

The Lebesgue Differentiation Theorem over $\mathbf{RCA}_0$: On Question 17 of Montalbán’s Open Questions in Reverse Mathematics

Alper Ferudun*

October 18, 2026

Abstract

Question 17 of Montalbán’s survey of open questions in reverse mathematics (Bull. Symbolic Logic 17 (2011)) asks whether the Lebesgue differentiation theorem (LDT) implies $\mathbf{WWKL}_0$ over $\mathbf{RCA}_0$; Simpson had observed, using Pathak’s work, that LDT is provable in $\mathbf{WWKL}_0$. The question does not fix how “for almost every x ” and the measure of an open set are formalised, and the answer depends on this. We prove in $\mathbf{RCA}_0$: for every L^1 function $f = (p_n)$ on $[0, 1]^d$, given by rational polynomials with $\|p_n - p_{n+1}\|_1 \leq 2^{-n}$, there are open sets U_N , uniformly Δ_1^0 in f , of measure at most $\varepsilon_N \rightarrow 0$ (for the extensional measure of Brown, Giusto and Simpson and for the measure $\bar{\mu}$ of Nies, Triplett and Yokoyama), such that every point outside some U_N is a Lebesgue point of f with $f(x) = \lim p_n(x)$, and every box of measure $> \varepsilon_N$ meets the complement of U_N . Hence $\mathbf{RCA}_0$ proves LDT under the null-cover reading of Yu and of Brown, Giusto and Simpson, under the reading that Lebesgue points exist and are dense, and under the randomness reading; under these readings LDT does not imply $\mathbf{WWKL}_0$. Under the outer reading of Nies, Triplett and Yokoyama, LDT is equivalent to $\mathbf{WWKL}_0$; the reversal uses their open set of measure < 1 whose complement consists of dyadic rationals, and it reflects the behaviour of measure in $\mathbf{RCA}_0$, not differentiation. We judge the null-cover reading to be the intended one, so that the answer is no. The method makes the maximal-function estimates of Pathak, Rojas and Simpson finite. Yu proved in $\mathbf{RCA}_0$ that the values $p_n(x)$ converge outside a Martin-Löf null set; our results are close to known techniques and may be known to experts, but we found no printed answer to the question. This is an unrefereed note.

1 Introduction

1.1 The question

In Section 3.4 of [Mon11], Montalbán recalls that Pathak [Pat09] proved the Lebesgue differentiation theorem at every Martin-Löf random point, that Simpson noticed that it follows that the theorem can be proved in $\mathbf{WWKL}_0$, and that Simpson asked about the reversal. The theorem is stated there as follows (our paraphrase): if f is an L^1 function on $[0, 1]^d$, “represented as a Cauchy limit of polynomials”, then for almost every $x \in [0, 1]^d$

$$f(x) = \lim_{Q \searrow x} \frac{\int_Q f}{\mu(Q)},$$

where Q ranges over the open cubes containing x . Call this statement LDT. Question 17 of [Mon11] reads: “Does LDT imply $\mathbf{WWKL}_0$?” The base theory is $\mathbf{RCA}_0$, as everywhere in the survey.

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: <https://github.com/AlperTheKing>.
Unrefereed preprint. CC BY 4.0.

The source fixes neither the value $f(x)$ of an L^1 function at a point, nor the measure of an open set (which in RCA_0 need not exist as a real and may depend on the code), nor the meaning of “for almost every x ”. We take $f(x) := \lim_n p_n(x)$ where the limit exists; this agrees with Pathak’s canonical representative [Pat09, Definition 3.4] at random points and is the convention of [BGS02, Remark 4.3]. We treat every reading of “almost every” that we found in this literature (Definition 2.3): the *null-cover* reading (N) of Yu and of Brown, Giusto and Simpson [Yu94, BGS02], the *existence* reading (E), the *randomness* reading (R) suggested by [Pat09, PRS14], and the *outer* reading (O) of Nies, Triplett and Yokoyama [NTY21, Definition 6.4]; and three measures of open sets: the intensional measure μ_{int} , the extensional measure μ_{ext} of [BGS02, Definition 2.1], and the measure $\bar{\mu}$ of [NTY21, Section 5].

1.2 Results

An L^1 function is a sequence $f = (p_n)$ of polynomials with rational coefficients with $\|p_n - p_{n+1}\|_1 \leq 2^{-n}$; a Lebesgue point of f is a point x where $f(x) = \lim p_n(x)$ exists and the averages of f over cubes containing x converge to $f(x)$ (Definition 2.1).

Theorem A (RCA_0). *Let $d \geq 1$, let $f = (p_n)$ be an L^1 function on $[0, 1]^d$, and put $c_d := 4^{d+1} + 2$ and $\varepsilon_N := (4c_d + 2) 2^{-\lceil N/2 \rceil}$. There are open sets $U_0 \supseteq U_1 \supseteq \dots$, given by a sequence of rational boxes that is Δ_1^0 in (p_n) , such that for every N :*

- (a) $\mu_{\text{int}}(U_N) \leq \varepsilon_N$, $\mu_{\text{ext}}(U_N) \leq \varepsilon_N$ and $\bar{\mu}(U_N) \leq \varepsilon_N$;
- (b) every $x \in [0, 1]^d$ that is not in U_N is a Lebesgue point of f , and $f(x) = \lim_n p_n(x)$;
- (c) for every rational box $C \subseteq [0, 1]^d$ the intensional measure of $C \cap U_N$ is a real number, computable from (p_n) , C and N ; and if $\mu(C) > \varepsilon_N$, then C contains a point that is not in U_N .

Corollary 1.1 (RCA_0). *LDT holds under the readings (N_{int}) , (N_{ext}) , (N_{bar}) , (E) (in the strong form: the Lebesgue points of every L^1 function are dense) and (R) . Hence under each of these readings LDT does not imply WWKL_0 over RCA_0 : the ω -model REC of the computable sets satisfies LDT but not WWKL_0 .*

Theorem B. *For each $m \in \{\text{int}, \text{ext}, \text{bar}\}$, LDT under the outer reading (O_m) is equivalent to WWKL_0 over RCA_0 . In every model of $\text{RCA}_0 + \neg \text{WWKL}_0$ it fails for one explicit L^1 function f on $[0, 1]^d$, a sum of the polynomials $(1 - (x_1 - r)^2)^M$ over dyadic rationals r .*

reading of “almost every x ”	over RCA_0	answer to Question 17
(N_{int}) null cover, intensional measure	provable; vacuous if $\neg \text{WWKL}_0$ (Remark 4.7)	no (degenerate)
(N_{ext}) , (N_{bar}) null cover	provable (Theorem A)	no
(E) Lebesgue points exist, are dense	provable (Theorem A)	no
(R) at points random relative to f	provable; vacuous in REC	no (weak reading)
(O_{int}) outer, intensional measure	$\leftrightarrow \text{WWKL}_0$, already for $f = 0$	yes (degenerate)
(O_{ext}) , (O_{bar}) outer	$\leftrightarrow \text{WWKL}_0$ (Theorem B)	yes, through measure

Table 1: The answer under each reading (definitions in Definition 2.3).

Two rows of Table 1 are degenerate: without WWKL_0 there are open sets of arbitrarily small intensional measure containing every point of $[0, 1]^d$ (Lemma 4.6), so every pointwise property holds almost everywhere in the sense (N_{int}) and none in the sense (O_{int}) . For μ_{ext} and $\bar{\mu}$ this does not happen, and the proof needs the located sets of Theorem A(c). The reading (R) is vacuous in REC, which contains no random points. Under the outer reading, (O_m) is equivalent to the statement that every open set containing all points with non-dyadic first coordinate has measure 1, in which differentiation does not occur (compare [NTY21, Remark 6.9]).

