

The Stress-Sign Stratification of Four Labeled Points in Three Dimensions

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Abstract

We classify the stress-sign stratification of all labeled four-point configurations in $\mathbb{R}^3$ for the complete graph K_4 . Coincident points and every affine-rank degeneration are included. There are exactly 65 connected strata: two of affine rank three, 25 of affine rank two, 37 of affine rank one, and the diagonal. We give their dimensions and the entire closure order, which has 800 strict incidences and 207 Hasse edges. Rank-two strata are indexed by mixed affine-dependence sign vectors modulo reversal; collinear strata are indexed by ordered coincidence partitions modulo reversal. Their boundary relation is a closed-interval intersection condition, and an explicit matrix perturbation realizes every asserted rank-two degeneration. In every ambient dimension $d \geq 4$ the two top strata merge, giving 64 strata with the same remaining incidence rules. This answers the $B_3(K_4)$ component of Karpenkov's Problem 1; the $B_2(K_6)$ and $B_3(K_5)$ components are not resolved here.

1 The question and the classification

For a labeled configuration $P = (p_1, p_2, p_3, p_4)$ in $\mathbb{R}^d$, a self-stress of the complete graph K_4 is a vector $w = (w_{ij})_{1 \leq i < j \leq 4} \in \mathbb{R}^6$, with $w_{ji} = w_{ij}$, satisfying

$$\sum_{j \neq i} w_{ij}(p_j - p_i) = 0 \quad (1 \leq i \leq 4). \quad (1)$$

Write $W(P)$ for this linear space. Following the configuration-space framework of Doray, Karpenkov and Schepers [1], two fibers are equivalent if there is a homeomorphism $h : W(P) \rightarrow W(Q)$ such that $\text{sgn}(h(w)_{ij}) = \text{sgn}(w_{ij})$ for every stress and every edge. A stratum of $B_d(K_4) = (\mathbb{R}^d)^4$ is a connected component of a fiber-equivalence class. The sign zero is part of this condition. All points remain labeled, coincidences are allowed, and no quotient by affine or Euclidean transformations is taken.

Karpenkov [3, Problem 1] asks for the combinatorics of $B_2(K_6)$, $B_3(K_4)$ and $B_3(K_5)$, meaning the strata, their dimensions and their adjacency. The planar classifications of $B_2(K_4)$ and $B_2(K_5)$ in [2] provide the preceding cases. Our result concerns the complete four-point component, together with its higher-dimensional extension. It does not assert a solution of the other two questions.

Let r denote the affine rank of a configuration. At rank two, the affine dependence is unique up to scale. Its sign vector $\sigma \in \{-, 0, +\}^4$ has both signs and is considered modulo simultaneous reversal. Write S_σ for the corresponding set, $k = |\text{supp } \sigma|$, and

$$P_\sigma = \{i : \sigma_i = +\}, \quad N_\sigma = \{i : \sigma_i = -\}.$$

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At rank one, the coincidence blocks form an ordered partition $\pi = (B_1, \dots, B_m)$ of $\{1, 2, 3, 4\}$ along the line, considered modulo reversal, with $m \geq 2$. Write L_π for this set. Finally, let D be the diagonal and let T_+ and T_- be the positive- and negative-determinant tetrahedra, using $\det[p_2 - p_1 \ p_3 - p_1 \ p_4 - p_1]$.

Theorem 1.1. *The sets in Table 1 are exactly the connected strata of $B_3(K_4)$. In particular there are 65 strata.*

Affine rank	Label	Number	Dimension	$\dim W(P)$
3	T_+, T_-	2	12	0
2	$S_\sigma, k = 4$	7	11	1
2	$S_\sigma, k = 3$	12	10	1
2	$S_\sigma, k = 2$	6	9	1
1	$L_\pi, m = 4$	12	8	3
1	$L_\pi, m = 3$	18	7	3
1	$L_\pi, m = 2$	7	6	3
0	D	1	3	6

Table 1: All labeled stress-sign strata in three dimensions.

