

Euler Numbers Modulo Powers of Two and Arnold's Sequence: Proof of Two Conjectures of Ramassamy

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Abstract

Let E_n be the Euler up/down numbers, $\sum_n E_n x^n/n! = \sec x + \tan x$, and let $e_{n,i}$ be the Entringer numbers, which form the Seidel–Entringer–Arnold triangle. Arnold observed, without proof, that the least 2-adic valuation m_i of the entries on the i -th diagonal of this triangle is weakly increasing in i , and he introduced the sequence $u_k = \max\{i : m_i < k\}$. In 2017 Ramassamy conjectured that the sequence $(E_n \bmod 2^k)_{n \geq 0}$ is periodic from the index u_k on but not from any earlier index, that its least period is 2^k for $k \neq 2$ and 2 for $k = 2$, and that (u_k) is the f -transform of $(2, 4, 4, 4)$ for an explicit doubling map f . We prove these two conjectures. With $h(j) = 2j - 2 - v_2(j)$, the 2-adic valuation of the tangent number E_{2j-1} , we show that $m_i = \min_{j \geq \lceil i/2 \rceil} h(j)$ for every i , which proves Arnold's observation, and that $u_k = 2 \max\{j : h(j) < k\}$. The statements about the period, and the value $2 \max\{j : h(j) < k\}$ of the preperiod, follow quickly from Stern's classical congruence for the Euler numbers and from the valuation of the tangent numbers. The new ingredients are the determination of m_i and u_k and the proof of the f -transform identity, which rests on the self-similarity $h(2^{a-1} + r) = h(r) + 2^a$ for $1 \leq r < 2^{a-1}$. The proofs are elementary, and computations serve only as consistency checks. This is an unrefereed note.

1 Introduction

The Euler up/down numbers E_n are defined by

$$\sum_{n \geq 0} E_n \frac{x^n}{n!} = \sec x + \tan x,$$

so that $(E_n)_{n \geq 0} = 1, 1, 1, 2, 5, 16, 61, 272, 1385, \dots$. We write $T_{2j-1} = E_{2j-1}$ ($j \geq 1$) for the tangent numbers and $S_{2m} = E_{2m}$ ($m \geq 0$) for the secant numbers. Knuth and Buckholtz [KB67] proved that for every integer $q \geq 1$ the sequence $(E_n \bmod q)_{n \geq 0}$ is eventually periodic. Following Ramassamy [Ram17], let $s(q)$ be the least number of initial terms whose deletion makes $(E_n \bmod q)_{n \geq 0}$ periodic, and let $d(q)$ be the least period of $(E_n \bmod q)_{n \geq s(q)}$.

A permutation σ of $[n] = \{1, \dots, n\}$ is *up/down* if $\sigma(1) < \sigma(2) > \sigma(3) < \dots$, that is, $\sigma(i-1) < \sigma(i)$ for even i and $\sigma(i-1) > \sigma(i)$ for odd i . By André's theorem [And79], E_n is the number of up/down permutations of $[n]$. For $1 \leq i \leq n$ the *Entringer number* $e_{n,i}$ [Ent66] is the number of up/down permutations σ of $[n]$ with $\sigma(n) = i$. These numbers satisfy $e_{1,1} = 1$ and, for $n \geq 2$ and $1 \leq i \leq n$,

$$e_{n,i} = \sum_{l < i} e_{n-1,l} \quad (n \text{ even}), \quad e_{n,i} = \sum_{l \geq i} e_{n-1,l} \quad (n \text{ odd}) \quad (1)$$

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(Lemma 2.1). Arranged in rows they form the Seidel–Entringer–Arnold triangle, whose rows 1 to 6 are 1; 0, 1; 1, 1, 0; 0, 1, 2, 2; 5, 5, 4, 2, 0; 0, 5, 10, 14, 16, 16. Let v_2 be the 2-adic valuation, with $v_2(0) = \infty$. For $i \geq 1$ let $D_i = (e_{n,i})_{n \geq i}$ be the i -th diagonal of the triangle parallel to its left boundary; in the indexing (1) it is the i -th column. (This reading is fixed by the example in [Ram17]: the valuations along D_1 are $0, \infty, 0, \infty, 0, \dots$) Put $m_i = \min\{v_2(e) : e \in D_i\}$. According to Ramassamy [Ram17, Sect. 3.2], Arnold [Arn91] observed, without proof, that (m_i) is weakly increasing and tends to infinity, and he introduced the sequence

$$u_k = \max\{i \geq 1 : m_i < k\} \quad (k \geq 1),$$

which is, up to a shift of the index, sequence A108039 of the OEIS [OEI26] (see Remark 6.1). In words, u_k is the number of diagonals that contain an entry not divisible by 2^k [Ram17, Sect. 3.2]. Ramassamy [Ram17] stated three conjectures. The first one, on odd prime powers, has been settled by Güleç in a recent preprint [Güleç26]; see Remark 6.2. The other two are as follows.

