

Volume-Minimising Projective Subspaces in Berger Projective Spaces: Three Further Cases of a Question of Gil-Medrano

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Abstract

Gil-Medrano asked whether, in the real projective space $\mathbb{RP}^{2n+1}$ with a Berger metric g_μ (the round metric with the Hopf fibres rescaled by $\sqrt{\mu}$), the projective subspaces coming from equatorial k -spheres are minimal, and whether they are the only volume-minimising k -cycles in their homology class. A recent preprint of G. Wheeler answers the first question, determines the volume minimisers among projective subspaces, proves that they minimise among all cycles in two regimes, and conjectures this in the remaining ones. This note is a partial answer to the second question: following a route proposed in that preprint, Crofton formulas adapted to the unitary group, we prove three further cases of the conjecture, for the classes of competitors named below. (I) For every n and every $\mu > 0$ the projective hyperplanes minimise volume; the proof is an exact Crofton formula with great circles and the positive weight $(1 + (\mu - 1)\kappa^2)^{-(n+1)}$, where κ is the cosine of the Kähler angle of the circle. (II) For $0 < \mu < 1$ and every even dimension $k = 2j \leq 2n$ the subspaces of type $\mathbb{C}^j \oplus \mathbb{R}$ minimise; the slices are common kernels of j pairs of Gaussian functionals correlated through the complex structure, mixed by an explicit positive measure. (III) In $\mathbb{RP}^5$, for every $\mu > 1$, the 3-dimensional subspaces of type $\mathbb{C} \oplus \mathbb{R}^2$ minimise; the measure is found by continuing the Gaussian model analytically to negative parameters. In each case we give the value of the minimum, prove the inequality for countably rectifiable sets that meet almost every complementary projective subspace (in particular for compact embedded C^1 submanifolds in the non-zero class mod 2 and for Lipschitz cycles mod 2), and prove uniqueness among compact embedded C^1 submanifolds. Together with Wheeler's theorems this determines the least volume and the minimisers, in these classes, for $0 < \mu < 1$ in all dimensions and for all μ in $\mathbb{RP}^5$. The case $\mu > 1$, $n < k < 2n$, $n \geq 3$ remains open. This is an unrefereed note.

1 Introduction

1.1 The problem

Let $n \geq 1$. We write $S = S^{2n+1}$ for the unit sphere of $\mathbb{C}^{n+1} = \mathbb{R}^{2n+2}$, $\langle \cdot, \cdot \rangle$ for the Euclidean inner product, J for the multiplication by i , g for the round metric, $\xi(p) = Jp$ for the Hopf vector field and $\eta = \langle \cdot, \xi \rangle$ for its dual one-form. For $\mu > 0$ the *Berger metric* is

$$g_\mu = g + (\mu - 1)\eta \otimes \eta,$$

so that $g_\mu(\xi, \xi) = \mu$, $g_\mu = g$ on the g -orthogonal complement of ξ , and ξ is g_μ -orthogonal to that complement; $g_1 = g$. The antipodal map is an isometry of g_μ , and $(\mathbb{RP}^{2n+1}, g_\mu)$ denotes the quotient. For a real linear subspace $V \subset \mathbb{R}^{2n+2}$ of dimension $k + 1$ let $S_V = S \cap V$ be the equatorial (great) k -sphere and $\mathbb{RP}(V) = S_V / \{\pm 1\}$ the corresponding projective subspace.

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In the list of open problems edited by Morgan and Pansu [MP21], Gil-Medrano recalls that for the round metric the $\mathbb{RP}(V)$ are the only volume minimisers in their homology class (Berger [Ber72], Fomenko [Fom72]), that in $(\mathbb{RP}^3, g_\mu)$ the equatorial projective planes are minimal but, for $\mu \neq 1$, not totally geodesic [BGM10], and that they are exactly the area-minimising projective planes there [GM16]; the proof in [GM16] uses the existence theorem of [BBEN10]. She then asks, for $2n + 1 > 3$ and $0 < k < 2n + 1$:

(Q1) are the projective subspaces $\mathbb{RP}(V)$ obtained from the k -dimensional equatorial spheres minimal submanifolds of $(\mathbb{RP}^{2n+1}, g_\mu)$?

(Q2) “Are they the only k -dimensional volume minimizing cycle in their homology classes?”

This is record AMR-067-0008 of the UnsolvedMath dataset [ula26]. Homology is taken with coefficients in $\mathbb{Z}_2$: for $0 \leq k \leq 2n + 1$ the group $H_k(\mathbb{RP}^{2n+1}; \mathbb{Z}_2)$ is $\mathbb{Z}_2$, generated by the class of any $\mathbb{RP}(V)$ with $\dim V = k + 1$, and for even k the $\mathbb{RP}(V)$ are not orientable.

The parameter μ is the *squared* length of the Hopf field; part of the literature writes τ^2 for it. We call the case $\mu > 1$ *stretched* and the case $0 < \mu < 1$ *squashed*.

1.2 Wheeler’s results

Question (Q1) was answered by Wheeler in the preprint [Whe26] (arXiv:2607.24001, version 1 of 27 July 2026, the only version at the time of writing; it is unrefereed). We quote four of its results, with its numbering; note that its N is our $n + 1$. A real subspace V is of *type* $\mathbb{C}^c \oplus \mathbb{R}^r$ if $V = C \oplus R$ is an orthogonal sum with $JC = C$, $\dim_{\mathbb{C}} C = c$, $\dim_{\mathbb{R}} R = r$ and $JR \perp V$; Wheeler calls these the linear CR subspaces. Such subspaces exist if and only if $c + r \leq n + 1$, and those of a given type form one orbit of $U(n + 1)$.

(W1) [Whe26, Theorems 1.1 and 4.1, Corollary 4.2]. For $\mu \neq 1$, $\mathbb{RP}(V)$ is minimal in $(\mathbb{RP}^{2n+1}, g_\mu)$ if and only if V is of type $\mathbb{C}^c \oplus \mathbb{R}^r$ for some c, r . So the answer to (Q1) is negative for general V when $1 \leq k \leq 2n - 1$, and positive for hyperplanes.

(W2) [Whe26, Theorem 5.3, Proposition 5.6]. Among all V of dimension $k + 1$ the volume of $\mathbb{RP}(V)$ is smallest exactly for the type $\mathbb{C}^c \oplus \mathbb{R}^{k+1-2c}$ with $c = \max\{0, k - n\}$ if $\mu > 1$, and with $c = \lfloor (k + 1)/2 \rfloor$ if $0 < \mu < 1$. For V of type $\mathbb{C}^c \oplus \mathbb{R}^r$, $\text{Vol}_{g_\mu}(\mathbb{RP}(V)) = v_k {}_2F_1(-\frac{1}{2}, c; \frac{k+1}{2}; 1 - \mu)$, where v_k is the volume of the round $\mathbb{RP}^k$.

(W3) [Whe26, Theorems 6.3 and 6.4, Corollary 6.5]. If $\mu > 1$ and $k \leq n$, every non-trivial k -cycle mod 2 has g_μ -mass at least v_k , the volume of the totally real $\mathbb{RP}(V)$. If $0 < \mu < 1$ and k is odd, the bound is $\sqrt{\mu} v_k$, the volume of the complex $\mathbb{RP}(V)$. The equality cases are described assuming the equality statement of the classical theorem for the round metric.

(W4) [Whe26, Conjecture 7.1]. In the remaining regimes, that is, $\mu > 1$ with $n < k \leq 2n$, and $0 < \mu < 1$ with k even, the least mass of a non-trivial k -cycle mod 2 is the least volume of a projective subspace, and every minimiser is a projective subspace of the type in (W2).

All $\mathbb{RP}(V)$ of one dimension are homologous. Hence, by (W2), an $\mathbb{RP}(V)$ that is not of the distinguished type is not volume minimising, and for $\mu \neq 1$ and $1 \leq k \leq 2n - 1$ the literal answer to (Q2) is negative too. The question that remains is the one formulated in (W4): does the distinguished orbit minimise among *all* cycles, and is there no other minimiser? Wheeler writes that he does not know how to prove (W4), and lists four possible routes. The first one is a Crofton formula for the group $U(n + 1)$, with weights on the orbits of that group in the real Grassmannian [Whe26, Section 7]. The three theorems below follow this route. We also note that the theorem of [GM16] in $\mathbb{RP}^3$, which is

quoted in [Whe26] as the first mixed case of (W4), concerns embedded projective planes and not all cycles of the class; Theorem I covers this case.

1.3 Results

Let $1 \leq k \leq 2n$. We denote by $\mathcal{G}_k$ the Grassmannian of real linear subspaces $W \subset \mathbb{R}^{2n+2}$ of codimension k and by ν its $O(2n+2)$ -invariant probability measure. The projective subspaces $\mathbb{RP}(W)$, $W \in \mathcal{G}_k$, have dimension $2n+1-k$; we call them the *complementary projective subspaces*.

Definition 1.1. A *k-dimensional competitor* is a countably k -rectifiable Borel set $N \subset \mathbb{R}^{2n+1}$ such that $N \cap \mathbb{RP}(W) \neq \emptyset$ for ν -almost every $W \in \mathcal{G}_k$. A *closed C^1 submanifold of the non-zero class* is a compact embedded C^1 submanifold N without boundary, of dimension k , whose fundamental class mod 2 has non-zero image in $H_k(\mathbb{R}^{2n+1}; \mathbb{Z}_2)$.

Rectifiable sets and their g_μ -volume Vol_{g_μ} are recalled in Section 2.2. Closed C^1 submanifolds of the non-zero class meet *every* complementary projective subspace (Lemma 2.3), so they are competitors, and so is the set of points of odd multiplicity of a Lipschitz cycle mod 2 (Proposition 3.8). Throughout, $v_k = \text{vol}(\mathbb{RP}^k) = \frac{1}{2} \text{vol}(S^k)$ for the round metrics.

Theorem I (hypersurfaces). *Let $n \geq 1$, $\mu > 0$, and let $V_0 \subset \mathbb{R}^{2n+2}$ be a real hyperplane.*

(a) *For every countably $2n$ -rectifiable Borel set $N \subset S^{2n+1}$,*

$$\text{Vol}_{g_\mu}(N) = \frac{\mu \text{vol}(S^{2n})}{2} \int_{\mathcal{G}_{2n}} (1 + (\mu - 1)\kappa(W)^2)^{-(n+1)} \#(N \cap W) d\nu(W),$$

where, for a real 2-plane W with orthonormal basis w_1, w_2 , $\kappa(W) = |\langle Jw_1, w_2 \rangle|$ is the cosine of its Kähler angle.

(b) *Every $2n$ -dimensional competitor $N \subset \mathbb{R}^{2n+1}$ satisfies*

$$\text{Vol}_{g_\mu}(N) \geq \text{Vol}_{g_\mu}(\mathbb{RP}(V_0)) = v_{2n} {}_2F_1\left(-\frac{1}{2}, n; n + \frac{1}{2}; 1 - \mu\right).$$

(c) *A closed C^1 hypersurface of the non-zero class with $\text{Vol}_{g_\mu}(N) = \text{Vol}_{g_\mu}(\mathbb{RP}(V_0))$ is a projective hyperplane.*

All real hyperplanes are of type $\mathbb{C}^n \oplus \mathbb{R}$ and congruent under $U(n+1)$. For hypersurfaces we also prove the inequality, with its equality case, in the class of sets of finite perimeter (Proposition 4.4); this is the natural class for cycles mod 2 of codimension one.

Theorem II (the squashed case in even dimension). *Let $0 < \mu < 1$, $j \geq 1$, $n \geq j$, and let V_0 be of type $\mathbb{C}^j \oplus \mathbb{R}$.*

(a) *Every $2j$ -dimensional competitor $N \subset \mathbb{R}^{2n+1}$ satisfies*

$$\text{Vol}_{g_\mu}(N) \geq \text{Vol}_{g_\mu}(\mathbb{RP}(V_0)) = v_{2j} {}_2F_1\left(-\frac{1}{2}, j; j + \frac{1}{2}; 1 - \mu\right).$$

(b) *A closed C^1 submanifold N of the non-zero class, of dimension $2j$, with $\text{Vol}_{g_\mu}(N) = \text{Vol}_{g_\mu}(\mathbb{RP}(V_0))$ is an $\mathbb{RP}(V)$ with V of type $\mathbb{C}^j \oplus \mathbb{R}$.*

The case $j = 1$ is the case of surfaces, and the case $j = n$ is the case $\mu < 1$ of Theorem I. The measure on $\mathcal{G}_{2j}$ that proves Theorem II is described in Theorem 5.1.

Theorem III (the first stretched case). *Let $n = 2$, $\mu > 1$, and let $V_0 \subset \mathbb{C}^3$ be of type $\mathbb{C} \oplus \mathbb{R}^2$.*

(a) Every 3-dimensional competitor $N \subset \mathbb{RP}^5$ satisfies

$$\text{Vol}_{g_\mu}(N) \geq \text{Vol}_{g_\mu}(\mathbb{RP}(V_0)) = v_3 \frac{2(\mu^{3/2} - 1)}{3(\mu - 1)} = v_3 {}_2F_1\left(-\frac{1}{2}, 1; 2; 1 - \mu\right), \quad v_3 = \pi^2.$$

(b) A closed C^1 submanifold N of the non-zero class, of dimension 3, with $\text{Vol}_{g_\mu}(N) = \text{Vol}_{g_\mu}(\mathbb{RP}(V_0))$ is an $\mathbb{RP}(V)$ with V of type $\mathbb{C} \oplus \mathbb{R}^2$.

The measure on $\mathcal{G}_3$ is $\varphi_\mu(z_W) d\nu(W)$ with an explicit positive function φ_μ of the Kähler invariant z_W of the 3-dimensional subspace W (Theorem 6.1).

Corollary 1.2 (Lipschitz cycles). *In the situation of Theorem I, II or III, with $k = 2n, 2j$ or 3 respectively, let $c = \sum_i \sigma_i$ be a singular k -chain mod 2 in $\mathbb{RP}^{2n+1}$ with Lipschitz simplices, with $\partial c = 0$ and $[c] \neq 0$ in $H_k(\mathbb{RP}^{2n+1}; \mathbb{Z}_2)$. Then the set M_c of points covered an odd number of times by c is a k -dimensional competitor. Hence $\text{Vol}_{g_\mu}(M_c)$, and a fortiori the g_μ -volume of c counted with multiplicity, is at least $\text{Vol}_{g_\mu}(\mathbb{RP}(V_0))$.*

The statement for arbitrary flat chains mod 2, which is the notion used in the definition of the homological systole in [Whe26], follows from Theorems I to III and from classical existence and monotonicity theorems of geometric measure theory. We give the derivation in Remark 3.9 and say there which statements we could check; we do not state it as a theorem. For hypersurfaces, Proposition 4.4 is a self-contained substitute.