Which reading is intended. This is a judgement, not a theorem. Simpson’s observation derives LDT from Pathak’s theorem, and in the work of Yu and of Brown, Giusto and Simpson on measure theory in WWKL_0 , which was the reference point at the time, almost-everywhere statements about L^1 functions are formalised by null covers: in WWKL_0 the approximants of an L^1 function converge on closed sets C_n with $\mu(X \setminus C_n) \leq 2^{-n}$, with the extensional measure [BGS02, Proposition 4.1, Definition 2.1]. The outer reading was introduced later [NTY21]. We therefore regard the null-cover reading (N_{ext}) as the most plausible intended one; under it the answer to Question 17 is *no*. Under the outer reading it is *yes*, for the reason just explained.

1.3 Relation to the literature

Pathak [Pat09, Theorem 5.1] proved the differentiation formula for the canonical representative of an L^1 -computable f at every Martin-Löf random point. Pathak, Rojas and Simpson [PRS14, Theorems 3.16 and 4.7] showed that Schnorr randomness suffices and characterises these points; their tests come from the Chebyshev and Hardy–Littlewood inequalities [PRS14, Lemmas 3.6 and 3.13], with measures made computable by Tarski’s quantifier elimination [PRS14, Lemma 3.3]. Rute [Rut13, Theorem 4.10] proved an effective differentiation theorem with an effective rate of almost-everywhere convergence. Brattka, Miller and Nies [BMN16, Section 1] note that the strength of the theorem can be studied, referring to [Mon11, Section 3.4]. The convergence of the approximants of an L^1 function outside a Martin-Löf null set was carried out in RCA_0 by Yu [Yu94, Lemma 2.1], as reported in [ADR12, Section 2] (computably in [Pat09, Lemma 3.2]); so the existence of $\lim p_n(x)$ in Theorem A(b) is not new, and Rute [Rut20, Section 3.5] expected the version with Schnorr null sets to be provable in RCA_0 . Nies, Triplett and Yokoyama [NTY21, Theorem 6.8] proved that almost-sure pseudo-differentiability of monotone and of bounded-variation functions, in the outer reading, is equivalent to WWKL_0 , while by their Remark 6.3 every rationally presented non-decreasing function is pseudo-differentiable at some point already in RCA_0 . This is the precedent of the pattern of Theorem A(c) and Theorem B, and the reversal of Theorem B reuses their open set. By the same theorem, however, already the existence of a point of pseudo-differentiability of every effectively uniformly continuous, absolutely continuous function with a rational presentation is equivalent to WWKL_0 [NTY21, Theorem 6.8(5)]; Remark 4.9 explains why this does not conflict with Theorem A(c). A related vacuity phenomenon without WWKL_0 is noted by Westrick [Wes25].

New here, to our knowledge, are the differentiation part of Theorem A in RCA_0 (the cube averages), the located exceptional sets with the bounds for μ_{ext} and $\bar{\mu}$ and the existence and density of Lebesgue points, and Theorem B for L^1 functions. The method is the maximal-function route of [Pat09, PRS14] made finite: Lemma 3.1 replaces the exceptional set of a polynomial by finitely many grid boxes, chosen by a finite Vitali selection as in the Baby Vitali Lemma [BGS02, Lemma 5.1]. All of this is close to known techniques and may be known to experts; the searches are described in the paragraph “Scope and priority” at the end.

2 Conventions in RCA_0

We argue in RCA_0 [Sim09]. Reals are fast Cauchy sequences of rationals, points of $[0, 1]^d$ are d -tuples of reals, and $|x - y| := \max_k |x_k - y_k|$. A *rational box* is a product $\prod_k [a_k, b_k]$ with rationals $a_k < b_k$, and μ is its volume. The measure of a finite union of rational boxes is a rational computed from the list, and it is finitely additive on almost disjoint unions, monotone and subadditive. We use the following facts.

(F1) For a polynomial $g = \sum_a c_a x^a$ in d variables with rational coefficients put $L(g) := 1 + \sum_a |c_a|(a_1 + \dots + a_d)$. Then $|g(x) - g(y)| \leq L(g)|x - y|$ for $x, y \in [0, 1]^d$ (write $x^a - y^a$ as a telescoping

sum of $a_1 + \dots + a_d$ differences of a single factor).

(F2) For a box $R \subseteq [0, 1]^d$ with real corners, $\int_R g$ and $\int_R |g|$ are reals given by midpoint Riemann sums, with error at most $L(g) \cdot \text{mesh} \cdot \mu(R)$; they are additive over finite almost disjoint partitions and monotone, and $a \leq g \leq b$ on R gives $a\mu(R) \leq \int_R g \leq b\mu(R)$; for polynomials the Riemann sums converge with this explicit error, so these reals exist in RCA_0 .

(F3) Let $t \in [0, 1]$ be real and $M \geq 1$ an integer. The least $j \in \{1, \dots, M\}$ with $t \leq j/M$ exists, and $(j-1)/M \leq t \leq j/M$; if $t < 1$, the least $j \in \{0, \dots, M-1\}$ with $t < (j+1)/M$ exists, and $j/M \leq t < (j+1)/M$. This is the least number principle for Π_1^0 and for Σ_1^0 formulas, which follows from Σ_1^0 induction; no comparison of reals has to be decided.

Definition 2.1. An L^1 function on $[0, 1]^d$ is a sequence $f = (p_n)$ of polynomials with rational coefficients such that $\|p_n - p_{n+1}\|_1 \leq 2^{-n}$ for all n , where $\|p\|_1 := \int_{[0,1]^d} |p|$. For a box C put $\int_C f := \lim_n \int_C p_n$ (the sequence is fast Cauchy). The value $f(x)$ is $\lim_n p_n(x)$ if this limit exists; otherwise $f(x)$ is undefined, and every equation $f(x) = y$ is false. A *cube* is a box $Q = \prod_k [a_k, a_k + s] \subseteq [0, 1]^d$ with real a_k and side $s = s(Q) > 0$. A point x is a *Lebesgue point* of f if $f(x)$ exists and for every rational $e > 0$ there is a rational $\delta > 0$ such that $|\int_Q f / \mu(Q) - f(x)| \leq e$ for every cube Q with $x \in Q$ and $s(Q) < \delta$.

The rate is that of [BGS02, Section 4]; under the convention $\|f - p_n\|_1 \leq 2^{-n}$ of [Pat09, PRS14] the tail $(p_{n+1})_n$ satisfies ours and has the same values and integrals. An open cube containing x has the same integral as its closure, a cube containing x ; cubes not contained in $[0, 1]^d$ are treated in Remark 4.8.

Definition 2.2. A (code of an) *open set* U is a sequence of rational boxes $B = \prod_k ((a_k, b_k) \cap [0, 1])$, relatively open in $[0, 1]^d$; a point is in U if it is in one of the boxes.

- (i) $\mu_{\text{int}}(U) \leq a$ means that every finite union of boxes of the code has measure at most a , and $\mu_{\text{int}}(U) = 1$ means that for every rational $e > 0$ some finite union has measure $> 1 - e$.
- (ii) [BGS02, Definition 2.1]: $\mu_{\text{ext}}(U)$ is the supremum of $\int h$ over the functions $h \in C([0, 1]^d)$ with $0 \leq h \leq 1$ and $h(x) = 0$ for every $x \in [0, 1]^d$ that is not in U , read comparatively ($\mu_{\text{ext}}(U) \leq a$: every such h has $\int h \leq a$; $\mu_{\text{ext}}(U) = 1$: for every $e > 0$ there is such an h with $\int h > 1 - e$).
- (iii) [NTY21, Section 5]: $\bar{\mu}(U)$ is the supremum of $\mu_{\text{int}}(V)$ over the codes V with the same points as U . For $d = 1$, Nies, Triplett and Yokoyama use codes $S \subseteq 2^{<\mathbb{N}}$ of $\pi^{-1}(U)$, where $\pi(Z) = \sum_n Z(n)2^{-n-1}$, with $\mu(S) := 1 - \lim_n |T_S \cap 2^n|/2^n$ and T_S the tree of strings without a prefix in S ; we treat these *binary codes* as well.