Conjecture 2 ([Ram17]). *For every $k \geq 1$, $s(2^k) = u_k$. Moreover $d(2^k) = 2^k$ for every $k \geq 1$ with $k \neq 2$, and $d(4) = 2$.*

For the third conjecture, let $\mathbb{Z}_+$ be the set of nonnegative integers. Ramassamy defines a map f from $\bigsqcup_{L \geq 1} \mathbb{Z}_+^L$ to itself that sends $\mathbb{Z}_+^L$ to $\mathbb{Z}_+^{2L}$. Let $x = (x_1, \dots, x_L)$. If $x_i = x_L$ for all i , then $f(x) = (x_1, \dots, x_L, 2x_L, \dots, 2x_L)$ with L copies of $2x_L$. Otherwise let $\ell = \max\{1 \leq i \leq L-1 : x_i \neq x_L\}$ (this index is called s in [Ram17]) and

$$f(x) = (x_1, \dots, x_L, x_1 + x_L, \dots, x_{\ell-1} + x_L, 2x_L, \dots, 2x_L),$$

with $L - \ell + 1$ copies of $2x_L$. For example $f(2, 4, 4, 4) = (2, 4, 4, 4, 8, 8, 8, 8)$ and $f(2, 4, 4, 4, 8, 8, 8, 8) = (2, 4, 4, 4, 8, 8, 8, 8, 10, 12, 12, 16, 16, 16, 16, 16)$. Since $f(x)$ begins with x , there is a unique infinite sequence, the f -transform of x , whose k -th term is the k -th term of $f^n(x)$ whenever $2^n L \geq k$ [Ram17, Lemma 3].

Conjecture 3 ([Ram17]). *Arnold’s sequence $(u_k)_{k \geq 1}$ is the f -transform of $(2, 4, 4, 4)$.*

Ramassamy reports numerical support for Conjecture 2 for $k \leq 12$ and for Conjecture 3 for $k \leq 512$. The two conjectures are the records AMR-090-0002 and AMR-090-0003 of the UnsolvedMath dataset [ula26].

Results. For $j \geq 1$ put

$$h(j) = 2j - 2 - v_2(j),$$

and for $i, k \geq 1$ put

$$G(i) = \min_{j \geq \lceil i/2 \rceil} h(j), \quad J(k) = \max\{j \geq 1 : h(j) < k\}.$$

By Lemma 2.2, $h(j) = v_2(T_{2j-1})$. The minimum and the maximum exist by Lemma 2.3.

Theorem 1.1. *For every $i \geq 1$ we have $m_i = G(i)$. Consequently (m_i) is weakly increasing and tends to infinity, $m_{2t-1} = m_{2t}$ for every $t \geq 1$, and for every $k \geq 1$*

$$\{i \geq 1 : m_i < k\} = \{1, 2, \dots, 2J(k)\}.$$

In particular $u_k = 2J(k) = 2 \max\{j \geq 1 : 2j - 2 - v_2(j) < k\}$, and u_k is also the number of diagonals D_i that contain an entry not divisible by 2^k .

j	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$h(j)$	0	1	4	4	8	9	12	11	16	17	20	20	24	25	28	26	32	33	36	36
$m_{2j-1} = m_{2j}$	0	1	4	4	8	9	11	11	16	17	20	20	24	25	26	26	32	33	36	36
k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
u_k	2	4	4	4	8	8	8	8	10	12	12	16	16	16	16	16	18	20	20	20
k	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
u_k	24	24	24	24	26	28	32	32	32	32	32	32	34	36	36	36	40	40	40	40

Table 1: The values $h(j) = v_2(T_{2j-1})$, the diagonal minima $m_{2j-1} = m_{2j} = G(2j)$ (Theorem 1.1) and Arnold's sequence $u_k = 2J(k)$. The values $u_1, \dots, u_{18}$ are those of [Ram17, Table 1].

Theorem 1.2. *For every $k \geq 1$ we have $s(2^k) = 2J(k) = u_k$. Moreover $d(2^k) = 2^k$ for every $k \geq 1$ with $k \neq 2$, and $d(4) = 2$. Thus Conjecture 2 holds.*

Theorem 1.3. *For every $a \geq 2$ we have $f(u_1, \dots, u_{2^a}) = (u_1, \dots, u_{2^{a+1}})$, and at this step the index ℓ in the definition of f equals $2^a - a - 1$. Since $(u_1, u_2, u_3, u_4) = (2, 4, 4, 4)$, the f -transform of $(2, 4, 4, 4)$ is $(u_k)_{k \geq 1}$. Thus Conjecture 3 holds.*

Table 1 lists the first values. The statements about $d(2^k)$ in Theorem 1.2, and the equality $s(2^k) = 2J(k)$, are quick consequences of two classical facts: Stern's congruence for the Euler numbers [Ste75] (Theorem 2.4) and the 2-adic valuation of the tangent numbers (Lemma 2.2). We claim no novelty for them. The new content is Theorem 1.1, which proves Arnold's observation and identifies u_k with $2J(k)$, and Theorem 1.3. Besides the recurrence (1), the proofs use only André's theorem, the expansion of $\tan x$ in Bernoulli numbers (that is, $T_{2j-1} = 2^{2j}(2^{2j} - 1)|B_{2j}|/(2j)$), the theorem of von Staudt and Clausen, and Stern's theorem; for the last one we include a short proof in Appendix A. For the equality $u_k = 2J(k)$ alone, which is all that Theorems 1.2 and 1.3 need, a lower bound and a single explicit entry suffice (Remark 3.3).

2 Preliminaries

Lemma 2.1. *The numbers $e_{n,i}$ satisfy (1), and $\sum_{i=1}^n e_{n,i} = E_n$ for every $n \geq 1$. Consequently, for $n \geq 2$ and $1 \leq i \leq n$,*

$$e_{n,i} = \sum_{l=i}^{n-1} e_{n-1,l} \quad (n \text{ odd}), \quad e_{n,i} = E_{n-1} - \sum_{l=i}^{n-1} e_{n-1,l} \quad (n \text{ even}). \quad (2)$$

In particular $e_{n,n} = 0$ for odd $n \geq 3$, and $e_{n,n} = E_{n-1}$ for even n .