Table 1 summarises the state of (Q2). In each row the minimisers are exactly the projective subspaces of the type shown, in the following sense: for the rows due to [Whe26] see (W3); for the rows proved here the inequality holds for all competitors and for Lipschitz cycles, and the minimisers among closed C^1 submanifolds of the non-zero class are exactly these projective subspaces.

regime	type of the minimisers	least volume/ v_k	source
$\mu > 1, 1 \leq k \leq n$	$\mathbb{R}^{k+1}$ (totally real)	1	[Whe26, Thm. 6.3]
$\mu > 1, k = 2n$	$\mathbb{C}^n \oplus \mathbb{R}$ (all hyperplanes)	$M_n(\mu)$	Theorem I
$\mu > 1, (n, k) = (2, 3)$	$\mathbb{C} \oplus \mathbb{R}^2$	$m(\mu)$	Theorem III
$\mu > 1, n < k < 2n, n \geq 3$	open		[Whe26, Conj. 7.1]
$\mu < 1, k$ odd	$\mathbb{C}^{(k+1)/2}$ (complex)	$\sqrt{\mu}$	[Whe26, Thm. 6.4]
$\mu < 1, k = 2j$ even	$\mathbb{C}^j \oplus \mathbb{R}$	$M_j(\mu)$	Theorem II

Table 1: The second question of Gil-Medrano in $(\mathbb{RP}^{2n+1}, g_\mu)$, $1 \leq k \leq 2n$, $\mu \neq 1$. Here $M_j(\mu) = {}_2F_1(-\frac{1}{2}, j; j + \frac{1}{2}; 1 - \mu)$ and $m(\mu) = \frac{2}{3}(\mu^{3/2} - 1)/(\mu - 1)$. In the open regime the conjectured type is $\mathbb{C}^{k-n} \oplus \mathbb{R}^{2n+1-k}$. For $\mu < 1$ and $k = 2n$ both Theorem I and Theorem II apply.

Consequently, together with the theorems of [Whe26]:

- for $0 < \mu < 1$ the minimisers are determined for every n and every k (odd k : [Whe26]; even k : Theorem II);
- in $\mathbb{RP}^5$, the first dimension of the question, they are determined for every k and every μ : $k = 1, 2$ with $\mu > 1$ and $k = 1, 3$ with $\mu < 1$ by [Whe26]; $k = 2$ with $\mu < 1$ by Theorem II; $k = 3$ with $\mu > 1$ by Theorem III; $k = 4$ by Theorem I;
- for $\mu > 1$ they are determined for $k \leq n$ [Whe26], for $k = 2n$ (Theorem I) and for $(n, k) = (2, 3)$ (Theorem III).

What remains open is the regime $\mu > 1, n < k < 2n$ with $n \geq 3$; the first cases are $k = 4$ and $k = 5$ in $\mathbb{RP}^7$. The proof of Theorem III does not extend to them as it stands (Section 7.4).

1.4 Method, and what is not new

The method is the classical integral-geometric proof of volume minimisation, which goes back to the round case [Ber72, Fom72] and was developed for homogeneous spaces by Lê [Lê93], on the basis of kinematic formulas as in Howard’s memoir [How93]. One looks for a positive measure $\mathbf{m}$ on the complementary subspaces whose *Crofton density* Ψ , defined by $\int \#(N \cap W) d\mathbf{m}(W) = \frac{2}{\text{vol}(S^k)} \int_N \Psi(T_p N)$, is at most the g_μ -volume density on all tangent planes, with equality on the tangent planes of the candidate. A cycle of the non-zero class meets almost every complementary subspace, the candidate meets it once, and the inequality follows (Proposition 3.5). For the round metric $\mathbf{m} = \nu$. The Berger metric has only the symmetry group $U(n+1)$, the Grassmannian $\mathcal{G}_k$ splits into orbits labelled by Kähler angles, and the measure must be weighted over these orbits. That such a weighted $U(n+1)$ -Crofton formula might prove (W4) is the first of the routes proposed in [Whe26, Section 7]; we carry it out in three cases by finding the measures explicitly.

Theorem I. In codimension one the Crofton density of a measure on lines is, at each point, the cosine transform of its density. For a Riemannian metric the volume density of hypersurfaces is the support function of an ellipsoid, the cosine transform is injective on even functions, and ellipsoids are zonoids with an explicit positive generating density (Lemma 4.1). So the pointwise density is forced, and positive; a Crofton formula exists if and only if this density is constant along great circles, which holds for g_μ because $\eta(\dot{\gamma})$ is constant along great circles. Thus Theorem I is an application of the classical Crofton mechanism with an explicit positive density, and *we claim no novelty for the mechanism*. The general fact that an even hypersurface integrand all of whose hyperplanes are extremal has a Crofton formula with a smooth, possibly signed, measure on lines is due to Gelfand and Smirnov [GS94] and Álvarez-Paiva and Fernandes [ÁPF07]; we quote it from [ÁPB06, last theorem of Section 5], where it is stated for integrands on $\mathbb{R}^n$. Ambrozio’s survey [Amb26] remarks that the Finsler metrics on $\mathbb{RP}^n$ whose hyperplanes minimise the Holmes–Thompson area in their homology class were classified in [ÁPF07] by positive measures on the Grassmannian of 2-planes, and that the Riemannian metrics among them are not easily recognised. *We could not read [GS94] and [ÁPF07]* (we found no openly accessible copy and have no subscription), and we cannot exclude that the inequality of Theorem I follows directly from a statement in these papers; see the paragraph “Scope and priority”. Metrics on spheres all of whose equators are minimal were characterised by Ambrozio, Marques and Neves [AMN25]; the Berger metrics are among them [Amb26, Whe26], and Remark 4.7 records what the present argument gives for that class. For $\mathbb{RP}^3$, related recent work includes [AM20], a weighted Crofton formula of Wang [Wan26] for the lengths of curves in $\mathbb{RP}^3$ with a surface Zoll metric, and [Mar26].

Theorem II. Here the slices are the common kernels of j independent pairs of Gaussian functionals $(a, bJa + \sqrt{1-b^2}a')$, $0 \leq b < 1$. For $b = 0$ this is the invariant measure ν , and for $b \rightarrow 1$ the slices become complex subspaces, for which the Crofton density is given by Tasaki’s Poincaré formula in terms of multiple Kähler angles [Tas03, KT01, BF11]. We compute the Crofton density for every b in closed form by a Kac–Rice computation (the link between Kac–Rice formulas and Crofton formulas is classical, see [EK95, Ste22]), prove that for each b it is largest on the tangent planes of the candidate among planes with the same Hopf component (Lemma 5.9), and find the positive mixing measure in b on $[0, \sqrt{1-\mu}]$ for which the mixture equals the Berger density on those planes (Lemma 5.11).

Theorem III. For $\mu > 1$ no mixture of invariant Gaussian models can work (Proposition 6.9). The measure was first found numerically, by a linear programme over the densities $\varphi(z_W)$. Its first-order term in $\mu - 1$ turned out to be the density of the Gaussian model “one correlated pair and one independent functional” with a *negative* value of $t = b^2$. Both the density of that model on $\mathcal{G}_3$ and its Crofton density are analytic in t , the Crofton formula continues to $t < 0$, the domination lemma holds there in the direction needed for $\mu > 1$, and an Abel integral equation gives a positive mixing measure on $t \in [1 - \mu, 0]$. For $t < -1$ the continued densities change sign, but the mixture φ_μ is

positive (Lemma 6.8).

We found no source for Theorems II and III: the only paper on the question that we know, [Whe26], lists these cases as open. This is a statement about a literature search, described in “Scope and priority”, and not a claim of priority. Our proofs do not use the results of [Whe26]; the facts needed about the candidates are proved in Section 2.

1.5 Organisation

Section 2 contains the preliminaries. Section 3 develops the general mechanism: the area formula for random linear slices, the principle described above, the lemma used for uniqueness, and Lipschitz cycles. Sections 4 to 6 prove Theorems I to III. Section 7 contains an exact weighted Crofton formula for all k , remarks on the complex endpoint and on limits, a numerical probe of the open cases, and open problems. Section 8 describes the computations; no proof depends on them.

2 Preliminaries

2.1 The Berger volume density

For $p \in S$ the tangent space $T_p S = p^\perp$ is the orthogonal sum of $\mathbb{R}\xi$ and the *horizontal space* $H_p = \{p, \xi\}^\perp$, which is J -invariant. For $u \in T_p S$ let $u^h = u - \eta(u)\xi$ be its horizontal part, and put

$$\omega(u, v) = \langle Ju, v \rangle \quad (u, v \in T_p S).$$

For a linear subspace $E \subset T_p S$ we put

$$x_E = |\operatorname{pr}_E \xi|^2 \in [0, 1], \quad J_\mu(E) = \sqrt{1 + (\mu - 1)x_E},$$

where pr_E is the orthogonal projection onto E .

Lemma 2.1. *Let $p \in S$ and $u, v \in T_p S$.*

- (a) $\omega(u, v) = \langle Ju^h, v^h \rangle$. Hence ω is a skew form on $T_p S$, $\omega(\xi, \cdot) = 0$, and $|\omega(u, v)| \leq |u^h| |v^h|$, with equality if and only if Ju^h and v^h are linearly dependent.
- (b) For every k -dimensional subspace $E \subset T_p S$ the g_μ -volume density of E is $J_\mu(E)$ times its g -volume density.

Proof. (a) Since $J\xi = -p$ and $\langle Ju^h, \xi \rangle = \langle Ju^h, Jp \rangle = \langle u^h, p \rangle = 0$, we get $\langle Ju, v \rangle = \langle Ju^h - \eta(u)p, v^h + \eta(v)\xi \rangle = \langle Ju^h, v^h \rangle$. The rest is the Cauchy–Schwarz inequality. (b) In a g -orthonormal basis $e_1, \dots, e_k$ of E the matrix of $g_\mu|_E$ is $I + (\mu - 1)(\eta(e_c)\eta(e_d))_{c,d}$, a rank-one perturbation of the identity, with determinant $1 + (\mu - 1)\sum_c \eta(e_c)^2 = 1 + (\mu - 1)x_E$. $\square$

Part (b) is [Whe26, Lemma 6.1]. It shows that $J_\mu(E) \geq 1$ for $\mu > 1$, with equality if and only if $\xi \perp E$, and $J_\mu(E) \geq \sqrt{\mu}$ for $\mu < 1$, with equality if and only if $\xi \in E$.

2.2 Rectifiable sets and their volume

We write $\mathcal{H}^k$ for the k -dimensional Hausdorff measure of the round metric on S . A Borel set $N \subset S$ is *countably k -rectifiable* if

$$N \subset N_0 \cup \bigcup_{i \geq 1} N_i, \quad \mathcal{H}^k(N_0) = 0,$$

where the N_i are embedded k -dimensional C^1 submanifolds of S . By [Sim83, Lemma 11.1] (applied in charts) this is equivalent to the definition by Lipschitz images: $\mathcal{H}^k$ -almost all of N is covered by countably many images of Lipschitz maps from subsets of $\mathbb{R}^k$. For $\mathcal{H}^k$ -almost every $p \in N$ the tangent plane $T_p N := T_p N_i$, $p \in N_i$, does not depend on the choice of the N_i : the points of $N_i \cap N_l$ at which the tangent planes of N_i and N_l differ lie, locally, in a C^1 submanifold of dimension $k - 1$ of N_i (implicit function theorem), and so form an $\mathcal{H}^k$ -null set. On an embedded C^1 submanifold, $\mathcal{H}^k$ is the Riemannian volume of the round metric. We define

$$\text{Vol}_{g_\mu}(N) = \int_N J_\mu(T_p N) d\mathcal{H}^k(p);$$

by Lemma 2.1(b) this is the Riemannian g_μ -volume when N is a C^1 submanifold. For a set $N \subset \mathbb{RP}^{2n+1}$ we write $\tilde{N}$ for its preimage in S ; N is called countably k -rectifiable if $\tilde{N}$ is, and $\text{Vol}_{g_\mu}(N) = \frac{1}{2} \text{Vol}_{g_\mu}(\tilde{N})$.

The volume of the round sphere is $\text{vol}(S^k) = 2\pi^{(k+1)/2}/\Gamma(\frac{k+1}{2})$, and $v_k = \frac{1}{2} \text{vol}(S^k)$.

2.3 Linear subspaces

For a real subspace $V \subset \mathbb{R}^{2n+2}$ let P_V be the orthogonal projection onto V and $K_V = P_V J|_V : V \rightarrow V$. The operator K_V is skew-symmetric; its singular values are the cosines of the Kähler angles of V . It is called A_V in [Whe26].

Lemma 2.2. *Let V be a real subspace of dimension $k + 1$, $1 \leq k \leq 2n$, and $p \in S_V$.*

- (a) $T_p S_V = V \cap p^\perp$, and $\text{pr}_{T_p S_V} \xi = K_V p$. Hence $x_{T_p S_V} = |K_V p|^2$, and on $T_p S_V$ we have $\eta = \langle \cdot, K_V p \rangle$ and $\omega = \langle K_V \cdot, \cdot \rangle$.
- (b) If $V = C \oplus R$ is of type $\mathbb{C}^c \oplus \mathbb{R}^r$ and $p = p_C + p_R$, then $K_V p = Jp_C$, and

$$\frac{\text{Vol}_{g_\mu}(S_V)}{\text{vol}(S^k)} = \frac{\text{Vol}_{g_\mu}(\mathbb{RP}(V))}{v_k} = {}_2F_1\left(-\frac{1}{2}, c; \frac{k+1}{2}; 1 - \mu\right).$$

Proof. (a) $\langle Jp, p \rangle = 0$, so the projections of Jp onto V and onto $V \cap p^\perp$ coincide. The formulas for η and ω follow because the vectors involved lie in V . (b) $Jp_C \in C \subset V$ and $Jp_R \perp V$, so $K_V p = Jp_C$. By (a) and Lemma 2.1(b), $\text{Vol}_{g_\mu}(S_V) = \int_{S_V} \sqrt{1 + (\mu - 1)|p_C|^2} d\mathcal{H}^k$. If $c, r \geq 1$, then under the normalised measure of S^k the variable $s = |p_C|^2$ has the Beta distribution with parameters $(c, \frac{r}{2})$, and Euler's integral ${}_2F_1(a, b; c'; z) = B(b, c' - b)^{-1} \int_0^1 s^{b-1} (1-s)^{c'-b-1} (1-zs)^{-a} ds$ with $a = -\frac{1}{2}$, $b = c$, $c' = \frac{k+1}{2}$, $z = 1 - \mu$ gives the formula. If $r = 0$ the integrand is $\sqrt{\mu}$, and if $c = 0$ it is 1, in agreement with the formula. $\square$

Part (b) is [Whe26, Propositions 5.1 and 5.6]. The three cases used below are

$$\begin{aligned} \mathbb{C}^n \oplus \mathbb{R} \text{ and } \mathbb{C}^j \oplus \mathbb{R} : \quad & M_j(\mu) := {}_2F_1\left(-\frac{1}{2}, j; j + \frac{1}{2}; 1 - \mu\right) \quad (j = n \text{ for hyperplanes}), \\ \mathbb{C} \oplus \mathbb{R}^2 : \quad & m(\mu) := {}_2F_1\left(-\frac{1}{2}, 1; 2; 1 - \mu\right) = \int_0^1 \sqrt{1 + (\mu - 1)s} ds = \frac{2(\mu^{3/2} - 1)}{3(\mu - 1)}. \end{aligned}$$

A real hyperplane $V = \nu_0^\perp$ is of type $\mathbb{C}^n \oplus \mathbb{R}$, with $C = \{\nu_0, J\nu_0\}^\perp$ and $R = \mathbb{R}J\nu_0$. A subspace $V = C \oplus \mathbb{R}u$ with C complex and $u \perp C$ is of type $\mathbb{C}^j \oplus \mathbb{R}$, because $Ju \perp C$ and $Ju \perp u$.