Write $X \preceq Y$ when $X \subseteq \bar{Y}$. A sign vector τ is a *weakening* of σ if, after choosing its overall sign, each τ_i equals either 0 or σ_i . Only mixed sign vectors label rank-two strata.

Theorem 1.2. *The entire closure order is specified by the following rules, with reflexive incidences understood.*

1. Every lower-rank stratum lies below each of T_+ and T_- . Neither of these two top strata lies below the other.
2. $S_\tau \preceq S_\sigma$ if and only if τ is a weakening of σ .
3. To test $L_\pi \preceq S_\sigma$, choose any scalar coordinates x_i realizing π . The necessary and sufficient condition is

$$\left[\min_{i \in P_\sigma} x_i, \max_{i \in P_\sigma} x_i \right] \cap \left[\min_{j \in N_\sigma} x_j, \max_{j \in N_\sigma} x_j \right] \neq \emptyset. \quad (2)$$

4. $L_\rho \preceq L_\pi$ if and only if ρ is obtained by merging adjacent blocks of π , up to reversal.
5. D lies below every stratum. There are no other incidences.

Thus every stratum closure is a union of strata. There are 800 strict incidences and 207 edges in the Hasse diagram.

2 Fibers and connected components

2.1 Stress matrices

Let $A(P)$ have columns $(1, p_i)^T$. Associate to a stress its symmetric matrix Ω , with $\Omega_{ij} = -w_{ij}$ for $i \neq j$ and $\Omega_{ii} = \sum_{j \neq i} w_{ij}$. Then $\Omega \mathbf{1} = 0$, and (1) is equivalent to $A(P)\Omega = 0$. Conversely,

a symmetric matrix annihilated on the left by $A(P)$ has the required row sums and determines a unique stress.

If the columns of Z form a basis of $\ker A(P)$, every such matrix is uniquely

$$\Omega = ZQZ^T, \quad Q = Q^T. \quad (3)$$

Indeed, its image lies in $\ker A(P)$, and symmetry makes the orthogonal complement of that space part of its kernel. A left inverse of Z then extracts Q . Since $\dim \ker A(P) = 3 - r$,

$$\dim W(P) = \frac{(3-r)(4-r)}{2}. \quad (4)$$

These dimensions are 0, 1, 3, 6 for ranks 3, 2, 1, 0, respectively. Invariance of dimension for Euclidean spaces therefore excludes fiber equivalence between different affine ranks. The calculation holds on all coincidence loci, not only at generic configurations.

2.2 Ranks three and zero

At rank three the fiber is zero. The configuration space of that rank is $\mathbb{R}^3 \times \mathrm{GL}(3, \mathbb{R})$, with the first factor recording p_1 . Its two components are T_+ and T_- , each of dimension 12. At rank zero all six stresses are free, and the configuration set is $D \cong \mathbb{R}^3$.

2.3 Rank two

Write $\ker A(P) = \mathbb{R}\lambda$. Because $\sum_i \lambda_i = 0$, λ has both signs, and (3) gives

$$w_{ij} = -t\lambda_i\lambda_j \quad (t \in \mathbb{R}). \quad (5)$$

Equal dependence sign vectors give sign-preservingly equivalent lines by keeping the parameter t .

The converse also holds. The nonzero edge support of the stress line is the complete graph on $\mathrm{supp} \lambda$, so it determines this vertex set. If it has at least three vertices, the product of the signs of $-\lambda_i\lambda_j$ on a triangle is negative. Exactly one of the two directions of the stress line has that property. The pairwise products in that direction determine the signs of λ up to simultaneous reversal. If the support has two vertices, their coefficients have opposite signs because they sum to zero. Thus σ is exactly the fiber-equivalence invariant.