Proof. Let $n \geq 2$, and let σ be an up/down permutation of $[n]$ with $\sigma(n) = i$. Delete the last entry and replace every value $v > i$ by $v - 1$. This relabelling preserves the relative order of $\sigma(1), \dots, \sigma(n-1)$, so it gives an up/down permutation τ of $[n-1]$. If n is even, then $\sigma(n-1) < i$, so $\tau(n-1) = \sigma(n-1) \leq i-1$. If n is odd, then $\sigma(n-1) > i$, so $\tau(n-1) = \sigma(n-1) - 1 \geq i$. Conversely, let τ be an up/down permutation of $[n-1]$ with $\tau(n-1) \leq i-1$ if n is even, or with $\tau(n-1) \geq i$ if n is odd. Increase every value $v \geq i$ by one and append the value i . The result is an up/down permutation σ of $[n]$ with $\sigma(n) = i$, and the two constructions are inverse to each other. This proves (1). Summing over i , the sum $\sum_i e_{n,i}$ counts all up/down permutations of $[n]$, and this number is E_n by André's theorem [And79]; see also [Sta10, Sta12]. Row $n-1$ consists of $e_{n-1,1}, \dots, e_{n-1,n-1}$, whose sum is E_{n-1} , so (2) follows from (1). $\square$

Lemma 2.2. *For every $j \geq 1$ we have $v_2(T_{2j-1}) = h(j) = 2j - 2 - v_2(j)$.*

Proof. The Taylor series of $\tan x$ gives $T_{2j-1} = 2^{2j}(2^{2j}-1)|B_{2j}|/(2j)$, where B_{2j} is a Bernoulli number; see [KB67]. By the theorem of von Staudt and Clausen [IR90, Chapter 15], the denominator of B_{2j} is the product of the primes p with $(p-1) \mid 2j$. This product contains $p=2$ exactly once, so $v_2(B_{2j}) = -1$. Since $2^{2j}-1$ is odd, we get $v_2(T_{2j-1}) = 2j-1 - v_2(2j) = 2j-2 - v_2(j)$. $\square$

Lemma 2.3. (a) $h(j) \geq 0$ for every $j \geq 1$, $h(1) = 0$, $h(2) = 1$, and $h(j) \geq j$ for $j \geq 3$.

(b) For every $i \geq 1$ the minimum $G(i)$ is attained. The function G is nondecreasing, $G(2t-1) = G(2t)$ for every $t \geq 1$, and $G(i) \rightarrow \infty$.

(c) For every $k \geq 1$, $J(k)$ is well defined, $J(k) \geq 1$, $h(J(k)) < k$, and $h(j) \geq k$ for every $j > J(k)$.

Proof. (a) We have $v_2(j) \leq \log_2 j$. The values $h(1) = 0$, $h(2) = 1$ and $h(3) = 4$ are direct. For $j \geq 4$ we have $j-2 \geq \log_2 j$, and hence $h(j) \geq 2j-2 - \log_2 j \geq j$. For $j \geq 1$, $\log_2 j \leq 2j-2$ gives $h(j) \geq 0$.

(b) By (a), only finitely many j satisfy $h(j) \leq h(\lceil i/2 \rceil)$, so the minimum is attained. The sets $\{j : j \geq \lceil i/2 \rceil\}$ shrink as i grows, and $\lceil (2t-1)/2 \rceil = \lceil 2t/2 \rceil = t$. For $i \geq 5$, (a) gives $G(i) \geq \lceil i/2 \rceil$.

(c) The set $\{j : h(j) < k\}$ contains 1, and by (a) it contains no $j \geq \max\{3, k\}$. $\square$

Let $E_{2m}^* = (-1)^m S_{2m}$ be the signed Euler numbers, so that $\operatorname{sech} t = \sum_{m \geq 0} E_{2m}^* t^{2m} / (2m)!$.

Theorem 2.4 (Stern [Ste75]). For all integers $m \neq n \geq 0$,

$$v_2(E_{2m}^* - E_{2n}^*) = 1 + v_2(m - n).$$

Z.-W. Sun gave a new proof [Sun05]. Refinements and extensions of Stern's congruence are in [Sun10, SW13, SL13]; see also [Liu08] for congruences of Euler numbers modulo powers of two. For the reader's convenience, Appendix A contains a short proof. The signs matter: for the unsigned secant numbers the analogous statement fails already for $m=1$ and $n=0$, since $S_2 = S_0 = 1$.

Corollary 2.5. (a) $S_{2m} \equiv 1 \pmod{4}$ for every $m \geq 0$.

(b) For $k \geq 2$ and $m \geq 0$, $S_{2m+2^k} \equiv S_{2m} \pmod{2^k}$.

(c) For $k \geq 3$ and $m \geq 0$, $S_{2m+2^{k-1}} \not\equiv S_{2m} \pmod{2^k}$.

Proof. (a) We have $S_0 = 1$. Let $m \geq 1$. Theorem 2.4 with $n=0$ gives $v_2(E_{2m}^* - 1) = 1 + v_2(m)$. If m is odd, then $E_{2m}^* \equiv 3 \pmod{4}$ and $S_{2m} = -E_{2m}^* \equiv 1 \pmod{4}$. If m is even, then $S_{2m} = E_{2m}^* \equiv 1 \pmod{4}$.