Here $C([0, 1]^d)$ consists of the continuous functions with a modulus of uniform continuity; by [BGS02, Section 2] these correspond in RCA_0 to the points of the space $C(X)$, $X = [0, 1]^d$, used there, so (ii) is their measure. For such h and a box C the integral $\int_C h$ exists (Riemann sums), and it is additive and monotone.

Definition 2.3. Let f be an L^1 function on $[0, 1]^d$ and $m \in \{\text{int}, \text{ext}, \text{bar}\}$.

- (N_m) There are open sets $W_0, W_1, \dots$ with $\mu_m(W_k) \leq 2^{-k}$ such that every $x \in [0, 1]^d$ that is not in some W_k is a Lebesgue point of f .
- (E) f has a Lebesgue point; in the strong form, every rational box of positive measure contains one.
- (R) Every $x \in [0, 1]^d$ that is Martin-Löf random relative to (p_n) is a Lebesgue point of f ; here x is Martin-Löf random relative to X if for every X -computable sequence of codes (S_k) with $\mu_{\text{int}}(S_k) \leq 2^{-k}$ there is a k with x not in S_k (the definition of [NTY21, Section 5], for box codes).

(O_m) Every open set containing all Lebesgue points of f has μ_m -measure 1.

LDT under a reading means: for every $d \geq 1$ and every L^1 function on $[0, 1]^d$ the statement holds.

Reading (N) with the rate 2^{-k} is used in [BGS02, Proposition 4.1] and by Simić [Sim04]; Yu's null G_δ sets [Yu94], without a rate, are discussed in [Rut20, Section 3.5] and are implied. Reading (O) is [NTY21, Definition 6.4].

3 Finite exceptional sets

The following lemma is a finite form of the weak-type estimate for the maximal function used in [Pat09, PRS14].

Lemma 3.1 (RCA₀). *From a polynomial g in d variables with rational coefficients, a rational $\lambda > 0$ and $n \in \mathbb{N}$ one can compute a finite list $V = V(g, \lambda, n)$ of rational boxes, relatively open in $[0, 1]^d$, such that, with $c_d = 4^{d+1} + 2$,*

$$(i) \quad \mu(\bigcup V) \leq c_d \|g\|_1 / \lambda + 2^{-n};$$

$$(ii) \quad \text{if } x \in [0, 1]^d \text{ lies in no box of } V, \text{ then } |g(x)| \leq 3\lambda/4, \text{ and } \int_Q |g| \leq \lambda \mu(Q) \text{ for every cube } Q \text{ with } x \in Q.$$

Construction. Let m be least with $L(g)2^{-m} \leq \lambda/4$, and put $h := 2^{-m}$ and $\eta := h/(4d)$; so $1/\eta$ is an integer. The *grid cubes* are the cubes of side h with corners in $h\mathbb{Z}^d$ that are contained in $[0, 1]^d$; they tile $[0, 1]^d$. Let V_1 be the set of grid cubes D whose centre c_D satisfies $|g(c_D)| > 5\lambda/8$ (a comparison of rationals). Let F be the finite set of boxes $R = \prod_k [u_k, v_k]$ with $u_k < v_k$ in $\eta\mathbb{Z} \cap [0, 1]$ whose smallest side is at least h and whose largest side is at most $(1 + \frac{1}{2d})$ times the smallest. For $R \in F$ compute by (F2) a rational r_R with $|r_R - \int_R |g|| \leq \frac{\lambda}{8}\mu(R)$, and let V_2 be the set of $R \in F$ with $r_R > \frac{3\lambda}{8}\mu(R)$. Let $K := |V_1| + |V_2|$ and $\rho := 2^{-n}/(4dK)$, and let V consist of the boxes $\prod_k ((a_k - \rho, b_k + \rho) \cap [0, 1])$ for the boxes $\prod_k [a_k, b_k]$ in $V_1 \cup V_2$ (V is empty if $K = 0$).

Properties. (P1) If $D \in V_1$, then $|g| \geq |g(c_D)| - \frac{\lambda}{8} > \frac{\lambda}{2}$ on D ; if $D \notin V_1$, then $|g| \leq \frac{3\lambda}{4}$ on D . Indeed $|y - c_D| \leq h/2$ for $y \in D$, so $|g(y) - g(c_D)| \leq L(g)h/2 \leq \lambda/8$.

(P2) If $R \in V_2$, then $\int_R |g| > \frac{\lambda}{4}\mu(R)$; if $R \in F \setminus V_2$, then $\int_R |g| \leq \frac{\lambda}{2}\mu(R)$ (by the choice of r_R).

(P3) $\mu(\bigcup V_1) \leq 2\|g\|_1/\lambda$, since by (P1) and (F2) $\sum_{D \in V_1} \frac{\lambda}{2}\mu(D) \leq \sum_{D \in V_1} \int_D |g| \leq \|g\|_1$.

(P4) $\mu(\bigcup V_2) \leq 4^{d+1}\|g\|_1/\lambda$. List V_2 as $R^1, \dots, R^q$ by non-increasing largest side, and select R^t if it is disjoint from the boxes selected before (a primitive recursion on a finite list). If R^t is not selected, it meets a selected $R^{t'} = \prod_k [u_k, v_k]$ whose largest side w is at least every side of R^t ; hence $R^t \subseteq R^{t'*} := \prod_k [u_k - w, v_k + w]$. Since $w \leq (1 + \frac{1}{2d})(v_k - u_k)$ for every k , $\mu(R^{t'*}) \leq (3 + \frac{1}{d})^d \mu(R^{t'}) \leq 4^d \mu(R^{t'})$. The selected boxes are pairwise disjoint unions of cells of the η -grid, so $\sum_{\text{sel}} \int_R |g| \leq \|g\|_1$, and by (P2)

$$\mu\left(\bigcup V_2\right) \leq \sum_{\text{sel}} \mu(R^*) \leq 4^d \sum_{\text{sel}} \mu(R) \leq \frac{4^{d+1}}{\lambda} \sum_{\text{sel}} \int_R |g| \leq \frac{4^{d+1}}{\lambda} \|g\|_1.$$

(P5) For a box with sides $s_k \leq 1$, $\prod_k (s_k + 2\rho) - \prod_k s_k \leq (1 + 2\rho)^d - 1 \leq 4d\rho$, because $(1 + a)^d \leq (1 - a)^{-d} \leq (1 - da)^{-1}$ (Bernoulli) and $2d\rho \leq \frac{1}{2}$. So the K enlarged boxes add at most $4d\rho K = 2^{-n}$ to the measure.

