. Moreover

$$\mu(E^c) \leq \mu(G^c) + \mu(\tilde{X}_{i-1} \in A, M_{i-2} \leq L) \leq \mu(G^c) + q\mathbb{E}C \leq 5/n$$

for large n , by Lemma 3.1 (or the uniformity of $\tilde{X}_1$ if $i=2$) and Lemma 3.3. By (2) and Jensen's inequality for the conditional law $\mu(\cdot \mid E)$,

$$\mathbb{E}N_i \geq (n-1)\mu[W_i \mathbf{1}_E] \geq \frac{n-1}{n-2} e^{-\frac{i-2}{n-2} - \frac{4i}{n^2}} \mu(E) \exp\left(-\frac{\alpha \mu[P \mathbf{1}_G]}{\mu(E)}\right).$$

Now $(i-2)/(n-2) \leq i/n + 3i/n^2$ for $n \geq 6$, so the first factors are at least $e^{-i/n-7Kn^{-1/2}}$. Further $\mu(E) \geq 1-5/n$ and $\alpha q = ((n-2)(n-1-L)(n-2-L))^{-1} = n^{-3}(1+O(L/n))$, so $\alpha \mu[P \mathbf{1}_G]/\mu(E) \leq \frac{i^2}{2n^3}(1+O(L/n)) = \frac{i^2}{2n^3} + O(K^2L/n)$. Hence

$$\mathbb{E}N_i \geq e^{-i/n-i^2/(2n^3)} (1 - O_K(n^{-1/2} \log n)).$$

Upper bound. Since $W_i \leq 1$, $\mathbb{E}N_i \leq (n-1)\mu[W_i \mathbf{1}_G] + (n-1)\mu(G^c)$. On G we have $h_{i-1} \leq 1/(n-1-L)$, and $c' \leq c_A(i-2) + k_{\tilde{X}_1}(i-2) \leq c_A(i-2) + L$. For large n , $\beta_1 \leq 2Kn^{-1/2}$ and $\beta_2 \leq 2/n$ in (3), and $(i-2)^2/(2n^3) \geq i^2/(2n^3) - 2i/n^3 \geq i^2/(2n^3) - Kn^{-1/2}$. So Lemma 3.4 gives on G

$$S_i \geq \frac{i}{n} + \frac{i^2}{2n^3} - 3Kn^{-1/2} - \frac{2L}{n} - \frac{2}{n} c_A(i-2),$$

and therefore

$$W_i \mathbf{1}_G \leq \frac{1}{n-1-L} e^{-i/n-i^2/(2n^3)} e^{3Kn^{-1/2}+2L/n} e^{(2/n)c_A(i-2)} \mathbf{1}_G.$$

If $i \geq 3$ then $G \subseteq \{M_{i-3} \leq L\}$, and Lemma 3.2(a) with $A' = A$, $a = C$, $N = i-2$ and $\lambda = 2/n$ (so $e^\lambda - 1 \leq 4/n$) gives

$$\tilde{\mathbb{E}}_T[e^{(2/n)c_A(i-2)} \mathbf{1}_G] \leq \exp\left(Cq(i-2)\frac{4}{n}\right) \leq \exp(8KCn^{-1/2});$$

for $i = 2$ the left side is at most 1. By Lemma 3.3 with $\theta = 8Kn^{-1/2} \leq 1$, $\mathbb{E}_T \exp(8KCn^{-1/2}) \leq \exp(\theta + e^\theta - 1) \leq e^{3\theta}$. With $(n-1)/(n-1-L) \leq 1 + 2L/n$ and (4),

$$\begin{aligned} \mathbb{E}N_i &\leq \left(1 + \frac{2L}{n}\right) e^{-i/n-i^2/(2n^3)} e^{27Kn^{-1/2}+2L/n} + n^{-1/2} e^{-i/n-i^2/(2n^3)} \\ &= e^{-i/n-i^2/(2n^3)} (1 + O_K(n^{-1/2} \log n)). \end{aligned}$$

This proves Theorem 1.1. $\square$

Remark 4.1. The choice of L balances two requirements: L must be large compared with the typical number $i/n \leq K\sqrt{n}$ of visits to a vertex, so that $\mu(G^c)$ is negligible, and L/n must be small. The Monte Carlo data of Section 7 suggest that at fixed $y = i/n^{3/2}$ the true relative error is of order $n^{-1/2}$, so the logarithm is probably not needed.