(b) Put $M = m + 2^{k-1}$. Since 2^{k-1} is even, $(-1)^M = (-1)^m$, so $S_{2M} - S_{2m} = (-1)^m (E_{2M}^* - E_{2m}^*)$. By Theorem 2.4 this difference has valuation $1 + (k-1) = k$.

(c) Put $M = m + 2^{k-2}$. Since $k \geq 3$, 2^{k-2} is even, and as in (b) the difference $S_{2M} - S_{2m}$ has valuation $1 + (k-2) = k-1 < k$. $\square$

3 The diagonal minima

For $n \geq 2$ and $1 \leq i \leq n$ put

$$H(i, n) = \min\{h(j) : \lceil i/2 \rceil \leq j \leq \lfloor n/2 \rfloor\},$$

with $\min \emptyset = \infty$. Clearly $H(l, n) \geq H(i, n)$ for $l \geq i$, and $H(i, n) \geq G(i)$.

Lemma 3.1. For all $n \geq 2$ and $1 \leq i \leq n$ we have $v_2(e_{n,i}) \geq H(i, n)$. Consequently $v_2(e_{n,i}) \geq G(i)$ for all $n \geq i \geq 1$, that is, $m_i \geq G(i)$.

Proof. We use induction on n . Row 2 is $(0, 1)$, and $H(1, 2) = H(2, 2) = h(1) = 0$. Let $n \geq 3$ and assume the claim for $n - 1$. For $i \leq l \leq n - 1$, the induction hypothesis and the monotonicity of H in its first argument give $v_2(e_{n-1,l}) \geq H(l, n - 1) \geq H(i, n - 1)$. Hence the sum $\sum_{l=i}^{n-1} e_{n-1,l}$ in (2) has valuation at least $H(i, n - 1)$; it is 0 if $i = n$. If n is odd, then $\lfloor (n - 1)/2 \rfloor = \lfloor n/2 \rfloor$, so $H(i, n - 1) = H(i, n)$, and (2) gives the claim. If n is even, then $v_2(E_{n-1}) = h(n/2)$ by Lemma 2.2, and (2) gives

$$v_2(e_{n,i}) \geq \min\{h(n/2), H(i, n - 1)\} = H(i, n),$$

because $\lfloor (n - 1)/2 \rfloor = n/2 - 1$ and $\lceil i/2 \rceil \leq n/2$. The last statement follows from $H(i, n) \geq G(i)$ and from $v_2(e_{1,1}) = 0 = G(1)$. $\square$

Lemma 3.2. *Let $i \geq 3$, put $c = \lceil i/2 \rceil$, and let j^* be the least $j \geq c$ with $h(j) = G(i)$, which exists by Lemma 2.3(b). Then $2j^* \geq i$, so $e_{2j^*,i} \in D_i$, and $v_2(e_{2j^*,i}) = G(i)$.*

Proof. We have $2j^* \geq 2c \geq i$ and $2j^* \geq 2c \geq 4$. By (2) for the even row $2j^*$,

$$e_{2j^*,i} = T_{2j^*-1} - R, \quad R = \sum_{l=i}^{2j^*-1} e_{2j^*-1,l}.$$

By Lemma 2.2, $v_2(T_{2j^*-1}) = h(j^*) = G(i)$. By Lemma 3.1 for the odd row $2j^* - 1 \geq 3$, each term $e_{2j^*-1,l}$ of R has valuation at least $H(l, 2j^* - 1) \geq H(i, 2j^* - 1)$, where

$$H(i, 2j^* - 1) = \min\{h(j) : c \leq j \leq j^* - 1\}.$$

Every j with $c \leq j < j^*$ satisfies $h(j) \geq G(i)$, and $h(j) \neq G(i)$ by the choice of j^* . Hence $v_2(R) > G(i)$; if $j^* = c$, the range is empty and $R = 0$. Therefore $v_2(e_{2j^*,i}) = G(i)$. $\square$

Proof of Theorem 1.1. For $i \in \{1, 2\}$ we have $G(i) = h(1) = 0$ and $e_{1,1} = e_{2,2} = 1$, so $m_1 = m_2 = 0$. For $i \geq 3$, Lemma 3.2 gives $m_i \leq G(i)$ and Lemma 3.1 gives $m_i \geq G(i)$. So $m_i = G(i)$ for all i , and the stated properties of (m_i) follow from Lemma 2.3(b).

Let $k \geq 1$ and $J = J(k)$. If $i \leq 2J$, then $\lceil i/2 \rceil \leq J$, so $m_i = G(i) \leq h(J) < k$. If $i > 2J$, then $\lceil i/2 \rceil > J$, so every $j \geq \lceil i/2 \rceil$ has $h(j) \geq k$ by Lemma 2.3(c), and $m_i = G(i) \geq k$. Hence $\{i : m_i < k\} = \{1, \dots, 2J\}$. Its maximum and its cardinality are both $2J$, and a diagonal D_i contains an entry not divisible by 2^k exactly when $m_i < k$. $\square$

Remark 3.3. The equality $u_k = 2J(k)$ alone needs only Lemma 3.1 and one entry. Let $J = J(k)$. By Lemma 2.1, $e_{2J,2J} = E_{2J-1} = T_{2J-1}$, which has valuation $h(J) < k$, so $m_{2J} < k$. For $i > 2J$, Lemma 3.1 gives $m_i \geq G(i) \geq k$. Lemma 3.2 is needed for the monotonicity of (m_i) and for the description of u_k as a number of diagonals.