For a fixed representative σ , normalize the dependence by

$$C_\sigma = \left\{ \lambda : \mathrm{sgn} \lambda = \sigma, \quad \sum_{i \in P_\sigma} \lambda_i = 1, \quad \sum_{j \in N_\sigma} \lambda_j = -1 \right\}. \quad (6)$$

This is a product of two open simplices of total dimension $k - 2$. Let $H = \mathbf{1}^\perp \subset \mathbb{R}^4$, and center the configuration so that its matrix X satisfies $X\mathbf{1} = 0$. Its affine dependence space is $\mathbb{R}\lambda$ exactly when it induces an injection $H/\mathbb{R}\lambda \rightarrow \mathbb{R}^3$.

The two-dimensional quotient bundle over the contractible cell C_σ is trivial. Its injection space is connected: the space of full-rank 3 by 2 matrices retracts onto the connected Stiefel manifold $V_2(\mathbb{R}^3)$. Consequently S_σ is connected and has dimension

$$3 + (k - 2) + 6 = k + 7.$$

Local quotient frames give manifold charts, so this is the actual stratum dimension. This parameterization also realizes every mixed sign vector. For each support of size k , the number of sign vectors modulo reversal is $2^{k-1} - 1$. Hence the counts are

$$\binom{4}{k} (2^{k-1} - 1) = 7, 12, 6 \quad (k = 4, 3, 2).$$

2.4 Rank one

Write $p_i = c + x_i u$, where $u \neq 0$. A nonzero stress supported on the single edge ij exists exactly when $p_i = p_j$. Thus the stress-sign data recover the coincidence blocks. On a triangle of three distinct blocks, the stress space is a line, and the edge joining the two extreme points has the opposite sign to the other two edges. This identifies the middle block. Betweenness determines a finite linear order up to reversal, so equivalent fibers determine the same π . Restricting a stress to such a triangle means requiring all other coordinates to vanish, which is preserved by the fiber homeomorphism. For two blocks the coincidence partition alone is enough.

Conversely, suppose (x_i) and (y_i) realize the same oriented weak order. Define

$$w'_{ij} = \begin{cases} w_{ij} \frac{x_j - x_i}{y_j - y_i}, & x_i \neq x_j, \\ w_{ij}, & x_i = x_j. \end{cases} \quad (7)$$

All multipliers are positive. The scalar equilibrium equations for y become those for x , and the inverse uses reciprocal multipliers. This is a sign-preserving linear isomorphism, including within-block edges.

For a fixed representative $(B_1, \dots, B_m)$, parameterize L_π by the location $c \in \mathbb{R}^3$ of B_1 , the unit direction $u \in S^2$ toward B_2 , and $0 < t_2 < \dots < t_m$. This is a bijective parameterization with connected domain and dimension $m + 4$. Reversing the representative describes the same set, not another component. There are $m!S(4, m)/2$ ordered partitions modulo reversal, where $S(4, m)$ is a Stirling number of the second kind. This gives 12, 18, 7 for $m = 4, 3, 2$, proving Theorem 1.1.

3 Every boundary incidence

Lemma 3.1. *Let Q have affine rank at most two and let σ be mixed. Then $Q \in \overline{S_\sigma}$ if and only if there is a nonzero $\lambda_0 \in \ker A(Q)$ whose nonzero signs agree with σ .*

Proof. For necessity, normalize the affine dependences of an approaching sequence by (6). The compact closure of C_σ contains a subsequential limit λ_0 . Its positive mass is one and its negative mass is minus one, so it is nonzero. The equation $A(P)\lambda = 0$ passes to the limit.