Proof of Lemma 3.1. (i) follows from (P3), (P4) and (P5), with $c_d = 2 + 4^{d+1}$. For (ii), let x lie in no box of V ; then x lies in no box of $V_1 \cup V_2$, since each of these lies, within $[0, 1]^d$, in its enlargement. By (F3), applied in each coordinate with $M = 1/h$, x lies in some grid cube D ; as $D \notin V_1$, $|g(x)| \leq 3\lambda/4$ by (P1). Let $Q = \prod_k [a_k, a_k + s]$ be a cube containing x . If $s \leq h$, then $|g(y)| \leq |g(x)| + L(g)s \leq \lambda$

for $y \in Q$, so $\int_Q |g| \leq \lambda \mu(Q)$. If $s > h$, then by (F3) with $M = 1/\eta$ there are $u_k, v_k \in \eta\mathbb{Z} \cap [0, 1]$ with $u_k \leq a_k < u_k + \eta$ and $v_k - \eta \leq a_k + s \leq v_k$. The box $R := \prod_k [u_k, v_k]$ contains Q , and its sides lie in $[s, s + 2\eta]$; as $2\eta = h/(2d) < s/(2d)$, $R \in F$ and $\mu(R) \leq (1 + \frac{1}{2d})^d s^d \leq 2\mu(Q)$ (the inequality $(1 + \frac{1}{2d})^d \leq 2$ follows as in (P5)). Since $x \in R$ and x is in no box of V_2 , $R \notin V_2$, and by (P2) $\int_Q |g| \leq \int_R |g| \leq \frac{\lambda}{2} \mu(R) \leq \lambda \mu(Q)$. $\square$

Remark 3.2. The family V_1 can be omitted: every grid cube D belongs to F , and $D \in V_1$ implies $D \in V_2$ by (P1), (P2); and $D \notin V_2$ gives $(|g(x)| - L(g)h)\mu(D) \leq \int_D |g| \leq \frac{\lambda}{2} \mu(D)$, that is $|g(x)| \leq \frac{3\lambda}{4}$, for $x \in D$. Then $c_d = 4^{d+1}$ suffices. We keep V_1 because it makes (ii) immediate.

4 Proof of Theorem A

Let $f = (p_n)$ be an L^1 function on $[0, 1]^d$. Put $g_i := p_i - p_{i+1}$, so that $\|g_i\|_1 \leq 2^{-i}$, $\lambda_i := 2^{-\lfloor i/2 \rfloor}$, and $V_i := V(g_i, \lambda_i, i)$ (Lemma 3.1). Let U_N be the open set whose code lists the boxes of $V_N, V_{N+1}, \dots$ in this order; the sequence of codes is Δ_1^0 in (p_n) , and $U_{N+1} \subseteq U_N$. By Lemma 3.1(i), $\mu(\bigcup V_i) \leq c_d 2^{-\lfloor i/2 \rfloor} + 2^{-i}$. Since $\sum_{i \geq N} 2^{-\lfloor i/2 \rfloor} \leq 4 \cdot 2^{-\lceil N/2 \rceil}$ and $\sum_{i \geq N} 2^{-i} \leq 2 \cdot 2^{-\lceil N/2 \rceil}$, and with $\tau_n := 4 \cdot 2^{-\lfloor n/2 \rfloor}$,

$$\sum_{i \geq N} \mu\left(\bigcup V_i\right) \leq \varepsilon_N, \quad \sum_{i \geq n} \lambda_i \leq \tau_n. \quad (1)$$

Lemma 4.1 (RCA₀). $\mu_{\text{int}}(U_N) \leq \varepsilon_N$.

Proof. A finite set of boxes of U_N lies in $V_N \cup \dots \cup V_M$ for some M , and its union has measure at most $\sum_{i=N}^M \mu(\bigcup V_i) \leq \varepsilon_N$. $\square$

Lemma 4.2 (RCA₀). *Let $x \in [0, 1]^d$ be in no box of U_N . Then $f(x) = \lim_n p_n(x)$ exists, $|f(x) - p_n(x)| \leq \tau_n$ for $n \geq N$, and for every cube Q containing x and every $n \geq N$*

$$\left| \frac{\int_Q f}{\mu(Q)} - f(x) \right| \leq 2\tau_n + L(p_n) s(Q).$$

In particular x is a Lebesgue point of f .

Proof. For $i \geq N$, x is in no box of V_i , so by Lemma 3.1(ii) $|g_i(x)| \leq \lambda_i$ and $\int_Q |g_i| \leq \lambda_i \mu(Q)$ for every cube $Q \ni x$. For $N \leq n < m$, (1) gives $|p_n(x) - p_m(x)| \leq \sum_{i=n}^{m-1} |g_i(x)| \leq \tau_n$; this is an explicit modulus, so the limit $f(x)$ exists and $|f(x) - p_n(x)| \leq \tau_n$. Likewise $|\int_Q p_m - \int_Q p_n| \leq \tau_n \mu(Q)$, and letting $m \rightarrow \infty$, $|\int_Q f - \int_Q p_n| \leq \tau_n \mu(Q)$. By (F1), $|\int_Q p_n - p_n(x)\mu(Q)| \leq L(p_n) s(Q) \mu(Q)$. Combining the three estimates gives the inequality. Given a rational $e > 0$, take $n \geq N$ with $\tau_n \leq e/4$ and $\delta := e/(2L(p_n))$. $\square$

Located measure. For a rational box $C \subseteq [0, 1]^d$ and $M \geq N$ let $\varphi_M(C) := \mu(C \cap \bigcup_{i=N}^M V_i)$, a rational. For $M < M'$, $0 \leq \varphi_{M'}(C) - \varphi_M(C) \leq \varepsilon_{M+1}$ by (1); so $\varphi(C) := \lim_M \varphi_M(C)$ is a real with an explicit modulus, uniformly in C , and it is the intensional measure of $C \cap U_N$. The *children* of a box are the 2^d boxes obtained by halving every side. From the finite stages: (L1) $\varphi(C)$ is the sum of φ over the children of C ; (L2) $0 \leq \varphi(C) \leq \min(\mu(C), \varepsilon_N)$; (L3) if C lies in one box of U_N , then $\varphi(C) = \mu(C)$. Put $\psi(C) := \mu(C) - \varphi(C) \geq 0$; by (L1), ψ is additive over children.

Lemma 4.3 (RCA₀). *Every rational box $C \subseteq [0, 1]^d$ with $\mu(C) > \varepsilon_N$ contains a point that is in no box of U_N , and hence a Lebesgue point of f .*

Proof. Put $C_0 := C$ and $\beta_0 := \mu(C) - \varepsilon_N$, so $0 < \beta_0 \leq \psi(C_0)$ by (L2). Given C_j and a rational β_j with $0 < \beta_j \leq \psi(C_j)$, compute rationals a_t with $|a_t - \psi(C^t)| \leq \beta_j/(4 \cdot 2^d)$ for the children C^t of C_j , let C_{j+1} be a child with the largest a_t (the first one in a fixed order), and put $\beta_{j+1} := \beta_j/(2 \cdot 2^d)$. Some child has $\psi \geq \psi(C_j)/2^d \geq \beta_j/2^d$, so $\psi(C_{j+1}) \geq \beta_j/2^d - 2\beta_j/(4 \cdot 2^d) = \beta_{j+1}$. The sequence (C_j, β_j) is defined by primitive recursion from (p_n) , so it exists; the corners of the C_j converge with an explicit modulus to a point x that lies in every C_j . Suppose x lies in a box B of some V_i , $i \geq N$. As B is relatively open, there is a rational $r > 0$ such that every $y \in [0, 1]^d$ with $|y - x| < r$ lies in B ; for j so large that the sides of C_j are smaller than r , $C_j \subseteq B$, and $\psi(C_j) = 0$ by (L3), a contradiction. So x is in no box of U_N , and Lemma 4.2 applies. $\square$

Lemma 4.4 (RCA₀). *Let $h \in C([0, 1]^d)$ with $0 \leq h \leq 1$ and $h(x) = 0$ for every $x \in [0, 1]^d$ that is not in U_N . Then $\int h \leq \varepsilon_N$. So $\mu_{\text{ext}}(U_N) \leq \varepsilon_N$.*