4 Periodicity modulo 2^k

Lemma 4.1. *Let $(a_n)_{n \geq 0}$ be a sequence. For $N \geq 0$ let $P(N)$ be the set of integers $t \geq 1$ such that $a_{n+t} = a_n$ for all $n \geq N$.*

(a) $P(N) \subseteq P(N + 1)$.

(b) If $P(N) \neq \emptyset$ and $N' \geq N$, then $P(N') = P(N)$.

(c) If $P(0) \neq \emptyset$, then $P(0)$ is the set of positive multiples of its least element.

Consequently, if the sequence is eventually periodic, its preperiod is the least N with $P(N) \neq \emptyset$, and its least period is $\min P(N)$ for every N at least the preperiod.

Proof. (a) is clear. (b) Let $t \in P(N')$ and $t' \in P(N)$, and let $n \geq N$. Choose M with $n + Mt' \geq N'$. Then $a_n = a_{n+Mt'} = a_{n+Mt'+t} = a_{n+t}$, where the first and the last equality use $t' \in P(N)$ and the middle one uses $t \in P(N')$. So $t \in P(N)$, and (a) gives the reverse inclusion. (c) Let $p = \min P(0)$ and $t \in P(0)$, and write $t = qp + r$ with $0 \leq r < p$. For every $n \geq 0$ we have $a_{n+r} = a_{n+r+qp} = a_{n+t} = a_n$. If $r \geq 1$, then $r \in P(0)$, which contradicts the minimality of p . So $r = 0$. Conversely, every positive multiple of p lies in $P(0)$. $\square$

Proof of Theorem 1.2. Fix $k \geq 1$, and let $a_n \in \{0, 1, \dots, 2^k - 1\}$ be the residue of E_n modulo 2^k . Put $J = J(k)$ and $b_m = a_{2m}$. Let $P(N)$ be as in Lemma 4.1 for (a_n) , and let $P_b(N)$ be the corresponding sets for (b_m) .

(i) *Odd indices.* By Lemmas 2.2 and 2.3(c), $a_{2J-1} \neq 0$ and $a_{2j-1} = 0$ for all $j > J$. So $a_n = 0$ for every odd $n \geq 2J + 1$.

(ii) *Even indices.* By Corollary 2.5(a), every b_m is odd; in particular $b_m \neq 0$. If $k \leq 2$, then $b_m = 1$ for all m , again by Corollary 2.5(a). If $k \geq 3$, then $2^{k-1} \in P_b(0)$ by Corollary 2.5(b) and $2^{k-2} \notin P_b(0)$ by Corollary 2.5(c). By Lemma 4.1(c) the least element of $P_b(0)$ divides 2^{k-1} but not 2^{k-2} , so $P_b(0)$ is the set of positive multiples of 2^{k-1} .

(iii) *A period.* Let $t = 2^k$, or $t = 2$ if $k \leq 2$. Then $t \in P(2J)$. Indeed, let $n \geq 2J$. If n is odd, then $a_n = a_{n+t} = 0$ by (i), since t is even. If $n = 2m$, then $a_{n+t} = b_{m+t/2} = b_m = a_n$ by (ii).

(iv) *Every period is even.* Let $N \geq 2J$ and $t \in P(N)$. If t were odd, then for an even $n \geq N$ the index $n + t$ would be odd and at least $2J + 1$, so that $a_{n+t} = 0 \neq a_n$ by (i) and (ii). Hence $t = 2\tau$ with $\tau \geq 1$, and $b_{m+\tau} = b_m$ for all $m \geq N/2$. Since $P_b(0) \neq \emptyset$ by (ii), Lemma 4.1(b) gives $\tau \in P_b(0)$.

(v) *The preperiod.* Suppose that $t \in P(2J - 1)$. Then $t \in P(2J)$ by Lemma 4.1(a), so t is even by (iv), and $2J - 1 + t$ is odd and at least $2J + 1$. Hence $a_{2J-1} = a_{2J-1+t} = 0$, which contradicts (i). So $P(2J - 1) = \emptyset$, and by Lemma 4.1(a) $P(N) = \emptyset$ for every $N \leq 2J - 1$. Since $P(2J) \neq \emptyset$ by (iii), we get $s(2^k) = 2J$, which equals u_k by Theorem 1.1.

(vi) *The period.* By Lemma 4.1, $d(2^k) = \min P(2J)$. If $k \geq 3$, every $t \in P(2J)$ has the form 2τ with τ a positive multiple of 2^{k-1} , by (iv) and (ii), so $t \geq 2^k$; together with (iii) this gives $d(2^k) = 2^k$. If $k \leq 2$, then $2 \in P(2J)$ by (iii) and $1 \notin P(2J)$ by (iv), so $d(2) = d(4) = 2$. As $2 = 2^1$, this proves $d(2^k) = 2^k$ for $k \neq 2$ and $d(4) = 2$. $\square$

5 The f -transform

Write $F(k) = 2J(k)$, so that $u_k = F(k)$ by Theorem 1.1. For $a \geq 2$ put $c_a = 2^a - a - 1$; then $1 \leq c_a \leq 2^a - 3$.