For sufficiency, normalize λ_0 likewise, choose $\lambda_* \in C_\sigma$, and set

$$\lambda_t = (1 - t)\lambda_0 + t\lambda_* \quad (0 < t < 1).$$

Let X_0 be the centered 3 by 4 matrix of Q , and choose a linear map $J : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ whose restriction to H is an isomorphism. The orthogonal projection onto $H \cap \lambda_t^\perp$ is

$$R_t = I - \frac{\mathbf{1}\mathbf{1}^T}{4} - \frac{\lambda_t \lambda_t^T}{\lambda_t^T \lambda_t}. \quad (8)$$

Define

$$X_t = (X_0 + tJ)R_t. \quad (9)$$

The determinant of $(X_0 + tJ)|_H$ is a nonzero polynomial in t : its leading coefficient is $\det(J|_H) \neq 0$. Therefore this map is invertible for all sufficiently small positive t . Since R_t has rank two, X_t has rank two and satisfies $X_t \mathbf{1} = X_t \lambda_t = 0$. Its affine dependence is precisely $\mathbb{R}\lambda_t$. Add the centroid of Q to its columns to obtain a configuration in S_σ . Finally $R_t \rightarrow R_0$ and $X_0 R_0 = X_0$, so these configurations tend to Q . $\square$

Proof of Theorem 1.2, excluding the counts. First suppose that Q has rank two. Its affine kernel is a line, so Lemma 3.1 gives exactly the weakening rule. For a rank-one limit $q_i = c + x_i u$, the lemma asks for nonnegative weights of mass one on P_σ and on N_σ having the same barycenter. In one dimension this is exactly (2). The condition is constant on the entire weak-order stratum. The lemma proves sufficiency for every point of that stratum, including endpoint coincidences.

In a rank-one limit, coincidence blocks cannot split. If the limit still has rank one, a nonzero limiting difference fixes the local line orientation; the scalar coordinates then have weakly ordered limits. Only adjacent blocks can merge, up to reversal. Conversely, separating merged blocks slightly along that same line, in the prescribed order, realizes every such limit.

Every singular 3 by 3 matrix is a limit of invertible matrices of either determinant sign: use rank normal form and choose the signs of small missing diagonal entries. Hence each top stratum contains every lower-rank stratum in its closure, but not the opposite top stratum. Homotheties realize the diagonal as a limit from every stratum. Affine rank cannot increase in a limit. These arguments prove all rules and exclude further incidences; in particular, they prove the frontier property rather than presupposing it. $\square$

4 Incidence counts and the Hasse diagram

The full labeled Hasse diagram is the transitive reduction of Theorem 1.2. Table 2 summarizes its edges by stratum dimension. The rules retain labels; the table does not identify strata of the same dimension.

Upper dim.	Lower dim.	Edges	Upper dim.	Lower dim.	Edges
12	11	14	9	7	18
11	10	24	8	7	36
10	9	24	7	6	36
10	8	48	6	3	7

Table 2: All 207 Hasse edges. Dimension drops need not be one.

Each top stratum covers all seven support-four strata, giving 14 edges. A rank-two cover deletes exactly one nonzero dependence coefficient. Support four to support three gives $4 \cdot 3 + 3 \cdot 4 = 24$ edges: the sign split is either one versus three or two versus two. Support three to support two gives $12 \cdot 2 = 24$.

For a support-three signature, a distinct collinear limit requires the singleton-sign vertex to lie between the other two supported vertices. After choosing the order up to reversal, the fourth label has four possible insertion positions. Thus there are four four-block covers per signature, giving 48. For a support-two signature its two supported vertices must coincide. The maximal collinear limits have three blocks, in three orders up to reversal, giving $6 \cdot 3 = 18$ covers.

There are no other rank-two-to-rank-one covers. An interval intersection is witnessed either by equal opposite-sign vertices or by three distinct vertices with one between two of the opposite sign. The corresponding dependence has support two or three. Any additional coincidences can be removed while preserving this witness; if no opposite-sign coincidence occurs, the fourth point can be separated while retaining strict betweenness. Thus every remaining incidence factors through one of the covers just listed or through a proper weakening of the original signature.

Rank-one covers merge one adjacent pair of blocks. Their counts are $12 \cdot 3 = 36$ and $18 \cdot 2 = 36$. Each of the seven two-block strata covers the diagonal. Summing the eight entries gives 207.