Proof. Suppose $\int h > \varepsilon_N + 3\eta$ for a rational $\eta > 0$. For rational boxes C put $\Gamma(C) := \int_C h - \varphi(C)$ and $D(C) := \Gamma(C)/\mu(C)$. By (L1), Γ is additive over children, and since the children have equal measure, $D(C)$ is the average of D over the children of C . By (L2), $D([0, 1]^d) \geq \int h - \varepsilon_N > 3\eta$. Choose $C_0 := [0, 1]^d$ and, recursively, a child C_{j+1} of C_j whose value $D(C_{j+1})$ is within $\eta 2^{-j-1}$ of the largest value of D over the children (using rational approximations); then $D(C_{j+1}) \geq D(C_j) - \eta 2^{-j-1}$, so $D(C_j) > 2\eta$ for all j . Let x be the common point of the C_j . If x lay in a box of U_N , then as in the proof of Lemma 4.3 C_j would lie in that box for large j , $\varphi(C_j) = \mu(C_j)$ by (L3), and $\Gamma(C_j) \leq 0$ because $h \leq 1$, a contradiction; so x is not in U_N and $h(x) = 0$. On the other hand, $\varphi \geq 0$ and $h(y) \leq h(x) + \omega(|y - x|)$, where ω is a modulus of uniform continuity of h , give $2\eta < D(C_j) \leq h(x) + \omega(\text{side}(C_j))$ for all j , so $h(x) \geq 2\eta$. This contradiction shows $\int h \leq \varepsilon_N + 3\eta$ for every $\eta > 0$. $\square$

Corollary 4.5 (RCA₀). *$\bar{\mu}(U_N) \leq \varepsilon_N$, for box codes and, if $d = 1$, for binary codes.*

Proof. Let V be a code with the same points as U_N , let $\mathcal{F}$ be a finite set of boxes of V and $\theta > 0$ rational. For a box $B = \prod_k ((a_k, b_k) \cap [0, 1])$ let h_B be the product of the trapezoids that equal 1 on $[a_k + \theta, b_k - \theta]$, vanish outside (a_k, b_k) and are linear in between, and let $h := \max_{B \in \mathcal{F}} h_B$. Then h is Lipschitz with rational data, $0 \leq h \leq 1$, $\int h \geq \mu(\bigcup \mathcal{F}) - 2d\theta|\mathcal{F}|$, and $h(y) > 0$ only for y in a box of $\mathcal{F}$, hence in U_N . By Lemma 4.4, $\mu(\bigcup \mathcal{F}) \leq \varepsilon_N + 2d\theta|\mathcal{F}|$ for every θ ; so $\mu_{\text{int}}(V) \leq \varepsilon_N$. For $d = 1$ and a binary code S of $\pi^{-1}(U_N)$, fix n and let J be the union of the closed intervals $I_\rho = [0.\rho, 0.\rho + 2^{-n}]$, $\rho \in 2^n \setminus T_S$; then $1 - |T_S \cap 2^n|/2^n = \mu(J)$. A point of the interior of J is $\pi(Z)$ for some Z extending such a ρ (at a common end point of two adjacent intervals take $Z = \rho 1^\infty$), and ρ has a prefix in S ; so the interior of J lies in U_N . The function h built as above for the finitely many components of the interior of J gives $\mu(J) \leq \varepsilon_N$ by Lemma 4.4; so $\mu(S) \leq \varepsilon_N$. $\square$

Proof of Theorem A. Part (a) is Lemmas 4.1 and 4.4 and Corollary 4.5, part (b) is Lemma 4.2, and part (c) is the construction of φ and Lemma 4.3. $\square$

Proof of Corollary 1.1. (N_m): let $N(k)$ be least with $\varepsilon_{N(k)} \leq 2^{-k}$ and $W_k := U_{N(k)}$; use Theorem A(a),(b). (E): Lemma 4.3 with $C = [0, 1]^d$, or with any rational box C and N such that $\varepsilon_N < \mu(C)$. (R): (W_k) is a (p_n) -computable sequence of codes with $\mu_{\text{int}}(W_k) \leq 2^{-k}$, so a point that is Martin-Löf random relative to (p_n) is not in some W_k and is a Lebesgue point. (Since the measures of the W_k are computable from (p_n) by Theorem A(c), the same holds for points that are Schnorr random relative to (p_n) , as in [PRS14, Theorem 3.16].) The ω -model REC satisfies RCA₀, hence LDT under these readings; it does not satisfy WWKL₀, which is equivalent to the existence of Martin-Löf random reals relative to every set (Simpson and Yu [YS90], as reported in [NTY21, Section 5]), since no computable real is random. $\square$

In REC the reading (R) holds vacuously, since a computable point lies in the computable boxes of side 2^{-k} centred at its rational approximations of error 2^{-k-2} . The substantive readings are (N_{ext}), (N_{bar}) and (E), as the next lemma shows.

Lemma 4.6 (RCA₀). *If WWKL₀ fails, then for every $d \geq 1$ there are open sets $W'_0, W'_1, \dots$ with $\mu_{\text{int}}(W'_k) \leq 2^{-k}$, each containing every point of $[0, 1]^d$.*

Proof. Let T be a tree without path and $\alpha > 0$ rational with $|T \cap 2^n| \geq \alpha 2^n$ for all n . The set E of strings not in T whose parent is in T is prefix-free; every $Z \in 2^{\mathbb{N}}$ has a prefix in E , and every finite $F \subseteq E$ has $\sum_{\tau \in F} 2^{-|\tau|} \leq 1 - \alpha$ (the strings of length $\ell \geq \max_F |\tau|$ extending members of F are distinct and not in T). Let $E^{(i)}$ be the set of concatenations of i members of E . It is prefix-free, its finite subsets have weight at most $(1 - \alpha)^i$, and by Σ_1^0 induction on i every Z has a prefix in $E^{(i)}$ (apply the property of E to the tail of Z). Choose i_k with $(1 - \alpha)^{i_k} \leq 2^{-k-1}$, list the dyadic rationals of $[0, 1]$ as $r_0, r_1, \dots$, and let W'_k consist of the boxes $(0.\tau, 0.\tau + 2^{-|\tau|}) \times [0, 1]^{d-1}$, $\tau \in E^{(i_k)}$, and $((r_j - 2^{-k-j-3}, r_j + 2^{-k-j-3}) \cap [0, 1]) \times [0, 1]^{d-1}$, $j \in \mathbb{N}$. Finite unions have measure at most $2^{-k-1} + 2^{-k-1}$. If $x_1 = r_j$ for some j , then $x \in W'_k$; otherwise x_1 has a binary expansion Z , computable from x_1 , and x_1 lies in the open interval of the prefix of Z in $E^{(i_k)}$. $\square$

This is the iteration used in the proof of [NTY21, Lemma 6.6]. A similar vacuity of a measure statement without WWKL₀ is observed by Westrick [Wes25].

Remark 4.7. Lemma 4.6 makes (N_{int}) true for every pointwise property in every model of RCA₀ + $\neg$ WWKL₀, and (O_{int}) false there even for $f = 0$. So LDT under (N_{int}) has content only in models of WWKL₀, where it follows from Theorem A and presumably also from a formalisation of Pathak's theorem as in Simpson's observation, which [Mon11] reports without fixing a reading. For μ_{ext} and $\bar{\mu}$ the argument fails: an open set containing every point has $\mu_{\text{ext}} = \bar{\mu} = 1$ in RCA₀ (take $h \equiv 1$, or the code consisting of the box $[0, 1]^d$).