Lemma 5.1. *Let $a \geq 2$.*

$$(S1) \quad h(2^{a-1} + r) = h(r) + 2^a \text{ for } 1 \leq r < 2^{a-1}.$$

$$(S2) \quad h(2^{a-1}) = c_a \text{ and } h(2^a) = c_a + 2^a - 1.$$

$$(R1) \quad h(j) \leq 2^a - 2 \text{ for } 1 \leq j \leq 2^{a-1}.$$

$$(R2) \quad 2^a \leq h(j) \leq 2^{a+1} - 2 \text{ for } 2^{a-1} < j \leq 2^a.$$

$$(R3) \quad h(j) \geq 2^{a+1} \text{ for } j > 2^a.$$

Proof. (S1) Since $v_2(r) < a - 1$, we have $v_2(2^{a-1} + r) = v_2(r)$. (S2) is a direct computation. (R1) and the upper bound in (R2) follow from $h(j) \leq 2j - 2$. For the lower bound in (R2), write $j = 2^{a-1} + r$ with $1 \leq r < 2^{a-1}$ if $j < 2^a$, and use (S1) and $h(r) \geq 0$; for $j = 2^a$ use (S2) and $c_a \geq 1$. (R3) Write $j = 2^a + r$ with $r \geq 1$. We need $v_2(2^a + r) \leq 2r - 2$. If $v_2(r) \neq a$, then $v_2(2^a + r) =$

$\min\{a, v_2(r)\} \leq v_2(r) \leq \log_2 r \leq r - 1 \leq 2r - 2$. If $v_2(r) = a$, then $r \geq 2^a \geq 4$ and $2^a + r \leq 2r$, so $v_2(2^a + r) \leq \log_2(2r) = 1 + \log_2 r \leq 2r - 2$. $\square$

Lemma 5.2. *Let $a \geq 2$ and $1 \leq i \leq 2^a$.*

(a) $F(i) = 2^a$ if $i > c_a$, and $F(i) \leq 2^a - 2$ if $i \leq c_a$.

(b) $F(2^a + i) = F(i) + 2^a$ if $i < c_a$, and $F(2^a + i) = 2^{a+1}$ if $i \geq c_a$.

Proof. (a) By (R2) and (R3), every $j > 2^{a-1}$ has $h(j) \geq 2^a \geq i$. So $J(i) \leq 2^{a-1}$, with equality if and only if $h(2^{a-1}) = c_a < i$. In particular, for $i \leq c_a$,

$$J(i) = \max\{r < 2^{a-1} : h(r) < i\}.$$

(b) Let $k = 2^a + i$. By (R1), every $j \leq 2^{a-1}$ has $h(j) < k$, and by (R3), every $j > 2^a$ has $h(j) \geq k$. If $i \geq c_a$, then $h(2^a) = c_a + 2^a - 1 < k$, so $J(k) = 2^a$ and $F(k) = 2^{a+1}$. Let $i < c_a$. Then $h(2^a) \geq k$. For $j = 2^{a-1} + r$ with $1 \leq r < 2^{a-1}$, (S1) shows that $h(j) < k$ if and only if $h(r) < i$, and $r = 1$ qualifies because $h(1) = 0 < i$. Hence $J(k) = 2^{a-1} + \max\{r < 2^{a-1} : h(r) < i\} = 2^{a-1} + J(i)$, by the display in (a). So $F(2^a + i) = 2^a + F(i)$. $\square$

Proof of Theorem 1.3. Let $a \geq 2$, $L = 2^a$ and $x = (F(1), \dots, F(L))$. By Lemma 5.2(a), $x_L = 2^a$, and $x_i = 2^a$ exactly for $c_a < i \leq L$; moreover $x_1 \neq x_L$, since $1 \leq c_a$. Since $c_a \leq L - 3$, the index in the definition of f is $\ell = c_a = 2^a - a - 1$, and

$$f(x) = (x, x_1 + 2^a, \dots, x_{c_a-1} + 2^a, 2^{a+1}, \dots, 2^{a+1})$$

with $L - c_a + 1$ copies of 2^{a+1} . By Lemma 5.2(b), position $L + i$ of $f(x)$ holds $F(i) + 2^a = F(2^a + i)$ for $1 \leq i \leq c_a - 1$ and $2^{a+1} = F(2^a + i)$ for $c_a \leq i \leq L$. Hence $f(x) = (F(1), \dots, F(2^{a+1}))$.

Finally, $h(1) = 0$, $h(2) = 1$ and $h(j) \geq 4$ for $j \geq 3$ (Lemma 2.3(a) and $h(3) = 4$), so $J(1) = 1$ and $J(2) = J(3) = J(4) = 2$, that is, $(F(1), \dots, F(4)) = (2, 4, 4, 4)$. By induction on n , $f^n(2, 4, 4, 4) = (F(1), \dots, F(2^{n+2}))$ for every $n \geq 0$. So the f -transform of $(2, 4, 4, 4)$ is $(F(k))_{k \geq 1}$, which is $(u_k)_{k \geq 1}$ by Theorem 1.1. $\square$

6 Remarks

Remark 6.1 (OEIS A108039). The OEIS entry A108039 [OEI26] lists $u_1, \dots, u_{18}$ with offset 0, so its term $a(n)$ is $u_{n+1} = \max\{i : m_i \leq n\}$. By Theorem 1.1, $a(n) = 2 \max\{j : 2j - 2 - v_2(j) \leq n\}$. The name of the entry replaces every entry of the triangle by its 2-adic valuation and says that “the entries n in the new triangle do not occur beyond diagonal $a(n)$ ”. If this is read for the value exactly n , then the least such diagonal is 3 for $n = 2$ and for $n = 3$. The example of the entry, $a(3) = 4$, fits only the reading with the entries at most n , and under this reading $a(n) = u_{n+1}$. Indeed, every entry of D_4 has valuation 1. For $J \geq 2$, (1) and Lemma 2.1 give $e_{2J-1,1} = e_{2J-1,2} = S_{2J-2}$, $e_{2J-1,3} = S_{2J-2} - S_{2J-4}$, $e_{2J,1} = 0$, $e_{2J,2} = S_{2J-2}$ and $e_{2J,3} = 2S_{2J-2}$, and hence

$$e_{2J,4} = 3S_{2J-2} - S_{2J-4}, \quad e_{2J+1,4} = S_{2J} - 3S_{2J-2}.$$

Both are $\equiv 2 \pmod{4}$ by Corollary 2.5(a). On the other hand D_3 contains $e_{5,3} = 4$ and $e_{7,3} = 56$, of valuations 2 and 3, and $m_i \geq m_5 = 4$ for $i \geq 5$.