2.4 Cycles of the non-zero class meet every complementary subspace

Lemma 2.3. *Let $1 \leq k \leq 2n + 1$ and let $W \subset \mathbb{R}^{2n+2}$ be a real linear subspace of codimension k . Then $\mathbb{RP}^{2n+1} \setminus \mathbb{RP}(W)$ deformation retracts onto $\mathbb{RP}(W^\perp)$, a projective space of dimension $k - 1$. Consequently, if $X \subset \mathbb{RP}^{2n+1}$ does not meet $\mathbb{RP}(W)$, the map $H_k(X; \mathbb{Z}_2) \rightarrow H_k(\mathbb{RP}^{2n+1}; \mathbb{Z}_2)$ induced by the inclusion is zero. In particular, for $k \leq 2n$, a closed C^1 submanifold of the non-zero class and of dimension k meets every complementary projective subspace.*

Proof. For $w \in W$, $w' \in W^\perp \setminus \{0\}$ and $s \in [0, 1]$ put $h_s([w + w']) = [sw + w']$. This is well defined and continuous on the complement of $\mathbb{RP}(W)$, h_1 is the identity, h_0 has its image in $\mathbb{RP}(W^\perp)$, and h_s fixes $\mathbb{RP}(W^\perp)$. Since $H_k(\mathbb{RP}^{k-1}; \mathbb{Z}_2) = 0$, the second statement follows. For a compact C^1 manifold N without boundary, $[N]$ is the image of the fundamental class mod 2, which gives the last statement. $\square$

3 The mechanism

Throughout this section $1 \leq k \leq 2n$ and $d = 2n + 1 = \dim S$.

3.1 Linear slices

For $a = (a_1, \dots, a_k) \in (\mathbb{R}^{2n+2})^k$ let $W(a) = \{z \in \mathbb{R}^{2n+2} : \langle a_i, z \rangle = 0 \text{ for all } i\}$. If the a_i are linearly independent, then $W(a) \in \mathcal{G}_k$, and $W(a)^\perp$ is the span of the a_i . We regard a as a $k \times (2n + 2)$ matrix with rows a_i .

Lemma 3.1 (row spaces). *Let $1 \leq m < N$, let M^* be the open set of real $m \times N$ matrices of rank m , and let ν_m be the $O(N)$ -invariant probability measure on the Grassmannian of m -dimensional subspaces of $\mathbb{R}^N$. For each such subspace R let O_R be an $m \times N$ matrix with orthonormal rows spanning R , chosen Borel measurably. There is a constant $c_{m,N} > 0$ such that for every Borel function $f \geq 0$ on M^**

$$\int_{M^*} f(A) dA = c_{m,N} \int \int_{GL(m, \mathbb{R})} f(gO_R) |\det g|^{N-m} dg d\nu_m(R),$$

where dA and dg are Lebesgue measures. Consequently:

- (a) *a Borel set Z of subspaces is ν_m -null if and only if the set of $A \in M^*$ whose row space lies in Z is Lebesgue-null; and if, for an arbitrary set Z of subspaces, the latter set is Lebesgue-null, then Z is contained in a ν_m -null Borel set;*
- (b) *if a random matrix A has a density γ , the law of its row space has, with respect to ν_m , the density $R \mapsto c_{m,N} \int \gamma(gO_R) |\det g|^{N-m} dg$; if $\gamma > 0$ everywhere, this law has the same null sets as ν_m ;*
- (c) *if the entries of A are independent standard Gaussian variables, the row space of A has the law ν_m ;*
- (d) *if $f \geq 0$ is a Lebesgue-measurable function on M^* which depends only on the row space, $f(A) = \hat{f}(R)$ for the row space R of A , then $\hat{f}$ is measurable for the completion of ν_m , and the displayed formula holds for f .*

Proof. Denote the right-hand side by $\rho(f)$. It does not depend on the choice of O_R : another choice is θO_R with $\theta \in O(m)$, and $|\det g|^{N-m} dg$ is invariant under $g \mapsto g\theta$. Near a given subspace O_R can be chosen smooth in R ; then $(g, R) \mapsto gO_R$ is a diffeomorphism onto an open subset of M^* , with

inverse $A \mapsto (AO_{R(A)}^T, R(A))$, $R(A)$ the row space. Hence $\rho = \psi(A) dA$ with a continuous positive function ψ on M^* . For $h \in GL(m, \mathbb{R})$ and $Q \in O(N)$ both measures are transformed in the same way by $A \mapsto hAQ$: Lebesgue measure is multiplied by $|\det h|^N$, and so is ρ , because $d(hg) = |\det h|^m dg$, $|\det(hg)|^{N-m} = |\det h|^{N-m} |\det g|^{N-m}$, $O_R Q$ has orthonormal rows spanning the image RQ of R , and ν_m is invariant. Therefore $\psi(hAQ) = \psi(A)$. The group $GL(m, \mathbb{R}) \times O(N)$ acts transitively on M^* , so ψ is constant. (a) for Borel sets and (b) follow at once ($\int |\det g|^{N-m} dg = \infty$). For an arbitrary Z as in (a) let $B \subset M^*$ be a Borel null set which contains all A whose row space lies in Z , and let λ be a positive continuous probability density on $GL(m, \mathbb{R})$. By the formula for the indicator function of B , the Borel set Z_B of all R with $\int 1_B(gO_R) \lambda(g) dg > 0$ is ν_m -null; and $Z \subset Z_B$, because $gO_R \in B$ for all g if $R \in Z$. (c) The law of A is invariant under $A \mapsto AQ$, so the law of the row space is an $O(N)$ -invariant probability measure. (d) Choose a Borel function $f_B \geq 0$ and a Borel null set B with $f = f_B$ outside B , and λ, Z_B as before. For $R \notin Z_B$ we have $f_B(gO_R) = f(gO_R) = \hat{f}(R)$ for almost every g , hence $\hat{f}(R) = \int f_B(gO_R) \lambda(g) dg$, and the right-hand side is a Borel function of R . So $\hat{f}$ is measurable for the completion of ν_m , and neither side of the formula changes if f is replaced by f_B . $\square$

Since $W \mapsto W^\perp$ maps the invariant measure of the Grassmannian of k -dimensional subspaces to ν , we obtain: *if the random k -tuple a has a positive density on $(\mathbb{R}^{2n+2})^k$, then $W(a) \in \mathcal{G}_k$ almost surely and the law of $W(a)$ has the same null sets as ν ; if the a_i are independent standard Gaussian vectors, the law of $W(a)$ is ν .* By (a) and (d), a property of $W(a)$ which holds for almost every a holds for ν -almost every W , and a Lebesgue-measurable function of a which depends only on $W(a)$ is a function of W that is measurable for the completion of ν ; integrals with respect to ν and to the measures $\mathbf{m}$ below are understood in this sense.

3.2 The area formula for linear slices

Lemma 3.2 (area formula). *Let $N \subset S$ be a countably k -rectifiable Borel set and let $G: S \times (\mathbb{R}^{2n+2})^k \rightarrow [0, \infty]$ be a Borel function. Then*

$$\int_{(\mathbb{R}^{2n+2})^k} \sum_{p \in N \cap W(a)} G(p, a) da = \int_N \left[\int_{(p^\perp)^k} G(p, a) |\det(\langle a_i, e_c \rangle)_{i,c=1}^k| da \right] d\mathcal{H}^k(p),$$

where $e_1, \dots, e_k$ is an orthonormal basis of $T_p N$ and the inner integral is taken with respect to the Lebesgue measure of the linear space $(p^\perp)^k$. Moreover, if $Z \subset S$ and $\mathcal{H}^k(Z) = 0$, then $Z \cap W(a) = \emptyset$ for almost every a .

Proof. Let first N be a Borel subset of an embedded C^1 submanifold N_0 of dimension k , contained in an open set $O \subset S$ on which there is a smooth orthonormal frame $f_1, \dots, f_d$ of TS . Consider

$$\Theta: (N_0 \cap O) \times \mathbb{R}^{kd} \rightarrow (\mathbb{R}^{2n+2})^k, \quad \Theta(p, c) = \left(\sum_{l=1}^d c_i^l f_l(p) \right)_{i=1, \dots, k}.$$

This is a C^1 map between manifolds of the same dimension $k + kd = k(2n+2)$. Its image consists of the a with $a_i \perp p$ for all i , that is, with $p \in W(a)$, and for such a the coefficients c are unique. Hence the points of $\Theta^{-1}(a)$ over N correspond bijectively to the points of $N \cap W(a)$. We compute the Jacobian with respect to the product of $\mathcal{H}^k$ and Lebesgue measure. Split $(\mathbb{R}^{2n+2})^k = (p^\perp)^k \oplus (\mathbb{R}p)^k$. A variation of c is mapped isometrically into $(p^\perp)^k$. A variation $e \in T_p N_0$ of p has in the i -th slot the $\mathbb{R}p$ -component $\sum_l c_i^l \langle D_e f_l, p \rangle = -\sum_l c_i^l \langle f_l, e \rangle = -\langle a_i, e \rangle$ (differentiate $\langle f_l, p \rangle = 0$). So the differential is block triangular, and the Jacobian is $|\det(\langle a_i, e_c \rangle)|$. The area formula for C^1 maps ([Sim83, 8.4,

8.5]; [Fed69, 3.2.3]) with the integrand $G(p, \Theta(p, c))$ gives the identity for this N . Both sides are countably additive in N .

If $\mathcal{H}^k(Z) = 0$ and $Z \subset O$, then $Z \times \mathbb{R}^{kd}$ is a null set for the Hausdorff measure of dimension $k + kd$ (cover Z by sets Z_i of diameters δ_i with $\sum \delta_i^k$ small, and $Z_i \times [-\varrho, \varrho]^{kd}$ by about $(\varrho/\delta_i)^{kd}$ sets of diameter comparable to δ_i). The map Θ , defined on $O \times \mathbb{R}^{kd}$ by the same formula, is locally Lipschitz into a space of that dimension, so the image of $Z \times \mathbb{R}^{kd}$ is Lebesgue-null. This image is the set of a with $Z \cap W(a) \neq \emptyset$. Finitely many such sets O cover S , which gives the last statement of the lemma.

A countably k -rectifiable Borel set is the union of an $\mathcal{H}^k$ -null set and countably many disjoint Borel sets of the kind treated first, and its tangent planes are those of the submanifolds almost everywhere (Section 2.2). This proves the lemma. $\square$

In particular $a \mapsto \#(N \cap W(a))$ is Lebesgue measurable. If a is a random k -tuple with a continuous density γ , the lemma with $G(p, a) = \gamma(a)$ is a formula of Kac–Rice type:

$$\mathbb{E} \#(N \cap W(a)) = \int_N d_0(p) \mathbb{E} \left[|\det(\langle a_i, e_c \rangle)| \mid \langle a_i, p \rangle = 0 \text{ for all } i \right] d\mathcal{H}^k(p), \quad (1)$$

where $d_0(p)$ is the density at $0 \in \mathbb{R}^k$ of the random vector $(\langle a_i, p \rangle)_i$. Indeed, writing $a_i = \alpha_i p + a_i^\perp$, the joint density of $(\alpha, a^\perp)$ is $\gamma(\alpha p + a^\perp)$, so $d_0(p) = \int \gamma(a^\perp) da^\perp$ and $\gamma(a^\perp)/d_0(p)$ is the conditional density of $a^\perp$ given $\alpha = 0$.

3.3 Two Gaussian determinants

We write χ_i for a chi variable with i degrees of freedom (the norm of a standard Gaussian vector of $\mathbb{R}^i$); $\mathbb{E}\chi_i = \sqrt{2} \Gamma(\frac{i+1}{2})/\Gamma(\frac{i}{2})$. A complex random variable ζ is $\mathcal{CN}(0, s^2)$ if its real and imaginary parts are independent centred Gaussian variables of variance $s^2/2$; then $|\zeta|^2$ is s^2 times an exponential variable of mean 1. Double factorials are defined with $(-1)!! = 1$.

Lemma 3.3. (i) *If Γ_m is a real $m \times m$ matrix with independent standard Gaussian entries, then $\mathbb{E}|\det \Gamma_m| = \prod_{i=1}^m \mathbb{E}\chi_i = 2^{m/2} \Gamma(\frac{m+1}{2})/\Gamma(\frac{1}{2})$. Hence $(2\pi)^{-m/2} \mathbb{E}|\det \Gamma_m| = 2/\text{vol}(S^m)$, and $\mathbb{E}|\det \Gamma_{2r}| = (2r-1)!!$.*

(ii) *If Z is a complex $j \times l$ matrix, $l \leq j$, with independent $\mathcal{CN}(0, 1)$ entries, then $\mathbb{E} \det(Z^* Z) = j!/(j-l)!$.*

Proof. (i) $|\det \Gamma_m|$ is the product of the distances of the i -th column from the span of the previous ones. Given the previous columns, this distance is the norm of a standard Gaussian vector in a space of dimension $m - i + 1$. So the factors are independent chi variables with $m, m-1, \dots, 1$ degrees of freedom, and the product of the means telescopes. (ii) Likewise $\det(Z^* Z)$ is the product of the squared distances, which are independent sums of $j - i + 1$ exponential variables of mean 1. $\square$

3.4 Uniform slices with a weight

Let $m = d - k$. For $p \in S$ and two m -dimensional subspaces P, F of $T_p S$ with orthonormal bases (p_i) , (f_i) we put $|\langle P, F \rangle| = |\det(\langle p_i, f_i \rangle)|$, the cosine of the angle between P and F . We write $\mathbb{E}_F$ for the mean over F with respect to the invariant probability measure of the Grassmannian of m -dimensional subspaces of $T_p S$. For $W \in \mathcal{G}_k$ and $p \in S \cap W$, the subspace $W \cap p^\perp$ of $T_p S$ has dimension m .