Remark 4.8. If cubes need not lie in $[0, 1]^d$ (with f extended by 0, or with averages over $Q \cap [0, 1]^d$), add to U_N the relatively open boxes of points with a coordinate in $[0, 2^{-N}/(2d))$ or in $(1 - 2^{-N}/(2d), 1]$. They have total measure at most 2^{-N} , so Lemmas 4.1, 4.3 and 4.4 hold with $\varepsilon_N + 2^{-N}$, and a point outside them lies in no cube of side $< 2^{-N}/(2d)$ that leaves $[0, 1]^d$. If the value at x is defined as the limit of the averages, Lemma 4.2 shows that this limit exists and equals $\lim p_n(x)$ outside U_N .

Remark 4.9. By [NTY21, Theorem 6.8, $1 \Leftrightarrow 5$], every model of RCA₀ + $\neg$ WWKL₀ contains an effectively uniformly continuous, absolutely continuous function F on $[0, 1]$ with a rational presentation that is pseudo-differentiable, in the sense of [NTY21, Definition 6.1], at no point. In such a model no L^1 function g on $[0, 1]$ in the sense of Definition 2.1 satisfies $F(b) - F(a) = \int_{[a,b]} g$ for all rationals $0 \leq a < b \leq 1$: by Theorem A(c), g would have a Lebesgue point $x \in (0, 1)$, and the slopes $(F(b) - F(a))/(b - a)$ over rationals $a \leq x \leq b$ would converge to $g(x)$ as $b - a \rightarrow 0$, so F would be pseudo-differentiable at x . So Theorem A(c) does not contradict [NTY21]: it depends on the rate of L^1 convergence required in Definition 2.1, which the derivative of such a function need not have in RCA₀.

5 Proof of Theorem B

Lemma 5.1 (WWKL₀). *If a code V of an open set contains every point of $[0, 1]^d$, then $\mu_{\text{int}}(V) = 1$.*

Proof. Send $Z \in 2^{\mathbb{N}}$ to the point $x(Z)$ whose j -th coordinate ($j < d$) is $\sum_l Z(ld + j)2^{-l-1}$. For a string σ let D_σ be the closed dyadic box of the points $x(Z)$, $Z \succ \sigma$; then $\mu(D_\sigma) = 2^{-|\sigma|}$, and its sides are at most $2^{-\lfloor |\sigma|/d \rfloor}$. Let S be the set of strings σ such that D_σ lies in one of the first $|\sigma|$ boxes of V , and T the tree of strings without a prefix in S . T has no path: $x(Z)$ lies in a box B_i of V , which is

relatively open, so $D_{Z \upharpoonright n} \subseteq B_i$ and $Z \upharpoonright n \in S$ for all large n . As $|T \cap 2^n|/2^n$ is non-increasing, WWKL₀ gives for every $e > 0$ an n with $|T \cap 2^n| < e2^n$. The shortest prefixes in S of the strings of length n outside T form a prefix-free set P with $\sum_{\sigma \in P} 2^{-|\sigma|} = 1 - |T \cap 2^n|/2^n > 1 - e$, and the boxes D_σ , $\sigma \in P$, are almost disjoint and lie in the first n boxes of V . So these n boxes have union of measure $> 1 - e$. $\square$

For codes of open sets in $2^\mathbb{N}$ the corresponding statement is equivalent to WWKL₀ by Yu and Simpson [YS90], as reported in [ADR12, Section 3]; Lemma 5.1 transfers it to $[0, 1]^d$.

Proposition 5.2 (WWKL₀). *LDT holds under (O_m) for $m \in \{\text{int}, \text{ext}, \text{bar}\}$.*

Proof. Let f be an L^1 function and U an open set containing every Lebesgue point of f . By Theorem A(b) every point of $[0, 1]^d$ is in U or in U_N . Let V be any code with the points of U . The code listing the boxes of V and of U_N alternately contains every point, so by Lemma 5.1 for every $e > 0$ some finite union of its boxes has measure $> 1 - e$; the part from U_N has measure at most ε_N (Lemma 4.1). So $\mu_{\text{int}}(V) \geq 1 - e - \varepsilon_N$ for all e and N , that is $\mu_{\text{int}}(V) = 1$. Hence $\mu_{\text{int}}(U) = \bar{\mu}(U) = 1$, and $\mu_{\text{ext}}(U) = 1$ by the trapezoid functions of Corollary 4.5. For $d = 1$ and a binary code S of $\pi^{-1}(U)$, let S_N be the set of strings σ such that I_σ lies in one of the first $|\sigma|$ boxes of U_N ; it is a binary code of $\pi^{-1}(U_N)$. The tree $T_S \cap T_{S_N}$ has no path, so by WWKL₀ for every $e > 0$ there is an n with $|T_S \cap T_{S_N} \cap 2^n| < e2^n$; as $1 - |T_{S_N} \cap 2^n|/2^n \leq \varepsilon_N$ by Corollary 4.5, we get $|T_S \cap 2^n|/2^n \leq e + \varepsilon_N$, and $\mu(S) = 1$. $\square$

Lemma 5.3 (RCA₀; Nies, Triplett and Yokoyama). *Let T be a tree without path and $\alpha > 0$ rational with $|T \cap 2^n| \geq \alpha 2^n$ for all n . Let $T' := \{\sigma 0^k : \sigma \in T, k \in \mathbb{N}\}$, let E be the set of strings $\tau \notin T'$ whose parent is in T' , let $U := \bigcup_{\tau \in E} (0.\tau, 0.\tau + 2^{-|\tau|})$ and $U_d := U \times [0, 1]^{d-1}$. Then*

- (i) *every $x \in [0, 1]^d$ whose first coordinate is not a dyadic rational is in U_d ;*
- (ii) *no x whose first coordinate is $0.\sigma$ with $\sigma \in T$ is in U_d ;*
- (iii) *$\mu_{\text{int}}(U_d) \leq 1 - \alpha$, $\mu_{\text{ext}}(U_d) \leq 1 - \alpha$ and $\bar{\mu}(U_d) \leq 1 - \alpha$ (for box codes, and for $d = 1$ for binary codes).*

Proof. Membership in T' is decided by removing final zeros, and T' is a tree. A path of T' with infinitely many 1s would be a path of T (an element of T' ending in 1 is in T), so the paths of T' end in 0^∞ . (i) A non-dyadic $z = x_1$ has a unique expansion Z , which is not a path of T' ; for the least n with $Z \upharpoonright n \notin T'$ (so $n \geq 1$), $\tau := Z \upharpoonright n \in E$, and z lies in the open interval of τ . (ii) If $0.\sigma$ lay in the open interval of some $\tau \in E$, then $\sigma 0^\infty$ would extend τ , so τ would be a prefix of some $\sigma 0^k \in T'$, hence in T' . (iii) E is prefix-free (a proper prefix of $\tau' \in E$ is a prefix of the parent of τ' , so it is in T'). For finite $F \subseteq E$ and $\ell \geq \max_F |\tau|$, the strings of length ℓ extending members of F are distinct and not in $T' \supseteq T$, so $\sum_F 2^{-|\tau|} \leq 1 - \alpha$; this is the bound for μ_{int} . Let $h \in C([0, 1]^d)$, $0 \leq h \leq 1$, vanish at the points outside U_d . For $\sigma \in T \cap 2^n$, every y in the slab $I_\sigma \times [0, 1]^{d-1}$ is within 2^{-n} of a point with first coordinate $0.\sigma$, where h vanishes by (ii); so $h \leq \omega(2^{-n})$ on these slabs, whose total measure is at least α , and $\int h \leq 1 - \alpha + \omega(2^{-n})$ for every n , where ω is a modulus of uniform continuity of h . This is the bound for μ_{ext} , and the bound for $\bar{\mu}$ follows from it as in Corollary 4.5. $\square$

This is the set constructed in the proof of [NTY21, Theorem 6.8] ($\neg 1 \Rightarrow \neg 3$), where the bound $k < |\tau|$ is put into the definition of T' , which does not matter. Since the intervals are open, T itself can also be used in place of T' : if $0.\sigma$ with $\sigma \in T$ lay in the open interval of a string $\tau \notin T$ whose parent is in T , then τ would be a prefix of some $\sigma 0^k$ and, not being in T , would extend σ ; so $0.\sigma = 0.\tau$ would be the left end point of that interval.