5 Proof of Theorem 1.2

Let $n \geq 23$ and $i \geq 2$. By (2), $\mathbb{E}N_i e^{i/n} = (n-1) e^{i/n} \mu[W_i]$.

Range I: $2 \leq i \leq n^2/64$. Let $L := \lfloor n/4 \rfloor$, so that $1 \leq L \leq n-3$, $q = 1/(n-2-L) \leq 2/n$ (as $n \geq 8$) and $1/(n-1-L) \leq 2/n$, and let $G := \{M_{i-1} \leq L\}$. On G the walk is not absorbed before time $i-1$, $h_{i-1} \leq 2/n$, and $c' \leq c_A(i-2) + L \leq c_A(i-2) + n/4$. Since $i-2 \leq n^2/64$ and $n \geq 10$, we have $\beta_1 \leq 1$ and $\beta_2 \leq 2/n$ in (3). So Lemma 3.4 gives $S_i \geq i/n + (i-2)^2/(2n^3) - 3/2 - (2/n)c_A(i-2)$ on G , and

$$(n-1) e^{i/n} \mu[W_i \mathbf{1}_G] \leq 2e^{3/2} e^{-(i-2)^2/(2n^3)} \mu[e^{(2/n)c_A(i-2)} \mathbf{1}_G].$$

As in Section 4, Lemma 3.2(a) with $\lambda = 2/n$ and $e^{2/n} - 1 \leq 4/n$ gives $\tilde{\mathbb{E}}_T[e^{(2/n)c_A(i-2)} \mathbf{1}_G] \leq \exp(8C(i-2)/n^2) \leq e^{C/8}$, and by Lemma 3.3, $\mathbb{E}e^{C/8} \leq \exp(1/8 + e^{1/8} - 1) < 1.2946$. Since $2e^{3/2} \cdot 1.2946 < 11.61 < 12$, the main term is at most $12e^{-(i-2)^2/(2n^3)}$. For the rest, Lemma 3.2(b) with $N = i-1$, $e(i-1)q/(L+1) \leq e(n^2/64)(2/n)(4/n) = e/8$ and $L+1 \geq n/4$ give

$$(n-1) e^{i/n} \mu(G^c) \leq (n-1)^2 e^{n/64} (e/8)^{n/4} \leq n^2 e^{n/64} (e/8)^{n/4}.$$

Range II: $i > n^2/64$. Then $i \geq 9$ because $n \geq 23$. We use $h_{i-1} \leq 1$. Every vertex is visited at most $n-1$ times and $|B| \leq C+1$, so $c' \leq (C+1)(n-1)$ and $m \geq m_* := \max\{0, i-2-(C+1)(n-1)\}$. By Lemma 3.4 and the monotonicity of f , $S_i \geq f(m_*)$ on $\{\tilde{X}_{i-1} \in U\}$, and $f(m) \geq m/n + m^2/(2n^3)$ for $m \geq 0$, because $(2n-5)/(2(n-2)^2) \geq 1/n$ for $n \geq 3$. In both cases below, $i-m_* \leq 2+(C+1)(n-1)$ (if $m_* = 0$ then $i \leq 2+(C+1)(n-1)$), so $(i-m_*)/n \leq C+2$.

- If $C+1 \leq i/(4n)$, then $(C+1)(n-1) < i/4$ and $m_* \geq 3i/4 - 2 \geq i/2$, so

$$\frac{i}{n} - S_i \leq \frac{i-m_*}{n} - \frac{m_*^2}{2n^3} \leq C+2 - \frac{i^2}{8n^3} < C+2 - \frac{n}{32768},$$

because $i^2/(8n^3) > (n^2/64)^2/(8n^3) = n/32768$.

- Otherwise $C+1 > i/(4n) > n/256$, so $C \geq \lfloor n/256 \rfloor$, and $i/n - S_i \leq (i-m_*)/n \leq C+2$.