Lemma 3.4 (weighted Crofton formula for the invariant measure). *Let $G \geq 0$ be a Borel function of pairs (p, F) , where $p \in S$ and $F \subset T_p S$ is an m -dimensional subspace. For every countably k -rectifiable Borel set $N \subset S$,*

$$\int_{\mathcal{G}_k} \sum_{p \in N \cap W} G(p, W \cap p^\perp) d\nu(W) = \frac{2}{\text{vol}(S^k)} \int_N \mathcal{A}G(p, T_p N) d\mathcal{H}^k(p),$$

where, for a k -dimensional subspace $E \subset T_p S$ with orthogonal complement $E^\perp$ in $T_p S$,

$$\mathcal{A}G(p, E) := \frac{\mathbb{E}_F[G(p, F) |\langle E^\perp, F \rangle|]}{\mathbb{E}_F[|\langle E^\perp, F \rangle|]}.$$

In particular ($G = 1$), $\int \#(N \cap W) d\nu(W) = 2 \mathcal{H}^k(N) / \text{vol}(S^k)$.

Proof. Let $a_1, \dots, a_k$ be independent standard Gaussian vectors, with density γ_0 ; the law of $W(a)$ is ν . Apply Lemma 3.2 to the function $G(p, W(a) \cap p^\perp) \gamma_0(a)$. The restriction of γ_0 to $(p^\perp)^k$ is $(2\pi)^{-k/2}$ times the standard Gaussian density there, so the inner integral is $(2\pi)^{-k/2} \mathbb{E}[G(p, F) |\det(\langle \tilde{a}_i, e_c \rangle)|]$, where $\tilde{a}$ is a $k \times d$ matrix with independent standard Gaussian entries (the rows are vectors of $T_p S \cong \mathbb{R}^d$) and F is the orthogonal complement of its row space in $T_p S$; indeed $W(\tilde{a}) \cap p^\perp = F$. Write $\tilde{a} = \mathbf{g} \circ$ with $\mathbf{g} = (\tilde{a} \tilde{a}^T)^{1/2}$ and $\circ$ a matrix with orthonormal rows. The law of $\tilde{a}$ is invariant under $\tilde{a} \mapsto \tilde{a}Q$, $Q \in O(d)$, which fixes $\mathbf{g}$ and maps $\circ$ to $\circ Q$. So, conditionally on $\mathbf{g}$, the row space is uniformly distributed: F is uniformly distributed and independent of $\mathbf{g}$. Moreover $|\det(\langle \tilde{a}_i, e_c \rangle)| = \det \mathbf{g} \cdot |\det(\circ e^T)|$, and $|\det(\circ e^T)|$, the cosine of the angle between the row space and $E = T_p N$, equals the cosine $|\langle E^\perp, F \rangle|$ of the angle between their orthogonal complements (complementary minors of an orthogonal matrix have the same absolute value). Hence the inner integral is $(2\pi)^{-k/2} \mathbb{E}[\det \mathbf{g}] \cdot \mathbb{E}_F[G(p, F) |\langle E^\perp, F \rangle|]$. For $G = 1$ it equals $(2\pi)^{-k/2} \mathbb{E}|\det \Gamma_k| = 2 / \text{vol}(S^k)$ by Lemma 3.3(i), since $(\langle \tilde{a}_i, e_c \rangle)$ is a standard Gaussian $k \times k$ matrix. The formula follows. $\square$

3.5 The principle

Proposition 3.5. *Let $\mu > 0$ and let V_0 be a real subspace of dimension $k + 1$. Let $\mathbf{m}$ be a finite positive Borel measure on $\mathcal{G}_k$ which is absolutely continuous with respect to ν , and let Ψ be a bounded Borel function on the bundle of tangent k -planes of S , such that*

(i) *for every countably k -rectifiable Borel set $N \subset S$,*

$$\int_{\mathcal{G}_k} \#(N \cap W) d\mathbf{m}(W) = \frac{2}{\text{vol}(S^k)} \int_N \Psi(T_p N) d\mathcal{H}^k(p);$$

(ii) $\Psi(E) \leq J_\mu(E)$ *for every tangent k -plane E ;*

(iii) $\Psi(T_p S_{V_0}) = J_\mu(T_p S_{V_0})$ *for every $p \in S_{V_0}$.*

Then:

(a) $\mathbf{m}(\mathcal{G}_k) = \text{Vol}_{g_\mu}(\mathbb{RP}(V_0)) / v_k$;

(b) *every k -dimensional competitor $N \subset \mathbb{RP}^{2n+1}$ satisfies $\text{Vol}_{g_\mu}(N) \geq \text{Vol}_{g_\mu}(\mathbb{RP}(V_0))$;*

(c) *if moreover ν is absolutely continuous with respect to $\mathbf{m}$, and N is a k -dimensional competitor with $\text{Vol}_{g_\mu}(N) = \text{Vol}_{g_\mu}(\mathbb{RP}(V_0))$, then N meets $\mathbb{RP}(W)$ in exactly one point for ν -almost every W , and $\Psi(T_p \tilde{N}) = J_\mu(T_p \tilde{N})$ for $\mathcal{H}^k$ -almost every $p \in \tilde{N}$.*

Proof. We have $\dim W + \dim V_0 = 2n + 3$, and the $W \in \mathcal{G}_k$ with $\dim(W \cap V_0) \geq 2$ form a proper algebraic subset, which is ν -null. So $\#(S_{V_0} \cap W) = 2$ for $\mathbf{m}$ -almost every W , and by (i) and (iii) $\text{Vol}_{g_\mu}(S_{V_0}) = \int_{S_{V_0}} \Psi = \text{vol}(S^k) \mathbf{m}(\mathcal{G}_k)$. This is (a). Let N be a competitor. Its preimage $\tilde{N}$ satisfies $\#(\tilde{N} \cap W) = 2 \#(N \cap \mathbb{RP}(W)) \geq 2$ for ν -almost every, hence $\mathbf{m}$ -almost every W . By (ii) and (i),

$$\text{Vol}_{g_\mu}(\tilde{N}) = \int_{\tilde{N}} J_\mu d\mathcal{H}^k \geq \int_{\tilde{N}} \Psi d\mathcal{H}^k = \frac{\text{vol}(S^k)}{2} \int \#(\tilde{N} \cap W) d\mathbf{m}(W) \geq \text{vol}(S^k) \mathbf{m}(\mathcal{G}_k) = \text{Vol}_{g_\mu}(S_{V_0}), \quad (2)$$

which is (b). In the case of equality both inequalities are equalities; the second gives $\#(\tilde{N} \cap W) = 2$ for $\mathbf{m}$ -almost every W , the first gives $\Psi = J_\mu$ almost everywhere on $\tilde{N}$. $\square$

By Lemma 3.4 the pair $(\mathfrak{m}, \Psi) = (\nu, 1)$ satisfies (i). This is the classical case.

Remark 3.6 (the two regimes of [Whe26]). By Lemma 2.1, $J_\mu \geq 1$ if $\mu > 1$ and $J_\mu \geq \sqrt{\mu}$ if $\mu < 1$. So Proposition 3.5 applies with $(\mathfrak{m}, \Psi) = (\nu, 1)$ and a totally real V_0 (which exists if $k \leq n$) when $\mu > 1$, and with $(\sqrt{\mu}\nu, \sqrt{\mu})$ and a complex V_0 (k odd) when $\mu < 1$. This is the argument of [Whe26, Theorems 6.3 and 6.4], here for k -dimensional competitors. With Lemma 3.7 below and part (c) one also gets the equality case among closed C^1 submanifolds of the non-zero class, without using the equality statement of the classical theorem: $N = \mathbb{RP}(V)$, and $J_\mu = 1$ on S_V means $K_V = 0$ (V totally real), while $J_\mu = \sqrt{\mu}$ means $|K_V p| = 1$ for all p (V complex).

3.6 Flats meeting a submanifold once

Lemma 3.7. *Let $1 \leq k \leq m-1$ and let $N \subset \mathbb{RP}^m$ be a non-empty compact embedded C^1 submanifold of dimension k without boundary. Suppose that almost every projective subspace of dimension $m-k$ meets N in at most one point. Then N is a projective subspace.*

“Almost every” refers to the invariant measure of the Grassmannian of $(m-k)$ -dimensional projective subspaces (flats). The proof only uses that no non-empty open set of flats consists of flats meeting N in at least two points.

Proof. Step 0. Let F_0 be a flat and $q \in F_0 \cap N$ a point at which F_0 is transversal to N , that is, $T_q F_0 \oplus T_q N = T_q \mathbb{RP}^m$. In an affine chart with $q = 0$, $T_q N = \mathbb{R}^k \times 0$ and $T_q F_0 = 0 \times \mathbb{R}^{m-k}$, the manifold N is near q the graph of a C^1 map ϕ with $\phi(0) = 0$, $d\phi(0) = 0$, and the flats near F_0 are the sets $\{(Ay + c, y)\}$ with (A, c) near 0. The intersection points are the solutions of $x - A\phi(x) - c = 0$, and by the implicit function theorem there are neighbourhoods $\mathcal{U}$ of F_0 and $\mathcal{O}$ of q such that every $F \in \mathcal{U}$ meets $N \cap \mathcal{O}$ in exactly one point, transversally.

Step 1. If $q \in N$ and F is a flat through q transversal to N at q , then $F \cap N = \{q\}$. Suppose $q' \in F \cap N$, $q' \neq q$. If F is transversal to N at q' too, Step 0 at q and at q' , with disjoint neighbourhoods, gives an open set of flats meeting N in at least two points: a contradiction. Otherwise, the flats through q' transversal to N at q' are dense among the flats through q' ; choose one, F_1 , so close to F that, by Step 0 at q , it meets N transversally at a point q_1 near q , $q_1 \neq q'$. Now the first case applies to F_1 , q_1 , q' .

Step 2. Let $q \neq q'$ be points of N and let $v \in T_q \mathbb{RP}^m$ be the direction of the projective line through q and q' . If $v \notin T_q N$, there is a linear complement of $T_q N$ containing v ; it is the tangent space at q of a flat F through q , which contains that line, hence q' , and is transversal to N at q . This contradicts Step 1. So $v \in T_q N$, which means that q' lies in the projective subspace Π_q of dimension k through q with $T_q \Pi_q = T_q N$. Hence $N \subset \Pi_q$.

Step 3. The inclusion of N into Π_q is an injective C^1 immersion between manifolds of the same dimension, hence open; N is compact, hence closed in Π_q ; and Π_q is connected. So $N = \Pi_q$. $\square$

Steps 0–2 are local: if R is an embedded C^1 submanifold of dimension k , not necessarily closed, and almost every flat meets R in at most one point, then $R \subset \Pi_q$ for every $q \in R$.

3.7 Lipschitz cycles

Proposition 3.8. *Let $c = \sum_i \sigma_i$ be a singular k -chain mod 2 in $\mathbb{RP}^{2n+1}$ with Lipschitz simplices $\sigma_i: \Delta^k \rightarrow \mathbb{RP}^{2n+1}$, such that $\partial c = 0$ and $[c] \neq 0$ in $H_k(\mathbb{RP}^{2n+1}; \mathbb{Z}_2)$. Put $\theta(q) = \sum_i \#\sigma_i^{-1}(q)$ and let M_c be the set of points q for which $\theta(q)$ is finite and odd. Then M_c is a k -dimensional competitor.*

Proof. For $l, m \geq 1$ the set of points with l preimages under σ_i at mutual distances at least $1/m$ is compact, so θ is a Borel function and M_c is a Borel set. It lies in the union of the Lipschitz images $\sigma_i(\Delta^k)$, so it is countably k -rectifiable. Each σ_i lifts to a Lipschitz map $\tilde{\sigma}_i: \Delta^k \rightarrow S$.

(α) For ν -almost every $W \in \mathcal{G}_k$: (1) $\mathbb{RP}(W)$ does not meet $\sigma_i(\partial\Delta^k)$; (2) every point s of $\sigma_i^{-1}(\mathbb{RP}(W))$ is an interior point at which σ_i is differentiable, with $d\sigma_i(s)$ injective and its image complementary to the tangent space of $\mathbb{RP}(W)$; (3) $\sigma_i^{-1}(\mathbb{RP}(W))$ is finite. (1) holds because $\sigma_i(\partial\Delta^k)$ is $\mathcal{H}^k$ -null (Lemmas 3.1 and 3.2). For (2), let Ω be an open subset of the interior of Δ^k which $\tilde{\sigma}_i$ maps into the domain of a frame (f_l) as in the proof of Lemma 3.2, and consider $\Theta_i(s, c) = (\sum_l c_m^l f_l(\tilde{\sigma}_i(s)))_{m \leq k}$ on $\Omega \times \mathbb{R}^{kd}$, a locally Lipschitz map between open subsets of Euclidean spaces of the same dimension. We have $\Theta_i(s, c) = a$ if and only if $\tilde{\sigma}_i(s) \in W(a)$ and c are the coordinates of a . The set of (s, c) such that $\tilde{\sigma}_i$ is not differentiable at s is null by Rademacher's theorem [Sim83, 5.2], and so is its image. At the other points Θ_i is differentiable, with Jacobian $|\det(\langle a_m, d\tilde{\sigma}_i(s)e_c \rangle)|$ by the computation in the proof of Lemma 3.2, and the image of the set where the Jacobian vanishes is null by the area formula for Lipschitz maps [Sim83, 12.4]. The Jacobian is non-zero if and only if $d\tilde{\sigma}_i(s)$ is injective and its image meets $W(a)$ only in 0. The two exceptional sets of a are unions of fibres of $a \mapsto W(a)$, so by Lemma 3.1(a) the properties hold for ν -almost every W . (3) By (1) and (2) the compact set $\sigma_i^{-1}(\mathbb{RP}(W))$ consists of isolated points.

(β) For such W the number $\sum_i \#\sigma_i^{-1}(\mathbb{RP}(W)) = \sum_{q \in \mathbb{RP}(W)} \theta(q)$ is odd. The set of points of $\mathbb{RP}(W)$ in the image of c is finite. As $\dim W \geq 2$, there is a unit vector $w_0 \in W$ such that the hyperplane $W' = W \cap w_0^\perp$ of W contains none of these points. Put $V' = W^\perp \oplus \mathbb{R}w_0$; then $\dim V' = k + 1$ and $V'^\perp = W'$. The linear projection $\phi: \mathbb{RP}^{2n+1} \setminus \mathbb{RP}(W') \rightarrow \mathbb{RP}(V')$, $[z] \mapsto [P_{V'}z]$, is a homotopy equivalence (Lemma 2.3 for W'), its fibre over $y = [w_0]$ is $\mathbb{RP}(W) \setminus \mathbb{RP}(W')$, and the kernel of its differential is the tangent space of the fibre. The chain c lies in the domain of ϕ , and its class there is not zero, because it is mapped to $[c] \neq 0$. Hence $\phi_\# c = \sum_i \phi \circ \sigma_i$ represents the non-zero element of $H_k(X; \mathbb{Z}_2) = \mathbb{Z}_2$, $X = \mathbb{RP}(V')$, which is the fundamental class. The map $H_k(X; \mathbb{Z}_2) \rightarrow H_k(X, X \setminus y; \mathbb{Z}_2) = \mathbb{Z}_2$ is an isomorphism [Hat02, Theorem 3.26]. The image of the class of $\phi_\# c$ is the sum of the classes of the simplices $\tau_i = \phi \circ \sigma_i$, which are relative cycles by (1). By excision around the finitely many points of $\tau_i^{-1}(y) = \sigma_i^{-1}(\mathbb{RP}(W))$, the class of τ_i is the sum of the local degrees mod 2 at these points, and each local degree is 1: at such a point s the map τ_i is differentiable with invertible differential $d\phi \circ d\sigma_i(s)$, so on a small sphere around s it is homotopic to its linearisation by a homotopy avoiding y . Therefore $\sum_i \#\sigma_i^{-1}(\mathbb{RP}(W))$ is odd.