For completeness the 800 strict incidences can also be counted without transitive-reduction software. There are $2 \cdot 63 = 126$ top-to-lower incidences and $24 + 24 + 24 = 72$ strict rank-two-to-rank-two incidences. A support-four signature has 24 collinear strata below it for a one-versus-three split and 28 for a two-versus-two split. Indeed, among the 37 weak orders, strictly separated positive and negative intervals account for 13 and 9 cases, respectively. For the first split these are the 13 ordered partitions of the three opposite-sign labels; for the second they are $3 \cdot 3$ ordered partitions within the two pairs, with the common reversal removed.

A support-three signature has $7 + 5 + 5 + 1 = 18$ collinear strata below it. The three supported vertices are respectively all distinct, have the singleton-sign vertex coincident with either opposite-sign vertex, or all coincide; inserting the fourth vertex gives the displayed counts. A support-two signature has six collinear strata below it, the nonconstant weak orders of three blocks up to reversal. Thus rank-two-to-rank-one incidences number

$$4 \cdot 24 + 3 \cdot 28 + 12 \cdot 18 + 6 \cdot 6 = 432.$$

There are also 25 rank-two-to-diagonal incidences, $108 = 36 + 36 + 36$ strict rank-one-to-rank-one incidences, and 37 rank-one-to-diagonal incidences. Therefore

$$126 + 72 + 432 + 25 + 108 + 37 = 800,$$

completing the count in Theorem 1.2.

5 Higher ambient dimensions

Corollary 5.1. *For every $d \geq 4$, the stratification of $B_d(K_4)$ has 64 strata. The labels and closure rules of Theorems 1.1 and 1.2 remain valid after merging the two top strata into one. The dimensions are $4d$ in rank three, $3d + k - 2$ in rank two, $2d + m - 2$ in rank one, and d on the diagonal.*

Proof. The stress spaces and their sign types do not change when the ambient space is enlarged. For $d > r$, the injection space $\text{Inj}(\mathbb{R}^r, \mathbb{R}^d)$ is connected. In particular, at rank three the two orientation components merge. The parameterizations above give the asserted dimensions, and full-rank d by 3 matrices are dense among all such matrices, supplying all top-to-lower limits.

In Lemma 3.1, choose $J|_H : H \rightarrow \mathbb{R}^d$ injective. A 3 by 3 minor of $X_0 + tJ$ has nonzero leading coefficient, so this map is injective on H for all sufficiently small positive t . The same projection construction proves every rank-two degeneration. The line-order and diagonal arguments are unchanged. $\square$

6 Exact data and scope

The accompanying standard-library Python program `check_four_points.py` enumerates all 65 labels, constructs rational representatives, and computes the closure order and its transitive reduction. The file `four_points_poset.json` contains the labeled strata, representative coordinates, all 800 strict incidences, and all 207 Hasse edges. An incidence pair `[a,b]` means that stratum `a` lies in the closure of stratum `b`. The labels `T:+`, `T:-`, `P:...` , `L:...` , and `D` encode respectively the top strata, dependence signs, ordered coincidence blocks, and diagonal.

All computations use exact rational arithmetic. They check stress dimensions from the equilibrium matrices, all closure transitivity conditions, 1,587 samples of (9), 1,200 integer-grid configurations, and 37 positive diagonal maps of the form (7). The finite tests supplement the proofs; they do not establish the classification of a continuum of configurations.

The planar classifications in [2] and the stress-sign framework in [1] are prior work. No absolute priority claim is made. The $B_2(K_6)$ and $B_3(K_5)$ components of [3, Problem 1] remain outside the result proved here.

AI assistance was used for mathematical exploration, drafting and implementation of the exact checks. This is an unrefereed preprint; the accompanying review is a self-audit, not independent peer review or proof-assistant verification.

References

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