Hence

$$\mathbb{E}N_i e^{i/n} \leq (n-1) \mu[e^{i/n-S_i} \mathbf{1}\{\tilde{X}_{i-1} \in U\}] \leq (n-1) \left(e^{-n/32768} \mathbb{E}e^{C+2} + \mathbb{E}[e^{C+2}; C \geq \lfloor n/256 \rfloor] \right).$$

By Lemma 3.3, $\mathbb{E}e^{C+2} \leq e^{2+e} < e^{4.72}$, and, since $C \geq 1$ and $\sum_{r \geq M} x^r/r! \leq (x^M/M!)e^x$,

$$\mathbb{E}[e^{C+2}; C \geq \lfloor n/256 \rfloor] \leq \sum_{c \geq \max\{\lfloor n/256 \rfloor, 1\}} \frac{e^{c+2}}{(c-1)!} \leq e^{3+e} \frac{e^{M_n}}{M_n!}.$$

Combining the two ranges proves the inequality of Theorem 1.2. The first term of ε_n tends to 0 because $(e/8)^{1/4}e^{1/64} < 0.78$, the second clearly, and the third because $e^{M_n}/M_n! \leq (e^2/M_n)^{M_n}$ decays faster than any exponential in n . If $i/n^{3/2} \rightarrow \infty$, then $(i-2)^2/(2n^3) \rightarrow \infty$ and $\mathbb{E}N_i e^{i/n} \rightarrow 0$. Finally, $\sum_{i>n^{3/2}} \mathbb{E}N_i \leq (12 + \varepsilon_n) \sum_{i>n^{3/2}} e^{-i/n} = O(ne^{-\sqrt{n}})$, as claimed after Corollary 1.3. $\square$

6 Proof of Corollary 1.3

(a) If $y < \infty$, apply Theorem 1.1 with $K > y$: then $\mathbb{E}N_i e^{i/n} = e^{-i^2/(2n^3)}(1 + o(1)) \rightarrow e^{-y^2/2}$. If $y = \infty$, apply Theorem 1.2. If $i/n^{3/2} \rightarrow 0$ the limit is 1. If $i/n^{3/2}$ does not tend to 0, some subsequence has $i/n^{3/2} \rightarrow y \in (0, \infty]$, along which the ratio tends to $e^{-y^2/2} < 1$. The uniform statement is Theorem 1.1 with $K = 1$ together with $1 - e^{-t} \leq t$.

(b) Let $\delta_n := C_1 n^{-1/2} \log n$ be the bound of Theorem 1.1 with $K = 1$. For $2 \leq i \leq n^{3/2}$,

$$|\mathbb{E}N_i - e^{-i/n}| \leq e^{-i/n}(1 - e^{-i^2/(2n^3)} + \delta_n) \leq e^{-i/n} \left(\frac{i^2}{2n^3} + \delta_n \right).$$

Put $r := e^{-1/n}$. Since $1 - r \geq 1/(n+1)$, we have $\sum_{i \geq 1} r^i \leq n$ and $\sum_{i \geq 1} i^2 r^i = r(1+r)/(1-r)^3 \leq 2(n+1)^3$. Hence $\sum_{2 \leq i \leq n^{3/2}} |\mathbb{E}N_i - e^{-i/n}| \leq (1+1/n)^3 + \delta_n n \leq 3 + \delta_n n$. Next, $|\mathbb{E}N_1 - e^{-1/n}| \leq 1$ and $\sum_{i>n^{3/2}} r^i \leq (n+1)e^{-\sqrt{n}} \leq 1$. Finally, since $\sum_i \mathbb{E}N_i = n-1$ and $\sum_{i \geq 1} r^i = 1/(e^{1/n} - 1) \geq n-1/2$,

$$\sum_{i>n^{3/2}} \mathbb{E}N_i = (n-1) - \sum_{2 \leq i \leq n^{3/2}} \mathbb{E}N_i \leq (n-1) - \sum_{2 \leq i \leq n^{3/2}} r^i + 3 + \delta_n n \leq (n-1) - \left(n - \frac{5}{2}\right) + 3 + \delta_n n.$$

Adding up, $\sum_{i \geq 1} |\mathbb{E}N_i - e^{-i/n}| \leq 10 + 2\delta_n n = O(n^{1/2} \log n)$. Only Theorem 1.1 was used.