By (α) and (β), for ν -almost every W some point $q \in \mathbb{RP}(W)$ has $\theta(q)$ finite and odd. $\square$

By the area formula, $\text{Vol}_{g_\mu}(M_c)$ is at most the g_μ -volume of c counted with multiplicity. Corollary 1.2 follows from Proposition 3.8 and Theorems I to III.

Remark 3.9 (flat chains mod 2; relies on cited theorems). Suppose that $(\mathfrak{m}, \Psi, V_0)$ satisfies the hypotheses of Proposition 3.5, as it will in Theorems I to III. Let T be a flat k -chain mod 2 of finite mass in $(\mathbb{RP}^{2n+1}, g_\mu)$ with $\partial T = 0$ and non-zero homology class ([Fle66], [Fed69, 4.2.26]). Then $\mathbf{M}_{g_\mu}(T) \geq \text{Vol}_{g_\mu}(\mathbb{RP}(V_0))$; that is, the homological systole $\text{sys}_k^{\mathbb{Z}_2}(g_\mu)$ of [Whe26, Definition 6.2] equals $\text{Vol}_{g_\mu}(\mathbb{RP}(V_0))$. The derivation uses the following facts. (a) The class of T contains a cycle T_0 of least mass, and T_0 is rectifiable: it is given by a countably k -rectifiable set with multiplicity one, whose g_μ -volume is $\mathbf{M}_{g_\mu}(T_0)$ (compactness and rectifiability for flat chains over a finite group [Fle66]; [Fed69, 4.2.26, 5.1.6]; the homology of flat chains mod 2 of a compact manifold is its singular homology with coefficients $\mathbb{Z}_2$, [Fed69, 4.4]). (b) A diffeomorphism isotopic to the identity preserves the class of T_0 and maps a set with multiplicity one to such a set, so the multiplicity-one varifold of T_0 is stationary in $(\mathbb{RP}^{2n+1}, g_\mu)$. After an isometric embedding into a Euclidean space it has bounded generalised mean curvature, and the monotonicity formula ([All72, Section 5]; [Sim83, 17.6–17.8]) shows that its density exists everywhere and is upper semicontinuous, hence at least 1 on the support. Therefore $\text{spt } T_0$ is a closed countably k -rectifiable set and $\text{Vol}_{g_\mu}(\text{spt } T_0) \leq \mathbf{M}_{g_\mu}(T_0)$ [Sim83, Remark 17.9(1), Theorem 3.2(1)]. (c) $\text{spt } T_0$ meets every $\mathbb{RP}(W)$: otherwise the homotopy formula for the deformation of Lemma 2.3 shows that T_0 is homologous to a rectifiable k -chain carried by the $(k-1)$ -dimensional $\mathbb{RP}(W^\perp)$, that is, to 0. (d) By Proposition 3.5(b) for $N = \text{spt } T_0$,

$$\mathbf{M}_{g_\mu}(T) \geq \mathbf{M}_{g_\mu}(T_0) \geq \text{Vol}_{g_\mu}(\text{spt } T_0) \geq \text{Vol}_{g_\mu}(\mathbb{RP}(V_0)).$$

The statements quoted from [Sim83] were compared with the scan of that text. We could not consult [Fed69, Fle66, All72]; the paragraph numbers of [Fed69] given here were not checked against the book. For this reason the statement is a remark and not a theorem. Nothing else in this note depends on it.

Remark 3.10 (uniqueness among all mass minimisers; a sketch). In the situation of Remark 3.9 assume moreover that ν is absolutely continuous with respect to $\mathbf{m}$, and let T_0 be a mass-minimising cycle of the non-zero class. Then $N = \text{spt } T_0$ has equality in (2), so ν -almost every $\mathbb{RP}(W)$ meets N exactly once. The density of T_0 is 1 at almost every point, and by Allard's regularity theorem ([All72]; [Sim83, Theorem 24.2]) N is a C^1 submanifold near every point of density one. So the set R of regular points is open in N and of full measure, hence dense. The proof of Lemma 3.7 applies to points $q \in R$ (in Step 1 one replaces q' by a nearby point of R) and shows that N lies in the projective subspace Π_q tangent to N at q . Since N meets every $\mathbb{RP}(W)$, and a generic $\mathbb{RP}(W)$ through a point $y \in \Pi_q$ meets Π_q only in y , it follows that $N = \Pi_q$. The type of the subspace is then determined as in the proofs of Theorems II and III. We have not written out the details; this remark is a sketch and is not used elsewhere. In codimension one, Proposition 4.4 gives a complete proof by other means.

4 Hypersurfaces: proof of Theorem I

In this section $d = 2n + 1$ and

$$c_d = \int_{S^{d-1}} |v_1| d\sigma(v) = \frac{2\pi^{(d-1)/2}}{\Gamma(\frac{d+1}{2})} = \frac{2\pi^n}{n!}, \quad w_\mu(\kappa) = (1 + (\mu - 1)\kappa^2)^{-(n+1)}.$$

Note that $\text{vol}(S^{2n+1}) = \pi c_d$ and that $\min\{1, \mu\}^{n+1} \leq w_\mu^{-1} \leq \max\{1, \mu\}^{n+1}$.

4.1 An ellipsoid is a zonoid with an explicit density

Lemma 4.1. *For a positive definite symmetric $d \times d$ matrix B and $\alpha \in \mathbb{R}^d$,*

$$\int_{S^{d-1}} |\langle v, \alpha \rangle| (v^T B v)^{-(d+1)/2} d\sigma(v) = c_d (\det B)^{-1/2} (\alpha^T B^{-1} \alpha)^{1/2}.$$

Proof. Compute $I(\alpha) = \int_{\mathbb{R}^d} |\langle x, \alpha \rangle| e^{-x^T B x/2} dx$ in two ways. In polar coordinates $x = rv$, with the substitution $s = r\sqrt{v^T B v}$, $I(\alpha) = I_d \int_{S^{d-1}} |\langle v, \alpha \rangle| (v^T B v)^{-(d+1)/2} d\sigma(v)$, where $I_d = \int_0^\infty s^d e^{-s^2/2} ds$. With $x = B^{-1/2}y$ and the rotation invariance of the Gaussian measure, $I(\alpha) = (\det B)^{-1/2} |B^{-1/2}\alpha| \int |y_1| e^{-|y|^2/2} dy = (\det B)^{-1/2} (\alpha^T B^{-1} \alpha)^{1/2} I_d c_d$. $\square$

We apply the lemma in $T_p S \cong \mathbb{R}^d$ to the matrix $B = I + (\mu - 1)\xi\xi^T$ of g_μ in a g -orthonormal basis. Then $v^T B v = 1 + (\mu - 1)\langle v, \xi \rangle^2$, $\det B = \mu$, and B^{-1} is the matrix of the dual metric; for a unit vector $\nu \in T_p S$ and $E = \nu^\perp$ we have $\nu^T B^{-1} \nu = (\mu + (1 - \mu)\langle \nu, \xi \rangle^2)/\mu = J_\mu(E)^2/\mu$. Hence, with $U_p S$ the unit sphere of $T_p S$,

$$\int_{U_p S} |\alpha(v)| w_\mu(\langle \xi, v \rangle) d\sigma(v) = \frac{c_d}{\sqrt{\mu}} |\alpha|_{g_\mu} \quad (\alpha \in T_p^* S), \quad \int_{U_p S} |\langle v, \nu \rangle| w_\mu(\langle \xi, v \rangle) d\sigma(v) = \frac{c_d}{\mu} J_\mu(\nu^\perp), \quad (3)$$

where $|\alpha|_{g_\mu}$ is the norm of the covector α for g_μ .

4.2 The Crofton formula

For a real 2-plane $W \subset \mathbb{R}^{2n+2}$ with orthonormal basis w_1, w_2 let $\kappa(W) = |\langle Jw_1, w_2 \rangle| \in [0, 1]$. The great circles are the sets $S \cap W$, $W \in \mathcal{G}_{2n}$.

Theorem 4.2. *Let $n \geq 1$ and $\mu > 0$.*

(a) *For every countably $2n$ -rectifiable Borel set $N \subset S$,*

$$\text{Vol}_{g_\mu}(N) = \frac{\mu \text{vol}(S^{2n})}{2} \int_{\mathcal{G}_{2n}} w_\mu(\kappa(W)) \#(N \cap W) d\nu(W). \quad (4)$$

(b) *Every $2n$ -dimensional competitor $N \subset \mathbb{RP}^{2n+1}$ satisfies $\text{Vol}_{g_\mu}(N) \geq \text{Vol}_{g_\mu}(\mathbb{RP}(V_0)) = v_{2n}M_n(\mu)$ for a real hyperplane V_0 , and $\mu \int_{\mathcal{G}_{2n}} w_\mu(\kappa(W)) d\nu(W) = M_n(\mu)$.*

(c) *A closed C^1 hypersurface N of the non-zero class with $\text{Vol}_{g_\mu}(N) = v_{2n}M_n(\mu)$ is a projective hyperplane.*

Proof. (a) Apply Lemma 3.4 with $k = 2n$, $m = 1$. A one-dimensional subspace F of $T_p S$ is spanned by a unit vector v , the plane $W = \mathbb{R}p \oplus F$ has $\kappa(W) = |\langle Jp, v \rangle|$, and we take $G(p, F) = w_\mu(\langle \xi, v \rangle)$. If ν is a unit normal of $E = T_p N$ in $T_p S$, then $|\langle E^\perp, F \rangle| = |\langle v, \nu \rangle|$, and by (3)

$$AG(p, E) = \frac{\int_{U_p S} |\langle v, \nu \rangle| w_\mu(\langle \xi, v \rangle) d\sigma(v)}{\int_{U_p S} |\langle v, \nu \rangle| d\sigma(v)} = \frac{J_\mu(E)}{\mu}.$$

So Lemma 3.4 gives $\int w_\mu(\kappa(W)) \#(N \cap W) d\nu(W) = \frac{2}{\mu \text{vol}(S^{2n})} \int_N J_\mu(T_p N) d\mathcal{H}^{2n}$.

(b), (c) The measure $\mathbf{m} = \mu w_\mu(\kappa(W)) d\nu(W)$ and the function $\Psi = J_\mu$ satisfy the hypotheses of Proposition 3.5 for every hyperplane V_0 : (i) is (4), and (ii), (iii) hold with equality on all tangent hyperplanes. As w_μ is bounded and bounded away from zero, $\mathbf{m}$ and ν have the same null sets. Part (b) follows from Proposition 3.5(a), (b) and Lemma 2.2(b). In the situation of (c), Proposition 3.5(c) shows that almost every projective line meets N exactly once, and Lemma 3.7 (with $m = 2n + 1$ and $k = 2n$) shows that N is a projective hyperplane. $\square$

Theorem 4.2 contains Theorem I.

4.3 The geodesic flow

Let $US = \{(p, v) : p \in S, v \in U_p S\}$ be the unit tangent bundle of the round sphere, with the measure $dL = dV_g(p) d\sigma_p(v)$ of total mass $\text{vol}(S^{2n+1}) \text{vol}(S^{2n}) = \pi c_d \text{vol}(S^{2n})$. For $(p, v) \in US$ let $\gamma_{p,v}(s) = \cos s \cdot p + \sin s \cdot v$ and $\varphi_s(p, v) = (\gamma_{p,v}(s), \gamma'_{p,v}(s))$. The flow φ_s commutes with $O(2n+2)$, which acts transitively on US and preserves L ; so L is invariant under φ_s , and also under $(p, v) \mapsto (p, -v)$. The function $\kappa(p, v) = \langle Jp, v \rangle$ is invariant under the flow: $\frac{d}{ds} \langle J\gamma, \gamma' \rangle = \langle J\gamma', \gamma' \rangle - \langle J\gamma, \gamma \rangle = 0$. In other words $g_\mu(\gamma', \gamma') = 1 + (\mu - 1)\kappa^2$ is constant along every great circle: g_μ is a Killing tensor of the round metric. The image of $L/L(US)$ under $(p, v) \mapsto \mathbb{R}p \oplus \mathbb{R}v$ is ν . Hence (4) is equivalent to

$$\text{Vol}_{g_\mu}(N) = \frac{\mu}{2\pi c_d} \int_{US} w_\mu(\kappa(p, v)) \# \{s \in [0, 2\pi) : \gamma_{p,v}(s) \in N\} dL(p, v). \quad (5)$$

This is the form in which the formula follows from Santaló's formula for the geodesic flow [San04].

Lemma 4.3. *For every $u \in C^1(S)$,*

$$\int_S |du|_{g_\mu} dV_{g_\mu} = \frac{\mu}{2\pi c_d} \int_{US} w_\mu(\kappa(p, v)) \int_0^{2\pi} |(u \circ \gamma_{p,v})'(s)| ds dL(p, v).$$

Proof. By the first identity in (3) with $\alpha = du_p$, and $dV_{g_\mu} = \sqrt{\mu} dV_g$,

$$\int_S |du|_{g_\mu} dV_{g_\mu} = \frac{\mu}{c_d} \int_{US} w_\mu(\kappa(p, v)) |du_p(v)| dL(p, v).$$

The measure L and the function κ are invariant under φ_s , so the right-hand side is unchanged if $|du_p(v)|$ is replaced by $|du(\gamma'_{p,v}(s))| = |(u \circ \gamma_{p,v})'(s)|$; average over $s \in [0, 2\pi)$. $\square$

4.4 Sets of finite perimeter

Cycles mod 2 of codimension one are boundaries of sets, and for these the inequality of Theorem I and its equality case can be proved directly, without rectifiability or regularity theory. For a measurable set $E \subset S$ let

$$\text{Per}_\mu(E) = \inf \left\{ \liminf_{j \rightarrow \infty} \int_S |du_j|_{g_\mu} dV_{g_\mu} : u_j \in C^1(S), u_j \rightarrow \chi_E \text{ in } L^1(S) \right\} \in [0, \infty],$$

where χ_E is the indicator function. This is the perimeter of E in the Riemannian manifold (S, g_μ) ; see Remark 4.5.