Lemma 5.4 (RCA₀). *Let $r_0, r_1, \dots$ list the dyadic rationals of $[0, 1]$. For $k \geq 0$ and $j \geq 1$ let $J := 2(j + k)$, $M := 2^{3J+6}$ and $P_{k,j}(t) := (1 - (t - r_k)^2)^M$, and let $p_n(x) := \sum \{P_{k,j}(x_1) : k \geq 0, j \geq 1, k + j \leq n + 1\}$. Then $f = (p_n)$ is an L^1 function on $[0, 1]^d$; if $x_1 = r_k$, then $p_n(x) \geq n + 1 - k$ for $n \geq k$, so $f(x)$ is undefined, and $\int_Q f/\mu(Q) \rightarrow \infty$ for cubes $Q \ni x$ with $s(Q) \rightarrow 0$.*

Proof. On $[0, 1]$, $0 \leq P_{k,j} \leq 1$ and $P_{k,j}(r_k) = 1$. With $\delta := 2^{-J-2}$, $P_{k,j} \leq 1$ where $|t - r_k| \leq \delta$ and $P_{k,j} \leq (1 - \delta^2)^M$ elsewhere; since $(1 - v)^M \leq 1/(1 + Mv)$ for $v \in [0, 1]$ (as $(1 - v)^M(1 + v)^M \leq 1$ and $(1 + v)^M \geq 1 + Mv$) and $M\delta^2 = 2^{J+2}$,

$$\int_0^1 P_{k,j} \leq 2\delta + \frac{1}{1 + 2^{J+2}} < 2^{-J-1} + 2^{-J-2} < 2^{-J}.$$

The difference $p_{n+1} - p_n$ is the sum of the $n + 2$ terms with $k + j = n + 2$, which are non-negative and have $J = 2(n + 2)$; so $\|p_{n+1} - p_n\|_1 < (n + 2)4^{-(n+2)} \leq 2^{-n}$. If $x_1 = r_k$, the terms with this k give $p_n(x) \geq n + 1 - k$. For a cube $Q \ni x$ of side $s \leq 1$, $|y_1 - r_k| \leq s$ on Q , so $P_{k,j} \geq (1 - s^2)^M \geq 1 - Ms^2 \geq \frac{1}{2}$ there whenever $Ms^2 \leq \frac{1}{2}$. As (p_n) is non-decreasing, $\int_Q f \geq \int_Q p_n$ for every n , and $\int_Q f/\mu(Q)$ is at least half the number of $j \geq 1$ with $2^{6(j+k)+6}s^2 \leq \frac{1}{2}$, which tends to infinity as $s \rightarrow 0$. $\square$

Proof of Theorem B. WWKL₀ implies LDT under (O_m) by Proposition 5.2. Conversely, assume RCA₀ + $\neg$ WWKL₀ and take T, α, U_d as in Lemma 5.3 and f as in Lemma 5.4. A point with dyadic first coordinate is not a Lebesgue point of f (the value is undefined, and the averages diverge), so by Lemma 5.3(i) the open set U_d contains every Lebesgue point of f , while $\mu_m(U_d) \leq 1 - \alpha < 1$ by Lemma 5.3(iii). $\square$

Remark 5.5. The proof shows that (O_m) is equivalent to the statement that every open set containing all points with non-dyadic first coordinate has μ_m -measure 1 (compare [NTY21, Remark 6.9], on open sets with countable complement). If only open cubes inside $[0, 1]^d$ are used and the value is the limit of the averages, boundary points of $[0, 1]^d$ carry no cubes; adding to U_d the strips of points with a coordinate within $\alpha/(8d)$ of 0 or 1 keeps all three measures at most $1 - \alpha/2$ (for μ_{ext} use $h \leq h(1 - \chi) + \chi$ with a continuous cut-off χ).

6 Computations and verification

No statement of this note depends on a computation. The program `check_note.py` of the reproducibility package (standard library, exact rational arithmetic, about half a minute) checks the constants of Lemma 3.1, (1) and Lemma 5.4 (with the exact value $\int_{-1}^1 (1 - u^2)^{4096} du = 0.0277\dots < \frac{1}{4}$ for the smallest case); Lemma 3.1 for $d = 1$ on 40 random polynomials, with part (ii) at all 3,721 points of a fine grid outside V and all grid intervals containing them; the Vitali step for $d = 2, 3$; the descent of Lemma 4.3; and the finite combinatorics of Lemmas 4.6, 5.1 and 5.3. Modified copies with deliberately wrong thresholds in Lemma 3.1 and a wrong definition of E in Lemma 5.3 were rejected by these tests; they are not part of the package. The package also contains, unchanged, the programs with which the results were first tested and those of the first two verification runs, with their recorded outputs and a table of their numbering; when the package was assembled, all of them were run again and reproduced their recorded outputs (one program now prints six additional lines, as explained in the README).

The folder `independent_run_2/` contains the programs of the verification run 2 described below, with their recorded outputs. Written from the text of this note, they check in exact arithmetic the constants of Lemmas 3.1 and 5.4 and (1) (with the exact integrals $\int_0^1 (1 - (t - r)^2)^{4096} dt$ for six values of r); they test Lemma 3.1 with an implementation of its own, for $d = 1$ on 101 polynomials, with part (ii) at 10,197 points outside V and for 498,642 intervals containing them in arbitrary position,

and for $d = 2$ on 36 polynomials, with all 277 million boxes of F , at 7,675 points outside V and for 207,225 cubes containing them; and they check the descents of Lemmas 4.3 and 4.4, the Vitali step for $d = 2, 3$, and the finite combinatorics of Lemmas 4.6, 5.1 and 5.3, Corollary 4.5, and Proposition 5.2. All checks passed, and for $d = 1$ constructions with V_2 omitted or with the threshold 2λ for V_2 were caught. That run also ran the whole package again from `source.zip` and obtained the recorded outputs.

Verification. The results were first obtained with written proofs and two sanity-check programs. Two independent verification runs, both AI-assisted, followed; each re-derived the proofs and wrote its own programs, and neither read the other’s report. The first run compared the statements with [Mon11] and the cited parts of [Pat09, PRS14, BGS02, ADR12, NTY21, Rut20], checked every constant in exact arithmetic and tested Lemma 3.1 in dimension 2. It found no mathematical error and required seven corrections: the credit to [Yu94, Lemma 2.1] and [Pat09, Lemma 3.2] with a narrower novelty claim, the reference to [NTY21, Remark 6.3], the binary codes in Proposition 5.2, a direct citation of [Pat09], the vacuity of (R) in REC, three minor points (the space $C([0, 1]^d)$, (F3), the bound in T'), and an explicit judgement on the intended reading. The second run tested the proofs with its own programs (Lemma 3.1 for $d = 1$ exactly on 297 polynomials and for $d = 2$ on 210, the Vitali step, Lemmas 4.3, 5.3 and 5.4), which found no counterexample and rejected deliberately wrong inputs, and it reproduced the outputs of the first programs. It observed the case split of Remark 4.7, found [Rut13] and the remark in [BMN16], and required four corrections, including the completion of the references. All required corrections were made; its optional simplifications are Remark 3.2 and the remark after Lemma 5.3. The exact program of this note was written and run for the final text.