(c) Given $\mathcal{C}$, a uniformly chosen excursion has length i with probability $N_i/(n-1)$, so $\mathbb{P}(\Lambda_n = i) = \mathbb{E}N_i/(n-1)$ and $\mathbb{E}\Lambda_n = \sum_i i \mathbb{E}N_i/(n-1) = n$. For $x \geq 0$ let $I := \lfloor xn \rfloor$. By (b),

$$\left| \mathbb{P}(\Lambda_n \leq xn) - \frac{1}{n-1} \sum_{i=1}^I e^{-i/n} \right| = O(n^{-1/2} \log n),$$

and $\frac{1}{n-1} \sum_{i=1}^I e^{-i/n} = \frac{1-e^{-I/n}}{(n-1)(e^{1/n}-1)} = (1-e^{-x}) + O(1/n)$ uniformly in $x \geq 0$, because $(n-1)(e^{1/n}-1) = 1 + O(1/n)$ and $0 \leq e^{-I/n} - e^{-x} \leq 1/n$. $\square$

7 Numerical evidence

The computations in this section are checks only; none of them is used in the proofs. The exact values come from complete enumerations; the Monte Carlo values are statistical estimates and are labelled as such.

Exact values. For $3 \leq n \leq 6$, $\mathbb{E}N_i$ is known exactly from complete enumeration and from the formula of Remark 2.5 (see the Verification paragraph below). For example $\mathbb{E}N_5 = 1624/3375$ for $n = 5$, and $\mathbb{E}N_2, \dots, \mathbb{E}N_5 = 5/6, 35/48, 5/8, 7405/13824$ for $n = 6$. The values agree with Remark 2.6 and satisfy $\sum_i \mathbb{E}N_i = n-1$ and $\sum_i i \mathbb{E}N_i = n(n-1)$.

Monte Carlo (tested, not proved). Formula (2) yields an unbiased importance-sampling estimator of $\mathbb{E}N_i$ for all i at once: sample a uniform tree and run the auxiliary walk, recording the weights W_i . It reproduces the exact values at $n = 5$ within about two standard errors, and it estimates the mean excursion length as n to within 0.02% for $60 \leq n \leq 6400$. Table 1 shows $r := \mathbb{E}N_i e^{i/n}$ at $i = yn^{3/2}$; the relative standard errors are below $2 \cdot 10^{-3}$. Direct simulation of complete uniform circuits, without

$y = i/n^{3/2}$	$n = 100$	$n = 400$	$n = 1600$	$n = 6400$	limit $e^{-y^2/2}$
0.25	0.991	0.978	0.973	0.971	0.969
0.5	0.909	0.893	0.887	0.885	0.882
1	0.617	0.611	0.609	0.608	0.607
1.5	0.305	0.317	0.321	0.323	0.325
2	0.105	0.122	0.129	0.133	0.135
3	0.0035	0.0070	0.0090	0.0101	0.0111

Table 1: Monte Carlo estimates of $r = \mathbb{E}N_i e^{i/n}$ at $i = yn^{3/2}$, and the limit given by Corollary 1.3(a).

importance sampling, gives $r \approx 0.991, 0.909, 0.616, 0.306$ at $y = 0.25, 0.5, 1, 1.5$ for $n = 100$ ($8 \cdot 10^6$ circuits), in agreement with the table.

The data suggest the following, none of which is proved here.

- At fixed y , the ratio $r/e^{-y^2/2}$ approaches 1 at rate about $n^{-1/2}$: the deviation roughly halves when n is quadrupled. The next term $-i^3/(3n^5) = -y^3/(3\sqrt{n})$ of the heuristic exponent below accounts for most of the deviation.
- For $i = xn$ with x fixed, $r = 1 + O(1/n)$. For example, at $n = 1600$ the values are 1.0008, 1.0000, 0.9970 and 0.9831 for $x = 1, 2, 4, 8$, while $\mathbb{E}N_i e^{i/n + i^2/(2n^3)}$ lies between 1.001 and 1.003.
- The heuristic formula $\mathbb{E}N_i \approx (1 - i/n^2)^n$, whose exponent is $-i/n - i^2/(2n^3) - i^3/(3n^5) - \dots$, is within about 2–5% of the estimates at $y = 3$ for $100 \leq n \leq 6400$.
- The sum in Corollary 1.3(b) seems to stay bounded: the estimates give $\sum_i |\mathbb{E}N_i - e^{-i/n}| \approx 2.2$ at both $n = 100$ and $n = 400$.

8 Concluding remarks

The argument uses little beyond the facts that the walk of Lemma 2.2 steps to v with an explicit hazard and that visit counts stay small. We expect that it can be adapted to other dense regular graphs. Example (b) of [AY14], on the Hamming cube, and the main torus conjecture of [AY14] are not addressed here. Two natural questions remain.