Proposition 4.4. *Let $n \geq 1$, $\mu > 0$, and let $E \subset S^{2n+1}$ be a measurable set such that $\chi_E(-x) = 1 - \chi_E(x)$ for almost every x . Then*

$$\text{Per}_\mu(E) \geq \text{Vol}_{g_\mu}(S_{V_0}) = 2 v_{2n} M_n(\mu)$$

for a real hyperplane V_0 . Equality holds if and only if E coincides, up to a null set, with an open hemisphere $\{x \in S : \langle x, \nu_0 \rangle > 0\}$.

Proof. We use two elementary facts. For a 2π -periodic C^1 function h and points $s_1 < s_2 < \dots < s_r < s_1 + 2\pi$ we have $\int_0^{2\pi} |h'| \geq \sum_{i=1}^r |h(s_{i+1}) - h(s_i)|$, where $s_{r+1} = s_1$. And, by the invariance of L under the flow, $\int_{US} \int_0^{2\pi} f(\gamma_{p,v}(s)) ds dL = 2\pi \text{vol}(S^{2n}) \int_S f dV_g$ for every Borel $f \geq 0$ on S . So null sets of S are met by almost every great circle in a set of parameters of measure zero, and convergence in $L^1(S)$ implies convergence in L^1 of the restrictions to great circles, on average.

The inequality. We may assume $\text{Per}_\mu(E) < \infty$. Choose $u_j \in C^1(S)$ with $u_j \rightarrow \chi_E$ in $L^1(S)$ and $\int_S |du_j|_{g_\mu} dV_{g_\mu} \rightarrow \text{Per}_\mu(E)$. After passing to a subsequence, for L -almost every (p, v) the functions $h_j = u_j \circ \gamma_{p,v}$ converge almost everywhere on $[0, 2\pi)$ to $f = \chi_E \circ \gamma_{p,v}$, and $f(s+\pi) = 1 - f(s)$ for almost every s . Fix such a (p, v) and put $A = \{s : f(s) = 1\}$; both A and its complement have measure π . Choose points $s_1 \in A$, $s_2 \notin A$ at which h_j converges to f . Then $\int_0^{2\pi} |h'_j| \geq 2|h_j(s_1) - h_j(s_2)| \rightarrow 2$. Hence

$$\liminf_{j \rightarrow \infty} F_j(p, v) \geq 2, \quad F_j(p, v) := \int_0^{2\pi} |(u_j \circ \gamma_{p,v})'(s)| ds, \quad (6)$$

for L -almost every (p, v) . By Lemma 4.3, Fatou's lemma and (5) for $N = S_{V_0}$, which almost every great circle meets twice,

$$\text{Per}_\mu(E) = \lim_j \frac{\mu}{2\pi c_d} \int_{US} w_\mu(\kappa) F_j dL \geq \frac{\mu}{2\pi c_d} \int_{US} w_\mu(\kappa) \cdot 2 dL = \text{Vol}_{g_\mu}(S_{V_0}).$$

Hemispheres. Let $E_0 = \{\langle x, \nu_0 \rangle > 0\}$ and $u_j(x) = \psi(j\langle x, \nu_0 \rangle)$ with a fixed non-decreasing smooth function ψ , $\psi = 0$ on $(-\infty, -1]$ and $\psi = 1$ on $[1, \infty)$. Then $u_j \rightarrow \chi_{E_0}$ in L^1 , and on every great circle $u_j \circ \gamma(s) = \psi(jc \cos(s - s_0))$ for some $c \geq 0$, s_0 , a function with total variation at most 2. By Lemma 4.3 and the last display, $\text{Per}_\mu(E_0) \leq \text{Vol}_{g_\mu}(S_{V_0})$, so equality holds for hemispheres.

The equality case. Let $\text{Per}_\mu(E) = \text{Vol}_{g_\mu}(S_{V_0})$ and u_j as above. Then $\int w_\mu(\kappa) (\liminf_j F_j - 2) dL \leq 0$ by Fatou's lemma, so by (6) $\liminf_j F_j = 2$ almost everywhere. Fix such a (p, v) and let $D \subset [0, 2\pi)$

be the set of full measure on which $h_j \rightarrow f$. If there were points $s_1 < s_2 < s_3 < s_4$ in D , in cyclic order, with $f = 1, 0, 1, 0$ at these points, the first elementary fact would give $\liminf_j F_j \geq 4$. Hence the sets $A \cap D$ and $D \setminus A$ are not interlaced on the circle, and so A coincides almost everywhere with an arc: cut the circle at a point $b_1 \in D \setminus A$, let $\alpha < \beta$ be the infimum and the supremum of $A \cap D$ in $(b_1, b_1 + 2\pi)$; a point $b \in D \setminus A$ with $\alpha < b < \beta$ would lie between two points of $A \cap D$, which together with b_1 would give an interlaced quadruple. The arc has length π . So *for almost every great circle γ , the set $\gamma^{-1}(E)$ is, up to a null set, a half circle.*

We now show that this property forces E to be a hemisphere. For a bounded Borel function Φ on $S \times S$,

$$\int_S \int_S \Phi(x, y) dV_g(y) dV_g(x) = \frac{1}{4\pi} \int_{US} \int_0^{2\pi} \int_0^{2\pi} \Phi(\gamma_{p,v}(s), \gamma_{p,v}(t)) |\sin(t-s)|^{2n} dt ds dL(p, v). \quad (7)$$

Indeed, in polar coordinates $y = \gamma_{x,v}(r)$ around x the left side is $\int_{US} \int_0^\pi \Phi(p, \gamma_{p,v}(r)) \sin^{2n} r dr dL$; by the invariance of L under φ_s it equals $\frac{1}{2\pi} \int_{US} \int_0^{2\pi} \int_0^\pi \Phi(\gamma_{p,v}(s), \gamma_{p,v}(s+r)) \sin^{2n} r dr ds dL$, and by the invariance under $(p, v) \mapsto (p, -v)$ the same holds with $s-r$ in place of $s+r$; the mean of the two expressions is the right side of (7). Take $\Phi(x, y) = \chi_E(x) \chi_E(y) \langle x, y \rangle$ and let $m_E = \int_E x dV_g(x)$. Since $\langle \gamma(s), \gamma(t) \rangle = \cos(t-s)$,

$$|m_E|^2 = \frac{1}{4\pi} \int_{US} \left[\iint_{A_{p,v} \times A_{p,v}} \cos(t-s) |\sin(t-s)|^{2n} dt ds \right] dL(p, v), \quad A_{p,v} = \gamma_{p,v}^{-1}(E).$$

If $A_{p,v}$ is a half circle, the inner integral is a number ι_n which does not depend on (p, v) . This is the case for almost every (p, v) , both for E and for a hemisphere E_0 . Hence $|m_E| = |m_{E_0}| =: m_0$, and $m_0 > 0$ because $m_{E_0} = (\int_{E_0} \langle x, \nu_0 \rangle dV_g) \nu_0$. Let $\nu_0 = m_E / |m_E|$ and $E_0 = \{\langle x, \nu_0 \rangle > 0\}$. Then

$$m_0 = \int_E \langle x, \nu_0 \rangle dV_g \leq \int_E \langle x, \nu_0 \rangle^+ dV_g \leq \int_S \langle x, \nu_0 \rangle^+ dV_g = \int_{E_0} \langle x, \nu_0 \rangle dV_g = m_0,$$

and equality in the two inequalities means that E and E_0 differ by a null set. $\square$

Remark 4.5 (perimeters and cycles mod 2). By the lower semicontinuity of the total variation and the approximation of functions of bounded variation by smooth functions [Sim83, Lemma 6.2], $\text{Per}_\mu(E)$ is the perimeter of E in (S, g_μ) in the usual sense, and by De Giorgi's theorem [Sim83, Theorem 14.3] it is the g_μ -volume of the reduced boundary of E . The latter is the mass of the flat chain ∂E mod 2. If T is a flat $2n$ -cycle mod 2 of finite mass in $\mathbb{RP}^{2n+1}$ with non-zero class, its lift to S is the boundary of a set E of finite perimeter, because $H_{2n}(S; \mathbb{Z}_2) = 0$, and E is antisymmetric as in Proposition 4.4, because otherwise E would be the lift of a set with boundary T . So Proposition 4.4 says that $\mathbf{M}_{g_\mu}(T) \geq v_{2n} M_n(\mu)$, with equality only for projective hyperplanes. This translation uses the identification of flat chains mod 2 of top dimension with measurable sets and of their boundary masses with perimeters ([Fed69, 4.2.26, 4.5], [Fle66]), which we quote without having been able to check it in these sources; Proposition 4.4 itself is self-contained.

4.5 Remarks

Remark 4.6 (the density is forced). By Lemma 3.4, a measure $\varrho(W) d\nu(W)$ on great circles has, at the point p and in the normalisation of Proposition 3.5, the Crofton density $\nu \mapsto \frac{1}{c_d} \int_{U_p S} |\langle \nu, \nu \rangle| \varrho(\mathbb{R}p \oplus \mathbb{R}\nu) d\sigma(\nu)$ on the hyperplane $\nu^\perp$, a cosine transform. The cosine transform is injective on even continuous functions on the sphere (a classical fact; see [Sch14, Section 3.5]). Together with Lemma 4.1 this shows that a measure on great circles with a continuous density represents the g_μ -volume of hypersurfaces if and only if its density is $\mu w_\mu(\kappa(W))$. For an arbitrary Riemannian metric g' on

the sphere, with matrix B_p in a g -orthonormal basis, the same reasoning gives the pointwise density $\det(B_p)(v^T B_p v)^{-(d+1)/2}$, which is always positive, and a Crofton formula with a continuous density exists if and only if this function on US is invariant under the geodesic flow of the round metric. So Theorem 4.2 is an application of the classical Crofton mechanism; what is specific to the Berger metric is that the density is explicit and invariant. As said in the introduction, the general statement that even hypersurface integrands with extremal hyperplanes have Crofton formulas with smooth signed measures on lines is due to [GS94, ÁPF07], and we could not read these two papers.

Remark 4.7 (metrics with minimal equators). Let g' be a Riemannian metric on S^d , write $dV_{g'} = F^{(d+1)/4} dV_g$ and $\mathfrak{k} = g'/F$. Then $\det(B_p)(v^T B_p v)^{-(d+1)/2} = \mathfrak{k}(v, v)^{-(d+1)/2}$. By the theorem of Ambrozio, Marques and Neves [AMN25, Section 9.4] (see also [Amb26]), all equators are minimal for g' if and only if $\mathfrak{k}$ is a Killing symmetric 2-tensor of the round metric, that is, if and only if $\mathfrak{k}(\gamma', \gamma')$ is constant along great circles. So for these metrics, which are invariant under the antipodal map, the proofs of this section apply without change: in $(\mathbb{RP}^d, g')$ the projective hyperplanes minimise volume among the competitors of dimension $d-1$, and they are the only minimisers among closed C^1 hypersurfaces of the non-zero class. This statement depends on the cited theorem; the Berger case does not.

Remark 4.8 (dimension three). For $n = 1$, $M_1(\mu) = I_\mu := \int_0^1 \sqrt{\mu + (1-\mu)t^2} dt$, and Theorem 4.2 gives the least area $2\pi I_\mu$. Gil-Medrano [GM16, Theorem 1 and Corollary 2] proved that the equatorial projective planes are exactly the surfaces of least area among embedded projective planes in $(\mathbb{RP}^3, g_\mu)$, with the value $\pi A(\mu)$; an elementary integration shows $A(\mu) = 2I_\mu$. Theorem 4.2 enlarges the class of competitors (all closed C^1 surfaces of the non-zero class, of any topology, and all competitors in the sense of Definition 1.1); it says nothing about minimal projective planes that are not minimising, which are also treated in [GM16].

Remark 4.9 (stability). Torralbo and Urbano [TU22, Proposition 2] proved that for $\mu \leq 1$ every compact orientable minimal hypersurface of the Berger sphere (S^{2n+1}, g_μ) is unstable; in particular the great hyperspheres are unstable there. This does not contradict Theorem I: the destabilising variation, which has constant normal speed, is not invariant under the antipodal map and does not descend to $\mathbb{RP}^{2n+1}$. The projective subspaces of mixed type are minimal but not totally geodesic [Whe26, Proposition 4.3]; the totally geodesic submanifolds of Berger spheres are classified in [TU22, Proposition 5].

5 The squashed case in even dimension: proof of Theorem II

Throughout this section $n \geq j \geq 1$.

5.1 Statement

Notation. For $p \in S$ and $0 \leq b < 1$ let $G_b = g - b^2 \eta \otimes \eta$ on $T_p S$; it is positive definite, since $G_b(u, u) = |u^h|^2 + (1 - b^2)\eta(u)^2$. For a $2j$ -dimensional subspace $E \subset T_p S$ let

$$\omega_1(E; b) \geq \cdots \geq \omega_j(E; b) \geq 0$$

be the singular values of the skew form $\omega|_E$ with respect to the inner product $G_b|_E$: there is a G_b -orthonormal basis $u_1, v_1, \dots, u_j, v_j$ of E with $\omega(u_a, v_a) = \omega_a(E; b)$ and ω vanishing on all other pairs

of basis vectors. Let e_r denote the elementary symmetric polynomials, and put

$$c_{j,r} = (2r-1)!! (2j-2r-1)!! \quad (0 \leq r \leq j), \quad F_j(\rho_1, \dots, \rho_j) = \sum_{r=0}^j c_{j,r} e_r(\rho_1^2, \dots, \rho_j^2),$$

$$A_j(t) = \sum_{r=0}^{j-1} c_{j,r} \binom{j-1}{r} t^r, \quad B_j(t) = \sum_{r=0}^{j-1} c_{j,r+1} \binom{j-1}{r} t^r,$$

$$\Phi_b^{(j)}(E) = \sqrt{1-b^2 x_E} \frac{F_j(b\omega_1(E; b), \dots, b\omega_j(E; b))}{(2j-1)!!}.$$

For example $F_1 = 1 + \rho^2$, $F_2 = 3 + e_1 + 3e_2$, $F_3 = 15 + 3e_1 + 3e_2 + 15e_3$; $A_1 = B_1 = 1$, $A_2 = 3 + t$, $B_2 = 1 + 3t$, $A_3 = 15 + 6t + 3t^2$, $B_3 = 3 + 6t + 15t^2$. For $0 < \mu < 1$ let $\tau = 1 - \mu$, let $\mathbf{m}_j$ be the measure on $[0, \sqrt{\tau}]$ with

$$\mathrm{d}\mathbf{m}_j(b) = (2j-1)!! \mu B_j(\tau) \tau^{-(j-1/2)} \frac{b^{2j-2}}{(1-b^2)^2 B_j(b^2)^2} \mathrm{d}b,$$

and let $\Psi^{(j)}(E) = \int_0^{\sqrt{\tau}} \Phi_b^{(j)}(E) \mathrm{d}\mathbf{m}_j(b)$.