The present text was written after these two runs. A further AI-assisted verification run, called run 2 in the package, then examined it as written. It read again the cited parts of [Mon11, Pat09, PRS14, BGS02, ADR12, NTY21, Rut13, Rut20, BMN16] and compared every statement about them with these sources; re-derived every proof line by line, among them the parts written after the first two runs (Lemma 4.6, Remark 4.7 and the binary codes in Corollary 4.5 and Proposition 5.2), and checked that Theorem A uses no more than Δ_1^0 comprehension and Σ_1^0 induction; checked the references with Crossref; repeated the literature search; and wrote the programs described above. It found no error in a proof. Its corrections: the covering statement of Yu and Simpson is reported in Section 3 of [ADR12], not in Section 2; [NTY21, Theorem 6.8(5)] is now cited, with Remark 4.9, which explains why it does not conflict with Theorem A(c); Remark 4.7 and the remark after Lemma 5.3 were reworted; and the description of the negative tests of `check_note.py` was corrected.

Scope and priority. The note answers Question 17 of [Mon11] for each reading in Table 1; that the null-cover reading is the intended one is our judgement, not a theorem. The question is Montalbán’s, and the derivation of LDT in $WWKL_0$ is Simpson’s, from Pathak’s theorem [Pat09]. We use the maximal-function method of [Pat09, PRS14] (with Rute’s effective version [Rut13, Theorem 4.10] as the closest computable statement), Yu’s convergence lemma [Yu94, Lemma 2.1] as reported in [ADR12], the measures and the finite Vitali selection of [BGS02], and the outer reading, the open set of Lemma 5.3 and the existence-versus-almost-surely pattern of [NTY21, Definition 6.4, Remark 6.3, Theorem 6.8, Remark 6.9]; Remark 4.9 compares Theorem A(c) with [NTY21, Theorem 6.8(5)]. To our knowledge only Lemma 3.1, the cube averages and parts (a) and (c) of Theorem A, and Theorem B for L^1 functions are not in print; they are close to these techniques and may be known to experts, and we claim no priority.

What was read: [Mon11] in the author’s preprint (Section 3.4); [Pat09] in the journal’s open version; [PRS14, BGS02] in the authors’ copies (the statements cited); [ADR12] (Sections 2 to 4), [NTY21] (Sections 5 and 6), [Rut20] (Section 3.5) and [BMN16] (Section 1) in their arXiv versions,

whose numbering we use, the journal versions not being compared; [Rut13] (Section 4.2); of [Wes25] the abstract. [Yu94, YS90, Sim04] were not read; what we say about them is taken from [ADR12, NTY21, Rut20].

The searches were made on 18 October 2026: arXiv, zbMATH and Crossref for the Lebesgue differentiation theorem together with reverse mathematics, WWKL_0 , second-order arithmetic or randomness; the works citing [Mon11] (104), [PRS14] (48), [Pat09] (18) and [NTY21] (15) in Semantic Scholar, by title; [MNZ16]; Nies’s Logic Blogs; Simpson’s lists of papers and talks; and a few web searches (OpenAlex refused the requests by its rate limit). No printed answer to the question was found in these sources, nor in the searches repeated on the same day by the verification run 2 (arXiv, Crossref, zbMATH, the citation counts of OpenAlex and one web search); a search that finds nothing is not a proof of novelty. A note of E. T. Dean on Pathak’s theorem (2009), listed by Semantic Scholar without a venue, could not be obtained.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

References

- [ADR12] Jeremy Avigad, Edward T. Dean, and Jason Rute. Algorithmic randomness, reverse mathematics, and the dominated convergence theorem. *Annals of Pure and Applied Logic*, 163(12):1854–1864, 2012. [arXiv:1106.0775](#), [doi:10.1016/j.apal.2012.05.010](#).
- [BGS02] Douglas K. Brown, Mariagnese Giusto, and Stephen G. Simpson. Vitali’s theorem and WWKL. *Archive for Mathematical Logic*, 41(2):191–206, 2002. [doi:10.1007/s001530100100](#).
- [BMN16] Vasco Brattka, Joseph S. Miller, and André Nies. Randomness and differentiability. *Transactions of the American Mathematical Society*, 368(1):581–605, 2016. [arXiv:1104.4465](#), [doi:10.1090/tran/6484](#).
- [MNZ16] Kenshi Miyabe, André Nies, and Jing Zhang. Using almost-everywhere theorems from analysis to study randomness. *The Bulletin of Symbolic Logic*, 22(3):305–331, 2016. [arXiv:1411.0732](#), [doi:10.1017/bsl.2016.10](#).
- [Mon11] Antonio Montalbán. Open questions in reverse mathematics. *The Bulletin of Symbolic Logic*, 17(3):431–454, 2011. Author’s preprint: <https://math.berkeley.edu/~antonio/papers/questionsRM.pdf>. [doi:10.2178/bsl/1309952320](#).
- [NTY21] André Nies, Marcus A. Triplett, and Keita Yokoyama. The reverse mathematics of theorems of Jordan and Lebesgue. *The Journal of Symbolic Logic*, 86(4):1657–1675, 2021. [arXiv:1704.00931](#), [doi:10.1017/jsl.2021.16](#).
- [Pat09] Noopur Pathak. A computational aspect of the Lebesgue differentiation theorem. *Journal of Logic and Analysis*, 1:Paper 9, 1–15, 2009. [doi:10.4115/jla.2009.1.9](#).
- [PRS14] Noopur Pathak, Cristóbal Rojas, and Stephen G. Simpson. Schnorr randomness and the Lebesgue differentiation theorem. *Proceedings of the American Mathematical Society*, 142(1):335–349, 2014. [doi:10.1090/S0002-9939-2013-11710-7](#).

- [Rut13] Jason Rute. *Topics in algorithmic randomness and computable analysis*. PhD thesis, Carnegie Mellon University, 2013. URL: https://www.andrew.cmu.edu/user/avigad/Students/rute_dissertation.pdf.
- [Rut20] Jason Rute. Algorithmic randomness and constructive/computable measure theory. In Johanna N. Y. Franklin and Christopher P. Porter, editors, *Algorithmic Randomness: Progress and Prospects*, volume 50 of *Lecture Notes in Logic*, pages 58–114. Cambridge University Press, 2020. [arXiv:1812.03375](#), [doi:10.1017/9781108781718.004](#).
- [Sim04] Ksenija Simić. *Aspects of ergodic theory in subsystems of second-order arithmetic*. PhD thesis, Carnegie Mellon University, 2004.
- [Sim09] Stephen G. Simpson. *Subsystems of Second Order Arithmetic*. Perspectives in Logic. Cambridge University Press, second edition, 2009. [doi:10.1017/CB09780511581007](#).
- [Wes25] Linda Westrick. Completely determined Borel sets and measurability. In *Higher Recursion Theory and Set Theory*, Lecture Notes Series, Institute for Mathematical Sciences, National University of Singapore, pages 373–397. World Scientific, 2025. [arXiv:2001.01881](#), [doi:10.1142/9789819806584_0015](#).
- [YS90] Xiaokang Yu and Stephen G. Simpson. Measure theory and weak König’s lemma. *Archive for Mathematical Logic*, 30(3):171–180, 1990. [doi:10.1007/BF01621469](#).
- [Yu94] Xiaokang Yu. Lebesgue convergence theorems and reverse mathematics. *Mathematical Logic Quarterly*, 40(1):1–13, 1994. [doi:10.1002/malq.19940400102](#).