Remark 6.2 (The period for general moduli). By [Ram17, Lemma 1], $d(q)$ is the least common multiple of the values $d(p^\alpha)$ over the prime powers p^α that exactly divide q . Knuth and Buckholtz [KB67] determined $d(p)$ for odd primes p . In a recent preprint [Güleç26], Güleç proved $d(p^r) = p^{r-1}d(p)$ for odd primes p , which is part (2) of Conjecture 1 of [Ram17]. He also proved $s(5^5) = 4$, which disproves part (1). Together with Theorem 1.2, these results determine $d(q)$ for every $q \geq 1$. The preperiods $s(p^r)$ for odd primes p are not determined in general.

Verification. The proofs above are complete without computer assistance. The following computations were used as consistency checks.

1. A self-contained program by the author, `verify_ramassamy.py` (standard library only, about 25 seconds), builds the triangle (1) in exact arithmetic for $n \leq 1000$. It checks the following.
 - The recurrence (1) against the definition of $e_{n,i}$ for $n \leq 9$, and the row sums against E_n computed independently from $\sec x + \tan x$.
 - Lemma 2.2 for $j \leq 500$, and Theorem 2.4 for all 125,250 pairs $0 \leq n < m \leq 500$.
 - Lemma 3.1 on all 500,499 entries with $2 \leq n \leq 1000$, Lemma 3.2 for the 998 columns $3 \leq i \leq 1000$, and Theorem 1.1 for $i \leq 1000$ and $k \leq 1000$.
 - Theorem 1.2 for $k \leq 13$, by a brute-force computation of the preperiod and the least period of $E_n \bmod 2^k$ for $n \leq 2^{14} + 64$.
 - Lemma 5.1 for $2 \leq a \leq 20$, Theorem 1.3 up to length 2^{20} (with the index $\ell = 2^a - a - 1$ at every step), and the formulas of Remark 6.1 and Appendix A.
2. The programs written when the results were found check the same statements in exact arithmetic for $n \leq 1200$, and the f -transform up to length 2^{20} . A C program computes $E_n \bmod 2^{64}$ for $n \leq 2^{20} + 256$ from (1). On this range the data agree with $s(2^k) = u_k$ and with the values of $d(2^k)$ in Theorem 1.2 for $k \leq 19$: the tail from u_k has the stated period, the tail from $u_k - 1$ does not, and for $k \geq 3$ half the stated period is not a period of the tail. Ramassamy's computations covered $k \leq 12$ for Conjecture 2 and $k \leq 512$ for Conjecture 3.
3. Two independent verification passes re-derived every proof by hand and wrote their own programs. One computed $E_n \bmod 2^{64}$ without the triangle, from the convolution identities given by $\operatorname{sech} t \cosh t = 1$ and $\tan t \cos t = \sin t$, for $n \leq 2^{16} + 200$, and confirmed $s(2^k) = u_k$ and $d(2^k)$ for $k \leq 14$. Both built the triangle for 3000 rows modulo a large power of 2. They confirmed $m_i = G(i)$ for $i \leq 2960$, the equality $u_k = 2J(k)$ for $k \leq 2920$ (in one pass), and the f -transform identity up to length 2^{22} . One of them reran all programs of item 2 and reproduced their outputs; the other reran one of the exact-arithmetic programs of item 2 for $n \leq 800$, with no failures.
4. A third independent pass reviewed the previous version of this note, checked every proof line by line and found no mathematical error. Its programs, written from the text before any of the programs above was read, are in the folder `referee2` of the release. They compute $E_n \bmod 2^{64}$ for $n \leq 2^{20} + 320$ by a third method, from the derivative polynomials of $\tan x$ and $\sec x$, and a brute-force search for the preperiod and the least period on this range agrees with $s(2^k) = u_k$ and with the values of $d(2^k)$ in Theorem 1.2 for $k \leq 19$. For $n \leq 131,232$ these residues coincide with those of the C program of item 2. The programs also check Lemmas 3.1 and 3.2 and Theorem 1.1 on the triangle with 3000 rows, Theorem 2.4 for all pairs $0 \leq n < m \leq 500$, and Theorem 1.3 up to length 2^{22} .