Planes of candidate type. A $2j$ -dimensional subspace $E \subset T_p S$ is of *candidate type* if $E = L \oplus E''$ is an orthogonal sum of a J -invariant subspace $L \subset H_p$ of real dimension $2j-2$ and a 2-plane E'' which either contains ξ or is spanned by $\sqrt{x}\xi + \sqrt{1-x}Jh$ and h , for a unit vector $h \in H_p$ orthogonal to L and some $x \in [0, 1)$. We shall see that these are exactly the tangent planes of the spheres S_V with V of type $\mathbb{C}^j \oplus \mathbb{R}$ (Lemma 5.8).

The Gaussian model. For $0 \leq b < 1$ let $a_1, a'_2, a_3, a'_4, \dots, a_{2j-1}, a'_{2j}$ be independent standard Gaussian vectors of $\mathbb{R}^{2n+2}$, let

$$a_{2i} = b J a_{2i-1} + \sqrt{1-b^2} a'_{2i} \quad (1 \leq i \leq j),$$

and $W = W(a_1, \dots, a_{2j})$. We call (a_{2i-1}, a_{2i}) a *J-correlated pair with parameter b* , write P_b for the law of W and $\mathbb{E}_b$ for the expectation.

Theorem 5.1. *Let $n \geq j \geq 1$.*

- (1) *For every $b \in [0, 1)$ the law P_b is a $U(n+1)$ -invariant probability measure on $\mathcal{G}_{2j}$ with the same null sets as ν , and for every countably $2j$ -rectifiable Borel set $N \subset S$*

$$\mathbb{E}_b \#(N \cap W) = \frac{2}{\mathrm{vol}(S^{2j})} \int_N \Phi_b^{(j)}(T_p N) \mathrm{d}\mathcal{H}^{2j}(p).$$

- (2) *Let $0 < \mu < 1$. Then $\mathbf{m}_j$ is a finite positive measure of total mass $M_j(\mu)$, and for every tangent $2j$ -plane E*

$$\Psi^{(j)}(E) \leq \sqrt{1-\tau x_E} = J_\mu(E),$$

with equality if and only if E is of candidate type.

- (3) *Let $0 < \mu < 1$ and let V_0 be of type $\mathbb{C}^j \oplus \mathbb{R}$. Every $2j$ -dimensional competitor $N \subset \mathbb{RP}^{2n+1}$ satisfies $\mathrm{Vol}_{g_\mu}(N) \geq \mathrm{Vol}_{g_\mu}(\mathbb{RP}(V_0)) = v_{2j} M_j(\mu)$.*
- (4) *Let $0 < \mu < 1$. A closed C^1 submanifold of the non-zero class, of dimension $2j$, with $\mathrm{Vol}_{g_\mu}(N) = v_{2j} M_j(\mu)$ is an $\mathbb{RP}(V)$ with V of type $\mathbb{C}^j \oplus \mathbb{R}$.*

Parts (3) and (4) are Theorem II. The proof occupies Sections 5.2 to 5.8.

5.2 Circular families

A family (ζ_s) of complex random variables is *circular* if the real and imaginary parts of its members are jointly centred Gaussian and $\mathbb{E}[\zeta_s \zeta_t] = 0$ for all s, t . Its law is determined by $C(s, t) = \mathbb{E}[\zeta_s \bar{\zeta}_t]$. We use four elementary facts. (C1) Complex linear combinations of members of a circular family form a circular family. (C2) If $C(s, t) = 0$ whenever s belongs to one subfamily and t to another, the two subfamilies are independent (all real covariances between them vanish). (C3) A member with $C(s, s) = \sigma^2$ is $\mathcal{CN}(0, \sigma^2)$; its density at 0 with respect to the Lebesgue measure of $\mathbb{C} = \mathbb{R}^2$ is $1/(\pi\sigma^2)$. (C4) If $C(0, 0) > 0$, then conditionally on $\zeta_0 = 0$ the family is circular with $C(s, t) - C(s, 0)C(0, t)/C(0, 0)$ in place of $C(s, t)$; indeed $\zeta_s - \frac{C(s, 0)}{C(0, 0)}\zeta_0$ is uncorrelated with, hence independent of, ζ_0 .

Lemma 5.2. *Let (a_1, a_2) be a J -correlated pair with parameter $b \in [0, 1)$ and $\ell(z) = \langle a_1, z \rangle + i\langle a_2, z \rangle$ for $z \in \mathbb{R}^{2n+2}$.*

- (a) $(\ell(z))_z$ is a circular family with $\mathbb{E}[\ell(z)\overline{\ell(w)}] = 2\langle z, w \rangle - 2ib\langle Jz, w \rangle$.
- (b) (a_1, a_2) has a positive density on $(\mathbb{R}^{2n+2})^2$, and its law is invariant under $U(n+1)$.
- (c) Let $p \in S$. The density of $\ell(p)$ at 0 is $1/(2\pi)$, and conditionally on $\ell(p) = 0$ the family $(\ell(u))_{u \in T_p S}$ is circular with

$$\mathbb{E}[\ell(u)\overline{\ell(v)} \mid \ell(p) = 0] = 2G_b(u, v) - 2ib\omega(u, v).$$

Proof. (a) We have $\mathbb{E}[a_1 a_1^T] = \mathbb{E}[a_2 a_2^T] = I$ and $\mathbb{E}[a_1 a_2^T] = bJ^T$, so $\mathbb{E}\langle a_1, z \rangle \langle a_2, w \rangle = b\langle Jz, w \rangle$ and $\mathbb{E}\langle a_2, z \rangle \langle a_1, w \rangle = -b\langle Jz, w \rangle$. This gives the formula and $\mathbb{E}[\ell(z)\overline{\ell(w)}] = 0$. (b) The covariance matrix of (a_1, a_2) is $I + b\mathcal{K}$ with the symmetric involution $\mathcal{K} = \begin{pmatrix} 0 & J^T \\ J & 0 \end{pmatrix}$; its eigenvalues are $1 \pm b > 0$. If U is unitary, then $Ua_2 = bJ(Ua_1) + \sqrt{1-b^2}Ua'_2$, and (Ua_1, Ua'_2) is again a pair of independent standard Gaussian vectors. (c) $C(p, p) = 2$, $C(u, p) = -2ib\langle Ju, p \rangle = 2ib\eta(u)$ and $C(p, v) = -2ib\eta(v)$ for $u, v \in T_p S$. By (C3) and (C4) the density is $1/(2\pi)$ and the conditional covariance is $2\langle u, v \rangle - 2ib\langle Ju, v \rangle - 2b^2\eta(u)\eta(v)$. $\square$

By (b) and the independence of the pairs, the $2j$ -tuple $a = (a_1, \dots, a_{2j})$ has a positive continuous density on $(\mathbb{R}^{2n+2})^{2j}$. Hence $W \in \mathcal{G}_{2j}$ almost surely, P_b has the same null sets as ν (Lemma 3.1), and P_b is $U(n+1)$ -invariant because $W(Ua) = UW(a)$. By (1) and Lemma 5.2(c),

$$\mathbb{E}_b \#(N \cap W) = (2\pi)^{-j} \int_N D_b(T_p N) d\mathcal{H}^{2j}(p), \quad D_b(E) := \mathbb{E} |\det M_E|, \quad (8)$$

where, for a tangent $2j$ -plane E with orthonormal basis $e_1, \dots, e_{2j}$, M_E is the real $2j \times 2j$ matrix with the rows $(\operatorname{Re} \tilde{\ell}_i(e_c))_c$ and $(\operatorname{Im} \tilde{\ell}_i(e_c))_c$, $i = 1, \dots, j$, and $\tilde{\ell}_1, \dots, \tilde{\ell}_j$ are independent circular families on $T_p S$ with the covariance of Lemma 5.2(c).

5.3 The normal form

Lemma 5.3. *For $\rho \in [0, 1]^j$ let $M^\pm$ be complex $j \times j$ matrices with independent entries, $M_{ia}^\pm \sim \mathcal{CN}(0, 1 \pm \rho_a)$, and*

$$\tilde{F}_j(\rho_1, \dots, \rho_j) = \mathbb{E} |\det_{\mathbb{R}}(z \mapsto M^+ z + M^- \bar{z})|,$$

the absolute value of the determinant of a real-linear map of $\mathbb{C}^j = \mathbb{R}^{2j}$. Then for every tangent $2j$ -plane E and every $b \in [0, 1)$ we have $b\omega_a(E; b) \leq 1$ for all a and

$$D_b(E) = \sqrt{1 - b^2 x_E} \tilde{F}_j(b\omega_1(E; b), \dots, b\omega_j(E; b)).$$

Proof. In the basis (e_c) the matrix of $G_b|_E$ is $I - b^2(\eta(e_c)\eta(e_d))$, with determinant $1 - b^2x_E$. Let $(f_c) = (u_1, v_1, \dots, u_j, v_j)$ be a G_b -orthonormal basis of E in which ω has its normal form, and $e_c = \sum_d f_d Q_{dc}$. Then $Q^T Q$ is the matrix of $G_b|_E$, so $|\det Q| = \sqrt{1 - b^2x_E}$, and $|\det M_E| = \sqrt{1 - b^2x_E} |\det M_f|$, where M_f is formed with the basis (f_c) . For one family $\tilde{\ell}$ and one block put $\lambda_a^\pm = \frac{1}{2}(\tilde{\ell}(u_a) \mp i\tilde{\ell}(v_a))$, so that $\tilde{\ell}(su_a + tv_a) = \lambda_a^+ z + \lambda_a^- \bar{z}$ for $z = s + it$. By Lemma 5.2(c), $\mathbb{E}|\tilde{\ell}(u_a)|^2 = \mathbb{E}|\tilde{\ell}(v_a)|^2 = 2$ and $\mathbb{E}[\tilde{\ell}(u_a)\tilde{\ell}(v_a)] = -2ib\omega_a$, where $\omega_a = \omega_a(E; b)$, and all covariances between different blocks vanish. Therefore

$$\mathbb{E}|\lambda_a^\pm|^2 = 1 \pm b\omega_a, \quad \mathbb{E}[\lambda_a^+ \overline{\lambda_a^-}] = 0,$$

and by (C1), (C2) all the variables $\lambda_{i,a}^\pm$ (pair i , block a) are independent, $\lambda_{i,a}^\pm \sim \mathcal{CN}(0, 1 \pm b\omega_a)$. In particular $b\omega_a \leq 1$. In the real coordinates (s_a, t_a) , M_f is the matrix of $z \mapsto M^+z + M^-\bar{z}$ with $M_{ia}^\pm = \lambda_{i,a}^\pm$. $\square$

5.4 One column block

We use the real inner product $\langle x, y \rangle = \operatorname{Re}(x^*y)$ on $\mathbb{C}^j = \mathbb{R}^{2j}$. In Lemma 5.3 the map $z \mapsto M^+z + M^-\bar{z}$ is $z \mapsto \sum_a c_a(z_a)$ with the independent *column blocks*

$$c_a(\zeta) = u_a\zeta + w_a\bar{\zeta} \quad (\zeta \in \mathbb{C}), \quad u_a = (M_{ia}^+)i \sim \mathcal{CN}(0, (1 + \rho_a)I_j), \quad w_a = (M_{ia}^-)i \sim \mathcal{CN}(0, (1 - \rho_a)I_j);$$

the block a contributes the two real columns $c_a(1), c_a(i)$.

Lemma 5.4. *Let $P \subset \mathbb{C}^j$ be a real 2-plane with orthonormal basis p_1, p_2 , let $\kappa_P = \langle ip_1, p_2 \rangle$, and let $c(\zeta) = u\zeta + w\bar{\zeta}$ with independent $u \sim \mathcal{CN}(0, (1 + \rho)I_j)$, $w \sim \mathcal{CN}(0, (1 - \rho)I_j)$, $0 \leq \rho \leq 1$. Then*

$$\mathbb{E} \left| \det \begin{pmatrix} \langle p_1, c(1) \rangle & \langle p_1, c(i) \rangle \\ \langle p_2, c(1) \rangle & \langle p_2, c(i) \rangle \end{pmatrix} \right| = 1 + \rho^2 \kappa_P^2.$$

Proof. The real-linear map $\phi(\zeta) = \langle p_1, c(\zeta) \rangle + i\langle p_2, c(\zeta) \rangle$ of $\mathbb{C}$ has the form $\phi(\zeta) = A\zeta + B\bar{\zeta}$, and the determinant in question is $|A|^2 - |B|^2$. From $\langle p, c(\zeta) \rangle = \frac{1}{2}(p^*u\zeta + \overline{p^*u}\bar{\zeta} + p^*w\bar{\zeta} + \overline{p^*w}\zeta)$ we get, with $q_\pm = p_1 \mp ip_2$,

$$A = \frac{1}{2}(q_+^*u + \overline{q_-^*w}), \quad B = \frac{1}{2}(q_+^*w + \overline{q_-^*u}).$$

Now $|q_\pm|^2 = 2 \pm 2\operatorname{Im}(p_1^*p_2) = 2(1 \pm \kappa_P)$ and $q_+^*q_- = |p_1|^2 - |p_2|^2 + 2i\langle p_1, p_2 \rangle = 0$. Hence the four variables $q_+^*u, q_-^*u, q_+^*w, q_-^*w$ are independent, with variances $2(1 + \rho)(1 \pm \kappa_P)$ and $2(1 - \rho)(1 \pm \kappa_P)$. So $A \sim \mathcal{CN}(0, 1 + \rho\kappa_P)$ and $B \sim \mathcal{CN}(0, 1 - \rho\kappa_P)$ are independent, and $|A|^2 - |B|^2$ has the law of $(1 + \rho\kappa_P)\varepsilon_1 - (1 - \rho\kappa_P)\varepsilon_2$ with independent exponential variables $\varepsilon_1, \varepsilon_2$ of mean 1. For $\alpha, \beta \geq 0$ the lack of memory of the exponential law gives $\mathbb{E}|\alpha\varepsilon_1 - \beta\varepsilon_2| = (\alpha^2 + \beta^2)/(\alpha + \beta)$, and here $\alpha + \beta = 2$. $\square$

Lemma 5.5. *The function $\tilde{F}_j$ is symmetric, and affine and non-decreasing in each of the variables ρ_a^2 . Consequently $\tilde{F}_j = \sum_{r=0}^j \tilde{c}_{j,r} e_r(\rho_1^2, \dots, \rho_j^2)$ with constants $\tilde{c}_{j,r}$, and*

$$\tilde{F}_j(\underbrace{1, \dots, 1}_l, 0, \dots, 0) = \sum_{r=0}^l \binom{l}{r} \tilde{c}_{j,r} \quad (0 \leq l \leq j). \quad (9)$$

Proof. Permuting the column blocks does not change the absolute value of the determinant, so $\tilde{F}_j$ is symmetric. Almost surely the $2j - 2$ columns of the blocks $a \geq 2$ are linearly independent; let V_1 be the volume of the parallelepiped they span and P the orthogonal complement of their span, a real 2-plane. By “base times height”, $|\det| = V_1 \cdot |\det(\langle p_m, c_1(1) \rangle, \langle p_m, c_1(i) \rangle)_{m=1,2}|$ for an orthonormal basis p_1, p_2 of P . The block c_1 is independent of the others, so by Lemma 5.4

$$\tilde{F}_j(\rho_1, \dots, \rho_j) = \mathbb{E}[V_1(1 + \rho_1^2 \kappa_P^2)] = \mathbb{E}[V_1] + \rho_1^2 \mathbb{E}[V_1 \kappa_P^2],$$

where the two expectations depend on $\rho_2, \dots, \rho_j$ only. A function f on $[0, 1]^j$ that is affine in each variable is of the form $\sum_S c_S \prod_{a \in S} t_a$, the sum over the subsets S of $\{1, \dots, j\}$, with $c_S = \sum_{T \subset S} (-1)^{|S|-|T|} f(1_T)$ (induction on j , using $f(t_1, t') = (1 - t_1)f(0, t') + t_1 f(1, t')$); if f is symmetric, c_S depends only on $|S|$. Finally $e_r(1, \dots, 1, 0, \dots, 0) = \binom{l}{r}$. $\square$

5.5 The closed form

Proposition 5.6. $\tilde{F}_j = F_j$. Consequently $D_b(E) = (2j - 1)!! \Phi_b^{(j)}(E)$, and F_j is strictly increasing in each ρ_a^2 .