Problem 8.1. *Find the second-order term in Theorem 1.1, and the behaviour of $\mathbb{E}N_i$ for $n^{3/2} \ll i \leq n(n-1)$, where Theorem 1.2 gives only an upper bound. How well does the heuristic $\mathbb{E}N_i \approx (1 - i/n^2)^n$ of Section 7 describe $\mathbb{E}N_i$ in this range?*

Problem 8.2. *Describe the joint law of the excursion lengths, for instance the fluctuations of N_i and the length of the longest excursion.*

Verification. The claims were checked as follows.

1. The finder’s program `exact_small.py` (exact rational arithmetic) enumerates all Eulerian circuits for $n = 4, 5$ and compares $\mathbb{E}N_i$ with two independent dynamic programs, one for the walk of Lemma 2.2 and one for the right side of Lemma 2.4, for every tree. All values agree exactly.
2. An independent referee wrote separate code: a brute force over all Eulerian circuits for $n = 3, 4, 5$ that counts excursions from every vertex; an exact integer evaluation of the formula of Remark 2.5, with the directed matrix-tree theorem, for $n = 3, \dots, 6$, which agrees with the

brute force for $n \leq 5$; and, for $n = 6$, dynamic programs over all 1296 trees for the literal walk and for Lemma 2.4, which agree with the exact values to within $2 \cdot 10^{-12}$ in floating point. At every reachable state these programs also check formula (1), the count of used out-arcs in Lemma 2.3(i), and that $j_s = n - 1$ occurs only at X_1 .

3. The author’s scripts check the closed forms of Remark 2.6 against these exact values. They verify $\mathbb{E}C = 2 - 2/n$ and $\mathbb{E}D_2 = 3(n - 1)(n - 2)/n^2$ by enumerating all trees for $n \leq 8$. They also check every numerical constant and elementary inequality of Sections 3 to 5, in exact rational arithmetic on finite ranges. For example, (3) is checked for all $n \leq 30$, all $i \leq n(n - 1)$ and all $c' \leq i - 2$.
4. Monte Carlo (tested only): the finder’s importance sampler, two independent importance samplers written by two referees, and two direct samplers of uniform circuits, with trees from the random-walk algorithm of Aldous and Broder [Ald90, Bro89] and from Wilson’s algorithm [Wil96]. The finder’s sampler and both of the second referee’s samplers were validated against the exact values for $n = 5$ or $n = 6$, and so was the first referee’s direct sampler. The two referees’ importance samplers reproduce Table 1 for $n \geq 400$ within statistical error. One direct sampler confirms it for $y \leq 1.5$ at $n = 100$; the other agrees with the finder’s sampler at $n = 60$.
5. Two independent referees checked all proofs line by line; the second rederived the supermartingale bound of Lemma 3.2, inequality (3) and all constants in Section 5. In the version they checked, the proof of Theorem 1.1 used $L = \lfloor n^{3/4} \rfloor$, which gives the error $O(n^{-1/4})$; the present choice $L = \lceil \sqrt{n} \log n \rceil$ changes only the error terms, and these were rederived for this note.

Scope and priority. Theorems 1.1 and 1.2 and Corollary 1.3 answer example (a) of Aldous and Yu [AY14, p. 4]: the conjectured asymptotics hold exactly in the range $i = o(n^{3/2})$, and they fail at and beyond the scale $n^{3/2}$. The main torus conjecture of [AY14], its two-dimensional variant and example (b) on the Hamming cube remain open. Aldous’s open-problem page [Ald] lists the whole topic “Random Eulerian circuits”, with his rating 2.2 on a conceptual–technical spectrum; this rating is not a problem number, and the page does not single out example (a). We searched arXiv, Crossref, zbMATH, OpenAlex and Semantic Scholar, and made web searches. The OpenAlex and Semantic Scholar queries, including a search for works citing [AY14], were refused in all our attempts (by rate limits, and by a temporary suspension of anonymous search), so no systematic citation search was possible; a web search for works citing [AY14] found nothing new. We found no result on excursion lengths of uniform Eulerian circuits of the complete graph, and no statement of the correction factor in Theorem 1.1. The related work we found concerns counting [MR98, Isa11, Isa13, CC13], sampling [KMUW96, TV01, Ana26], rotor-router walks [PP13] and Markov loops [LJ14]. This negative search is not a proof of priority.

AI-assisted tools were used to search the problem corpus and the literature, to find the proofs, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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