Scope and priority. Theorems 1.2 and 1.3 prove Conjectures 2 and 3 of [Ram17] as stated there, with the definitions of [Ram17]: the triangle (1), the diagonals D_i read as columns, and u_k defined as a maximum. By Theorem 1.1 nothing changes if u_k is read as a number of diagonals, as [Ram17] also describes it. The value of $d(2^k)$ and the equality $s(2^k) = 2J(k)$ follow quickly from Stern's theorem [Ste75] and Lemma 2.2, and we make no novelty claim for them. Conjecture 1 of [Ram17], on odd prime powers, is not treated here; it was settled by Güleç [Güleç26], see Remark 6.2. In September 2026 we searched arXiv, Crossref, OpenAlex, zbMATH, Semantic Scholar and the OEIS, and ran a

web search. Crossref, OpenAlex and zbMATH record no work citing [Ram17], and Semantic Scholar records only [Güleç26]. The OEIS entry A108039 lists $u_1, \dots, u_{18}$ without a formula. We found no earlier proof of Conjectures 2 or 3, or of Arnold's observation. We could not access Arnold's paper [Arn91], because the publisher's site blocked anonymous access, and we did not see the commentary in [Arn04]. Our description of [Arn91] follows [Ram17], which states that Arnold observed the monotonicity without proof. We read Arnold's follow-up survey [Arn92] in the Russian original. Its section on congruences treats odd primes only, and we found no statement there on the 2-adic valuations of the Entringer numbers. According to its zbMATH summary, Arnold's note [Arn93] also concerns odd moduli. The OpenAlex keyword searches were rate-limited and are incomplete. This negative search is not a proof of priority.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

A A proof of Stern's theorem

We include a short proof of Theorem 2.4 for completeness; the theorem is due to Stern [Ste75], and a different proof is in [Sun05]. Let Δ be the forward difference operator, $(\Delta g)(x) = g(x+1) - g(x)$, so that $(\Delta^k g)(0) = \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} g(j)$, and $\Delta^k g = 0$ if g is a polynomial of degree less than k .

Step 1. In $\mathbb{Q}[[t]]$ we have

$$\operatorname{sech} t = \frac{2e^t}{1+e^{2t}} = \frac{e^t}{1+\frac{1}{2}(e^{2t}-1)} = \sum_{k \geq 0} \left(-\frac{1}{2}\right)^k e^t (e^{2t}-1)^k,$$

where the series converges because $(e^{2t}-1)^k \in t^k \mathbb{Q}[[t]]$. Since $e^t (e^{2t}-1)^k = \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} e^{(2j+1)t}$, comparing the coefficients of $t^N/N!$ gives, for even N ,

$$E_N^* = \sum_{k \geq 0} \left(-\frac{1}{2}\right)^k (\Delta^k p_N)(0), \quad p_N(x) = (2x+1)^N.$$

Step 2. Let $N = 2n$. Since $(2x+1)^2 = 1 + 8\theta(x)$ with $\theta(x) = x(x+1)/2$, we have $p_{2n} = \sum_{r=0}^n \binom{n}{r} 8^r \theta^r$. The polynomial θ^r has degree $2r$, so

$$E_{2n}^* = \sum_{r=0}^n \binom{n}{r} b_r, \quad b_r = 8^r \sum_{k=0}^{2r} \left(-\frac{1}{2}\right)^k (\Delta^k \theta^r)(0),$$

and b_r does not depend on n . By binomial inversion, b_r is the r -th difference of the sequence $(E_{2n}^*)_{n \geq 0}$ at 0; in particular $b_1 = E_2^* - E_0^* = -2$ and $b_2 = E_4^* - 2E_2^* + E_0^* = 8$.

Step 3. Write $\theta^r = 2^{-r}(x^2+x)^r$ and $(x^2+x)^r = \sum_m c_m x^m$ with $c_m \in \mathbb{Z}$. Since $(\Delta^k x^m)(0) = k! S(m, k)$, where $S(m, k)$ is a Stirling number of the second kind, we get $(\Delta^k (x^2+x)^r)(0) = k! A_{r,k}$ with $A_{r,k} \in \mathbb{Z}$. Hence

$$b_r = 4^r \sum_{k=0}^{2r} (-1)^k \frac{k!}{2^k} A_{r,k},$$

and for $r \geq 1$ the term $k=0$ vanishes, because $(x^2+x)^r$ vanishes at 0. By Legendre's formula, $v_2(k!/2^k) = -s_2(k)$, where $s_2(k)$ is the sum of the binary digits of k . A number with s binary digits equal to 1 is at least $2^s - 1$, so $s_2(k) \leq \lfloor \log_2(2r+1) \rfloor$ for $k \leq 2r$. Therefore

$$v_2(b_r) \geq 2r - \lfloor \log_2(2r+1) \rfloor \quad (r \geq 1).$$

Step 4. Put $\beta_r = b_r/r!$, so that $E_{2n}^* = \sum_{r \geq 0} \beta_r (n)_r$ with $(n)_r = n(n-1) \cdots (n-r+1)$. For $r \geq 1$ we have $v_2(r!) = r - s_2(r) \leq r - 1$, hence $v_2(\beta_r) \geq r + 1 - \lfloor \log_2(2r+1) \rfloor$, and this is at least 2 for $r \geq 3$ because $2r+1 < 2^r$. Also $\beta_1 = -2$ and $\beta_2 = 4$. For every $r \geq 1$ the polynomial $(x)_r - (y)_r \in \mathbb{Z}[x, y]$ is divisible by $x - y$; write $(x)_r - (y)_r = (x - y)Q_r(x, y)$ with $Q_r \in \mathbb{Z}[x, y]$, where $Q_1 = 1$. For integers $m \neq n \geq 0$ we obtain

$$E_{2m}^* - E_{2n}^* = (m - n) \left(-2 + \sum_{r \geq 2} \beta_r Q_r(m, n) \right),$$

a finite sum. Every term $\beta_r Q_r(m, n)$ with $r \geq 2$ is a rational number of 2-adic valuation at least 2. So the second factor has valuation exactly 1, and $v_2(E_{2m}^* - E_{2n}^*) = 1 + v_2(m - n)$. $\square$

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