Proof. We compute the left side of (9). If $\rho_a = 1$, then $w_a = 0$ and the block is (u_a, iu_a) with $u_a \sim \mathcal{CN}(0, 2I_j)$: it is complex linear. If $\rho_a = 0$, the block is $(u_a + w_a, i(u_a - w_a))$, and $u_a \pm w_a \sim \mathcal{CN}(0, 2I_j)$ are independent; so the two columns are independent standard Gaussian vectors of $\mathbb{R}^{2j}$. Hence $\tilde{F}_j(1, \dots, 1, 0, \dots, 0) = \mathbb{E}|\det[U_{\mathbb{R}} | \Gamma]|$, where $U_{\mathbb{R}}$ is the real $2j \times 2l$ matrix of a complex $j \times l$ matrix $U = \sqrt{2}Z$, Z with independent $\mathcal{CN}(0, 1)$ entries, and Γ is an independent real $2j \times (2j - 2l)$ standard Gaussian matrix. By “base times height” the determinant is the $2l$ -dimensional volume of $U_{\mathbb{R}}$, which is $\det(U^*U) = 2^l \det(Z^*Z)$, times the absolute value of the determinant of the projection of Γ to the orthogonal complement of the image of U , a standard Gaussian square matrix of size $2j - 2l$ independent of U . By Lemma 3.3,

$$\tilde{F}_j(\underbrace{1, \dots, 1}_l, 0, \dots, 0) = 2^l \frac{j!}{(j - l)!} (2j - 2l - 1)!!.$$

The triangular system (9) has a unique solution, and it is $\tilde{c}_{j,r} = c_{j,r}$: indeed $(2r - 1)!! = 2^r (\frac{1}{2})_r$ and $(2j - 2r - 1)!! = (2j - 2l - 1)!! 2^{l-r} (j - l + \frac{1}{2})_{l-r}$ with rising factorials, and the Chu–Vandermonde identity $\sum_r \binom{l}{r} (x)_r (y)_{l-r} = (x + y)_l$ gives

$$\sum_{r=0}^l \binom{l}{r} (2r - 1)!! (2j - 2r - 1)!! = (2j - 2l - 1)!! 2^l (j - l + 1)_l = 2^l \frac{j!}{(j - l)!} (2j - 2l - 1)!!.$$

All $c_{j,r}$ are positive, and $\partial F_j / \partial (\rho_a^2) \geq c_{j,1} > 0$. The formula for D_b is Lemma 5.3. $\square$

Since $(2\pi)^{-j} (2j - 1)!! = 2 / \text{vol}(S^{2j})$, the formula (8) and Proposition 5.6 prove part (1) of Theorem 5.1.

5.6 Bounds, candidate planes and domination

Let E be a tangent $2j$ -plane, $x = x_E$, and $b \in [0, 1]$. Let $|u|_G = G_b(u, u)^{1/2}$ and

$$\omega_b^c(x) = \sqrt{\frac{1 - x}{1 - b^2 x}} \in [0, 1].$$

Lemma 5.7. $\omega_1(E; b) \leq 1$ and $\omega_j(E; b) \leq \omega_b^c(x_E)$.

Proof. Let Ω be the endomorphism of E with $\omega(u, v) = G_b(\Omega u, v)$; in the normal form basis $\Omega u_a = \omega_a v_a$ and $\Omega v_a = -\omega_a u_a$, so $\omega_j |u|_G \leq |\Omega u|_G \leq \omega_1 |u|_G$, and $|\Omega u|_G = \max_{v \neq 0} |\omega(u, v)| / |v|_G$. By Lemma 2.1, $|\omega(u, v)| \leq |u^h| |v^h| \leq |u|_G |v|_G$, which gives $\omega_1 \leq 1$. If $x > 0$, let u_0 be the g -unit vector of E in the direction of $\text{pr}_E \xi$. Then $\eta(u_0) = \sqrt{x}$, $|u_0^h|^2 = 1 - x$, $|u_0|_G^2 = 1 - b^2 x$ and $|\Omega u_0|_G \leq |u_0^h|$, so $\omega_j \sqrt{1 - b^2 x} \leq \sqrt{1 - x}$. If $x = 0$ the second bound is the first. $\square$

Lemma 5.8. (i) If E is of candidate type, then $(\omega_a(E; b))_a = (1, \dots, 1, \omega_b^c(x_E))$ for every $b \in [0, 1]$.

(ii) If V is of type $\mathbb{C}^j \oplus \mathbb{R}$, all tangent planes of S_V are of candidate type, and every value $x \in [0, 1]$ of x_E occurs among them. Conversely, a plane E of candidate type at p is the tangent plane at p of S_V for $V = \mathbb{R}p \oplus E$, which is of type $\mathbb{C}^j \oplus \mathbb{R}$.

(iii) If for some $b \in [0, 1)$ we have $\omega_a(E; b) = 1$ for $a < j$ and $\omega_j(E; b) = \omega_b^c(x_E)$, then E is of candidate type.

(iv) If $j = n$, every $2n$ -dimensional subspace of $T_p S$ is of candidate type.

Proof. (i) Let $E = L \oplus E''$. On the horizontal J -invariant space L we have $G_b = g$ and $\omega = \langle J \cdot, \cdot \rangle$, so the singular values on L are 1; and L is orthogonal to E'' for both G_b and ω , because $\langle l, w \rangle = 0$ and $\langle Jl, w \rangle = 0$ for $l \in L$, $w \in E''$. On E'' , with a g -orthonormal basis e_1, e_2 , the determinant of G_b is $1 - b^2 x$, $x = x_{E''} = x_E$, and $\omega(e_1, e_2)^2 = 1 - x$: if $\xi \in E''$ then $x = 1$ and $\omega(\xi, \cdot) = 0$; otherwise $\omega(\sqrt{x}\xi + \sqrt{1-x}Jh, h) = -\sqrt{1-x}$. So the last singular value is $\omega_b^c(x)$.

(ii) Let $V = C \oplus \mathbb{R}u$ and $p = p_C + tu \in S_V$. By Lemma 2.2, $\text{pr}_{T_p S_V} \xi = Jp_C$ and $x = |p_C|^2 = 1 - t^2$. If $p_C \neq 0$ and $t \neq 0$, let $L = C \cap \{p_C, Jp_C\}^\perp$, a horizontal complex subspace contained in $T_p S_V$, and

$$e'_1 = \frac{Jp_C}{|p_C|}, \quad e'_2 = \frac{tp_C - |p_C|^2 u}{|p_C|}.$$

Then $T_p S_V = L \oplus \text{span}(e'_1, e'_2)$, the unit vector e'_2 is horizontal, and a direct computation gives $e'_1 = \sqrt{x}\xi + tJe'_2$; so the plane is spanned by $\sqrt{x}\xi + \sqrt{1-x}Jh$ and $h = \pm e'_2$. If $t = 0$, then $\xi = Jp \in T_p S_V$ and $T_p S_V = (C \cap \{p, Jp\}^\perp) \oplus \text{span}(\xi, u)$. If $p_C = 0$, then $T_p S_V = C$ is complex and horizontal, $x = 0$.

Conversely, let $E = L \oplus E''$ be of candidate type at p and $V = \mathbb{R}p \oplus E$, so that $T_p S_V = V \cap p^\perp = E$. If $\xi \in E''$, then $E'' = \text{span}(\xi, h')$ with a unit vector $h' \in H_p$ orthogonal to L ; the space $C = L \oplus \text{span}(p, \xi)$ is complex, $V = C \oplus \mathbb{R}h'$, and Jh' is orthogonal to p , ξ , L and h' , hence to V . Otherwise put $c_0 = \sqrt{x}p + \sqrt{1-x}h$ and $u = \sqrt{1-x}p - \sqrt{x}h$. Then $Jc_0 = \sqrt{x}\xi + \sqrt{1-x}Jh \in E''$, so $C = L \oplus \text{span}(c_0, Jc_0)$ is complex, $V = C \oplus \mathbb{R}u$, and $Ju = \sqrt{1-x}\xi - \sqrt{x}Jh$ is orthogonal to p , h , L and Jc_0 , hence to V . In both cases V is of type $\mathbb{C}^j \oplus \mathbb{R}$.

(iii) For $a < j$, $1 = \omega(u_a, v_a) \leq |u_a^h| |v_a^h| \leq |u_a|_G |v_a|_G = 1$. Since $|u|_G^2 - |u^h|^2 = (1 - b^2)\eta(u)^2$ and $b < 1$, this forces $\eta(u_a) = \eta(v_a) = 0$, and then $v_a = Ju_a$ by the equality case of Lemma 2.1(a). So $L = \text{span}(u_a, v_a : a < j)$ is horizontal and complex, and $E'' = \text{span}(u_j, v_j)$ is its g -orthogonal complement in E , with $x_{E''} = x_E = x$. For a g -orthonormal basis e_1, e_2 of E'' , $\omega(e_1, e_2)^2 = \omega_j^2(1 - b^2 x) = 1 - x$. Choose e_1 in the direction of $\text{pr}_{E''} \xi$ (arbitrary if $x = 0$); then $e_1 = \sqrt{x}\xi + e_1^h$, e_2 is horizontal, $\langle e_1^h, e_2 \rangle = 0$ and $|e_1^h|^2 = 1 - x$. If $x = 1$, then $\xi \in E''$. If $x < 1$, $|\langle Je_1^h, e_2 \rangle| = |e_1^h| |e_2|$ forces $e_1^h = \pm \sqrt{1-x}Je_2$, and E'' has the required form with $h = \pm e_2$.

(iv) Let $E = \nu^\perp \subset T_p S$. If $\nu = \pm \xi$, then $E = H_p$ is complex and $x_E = 0$. Otherwise $\nu = \cos \alpha \xi + \sin \alpha h_0$ with a unit vector $h_0 \in H_p$ and $\sin \alpha \neq 0$, and $E = L \oplus \text{span}(\sin \alpha \xi - \cos \alpha h_0, Jh_0)$ with the complex space $L = H_p \cap \{h_0, Jh_0\}^\perp$; the last plane contains ξ if $\cos \alpha = 0$, and otherwise it has the required form with $h = \pm Jh_0$ and $x = \sin^2 \alpha$. $\square$

By (i) and (ii) we may write $\Phi_b^{(j)}(E_x^c)$ for the common value of $\Phi_b^{(j)}$ on the planes of candidate type with $x_E = x$.

Lemma 5.9 (domination). *For every tangent $2j$ -plane E and every $b \in [0, 1)$, with $x = x_E$ and $t = b^2$,*

$$\Phi_b^{(j)}(E) \leq \Phi_b^{(j)}(E_x^c) = \frac{A_j(t)(1 - tx) + t(1 - x)B_j(t)}{(2j - 1)!! \sqrt{1 - tx}},$$

and for $b > 0$ equality holds if and only if E is of candidate type.

Proof. F_j is symmetric and strictly increasing in each ρ_a^2 (Proposition 5.6), and by Lemma 5.7 $b\omega_a(E; b) \leq b$ for $a < j$ and $b\omega_j(E; b) \leq b\omega_b^c(x)$. With Lemma 5.8(i), (iii) this gives the inequality and the equality case. For the value, put $\rho_j^2 = t(1-x)/(1-tx)$; then $e_r(t, \dots, t, \rho_j^2) = \binom{j-1}{r} t^r + \binom{j-1}{r-1} t^{r-1} \rho_j^2$, so $F_j(b, \dots, b, \rho_j) = A_j(t) + \rho_j^2 B_j(t)$. $\square$

5.7 The mixing measure

Lemma 5.10. *For every $j \geq 1$, $A_j = (2j-1)B_j - 2(j-1)tB_j - 2t(1-t)B'_j$.*

Proof. Let $[B_j]_r = c_{j,r+1} \binom{j-1}{r}$ be the coefficient of t^r in B_j (zero for $r = -1$ and $r = j$). The coefficient of t^r on the right is $(2j-1-2r)[B_j]_r - 2(j-r)[B_j]_{r-1}$. Since $(2j-2r-1)c_{j,r+1} = (2r+1)c_{j,r}$, this equals $c_{j,r}[(2r+1)\binom{j-1}{r} - 2(j-r)\binom{j-1}{r-1}] = c_{j,r}\binom{j-1}{r}$, because $r\binom{j-1}{r} = (j-r)\binom{j-1}{r-1}$. $\square$

Lemma 5.11. *Let $0 < \mu < 1$. The measure $\mathbf{m}_j$ is finite and has a continuous density which is positive on $(0, \sqrt{\tau}]$, and for every $x \in [0, 1]$*

$$\int_0^{\sqrt{\tau}} \Phi_b^{(j)}(E_x^c) d\mathbf{m}_j(b) = \sqrt{1-\tau x}.$$

Proof. The polynomial B_j has positive coefficients and $1-b^2 \geq \mu$ on $[0, \sqrt{\tau}]$, which gives the first statement. Put $h_x(b) = b^{2j-1} \sqrt{1-b^2 x} / ((1-b^2)B_j(b^2))$. Logarithmic differentiation, with $t = b^2$, gives

$$h'_x(b) = b^{2j-2} \frac{(1-tx)((2j-1)(1-t)B_j + 2tB_j - 2t(1-t)B'_j) - tx(1-t)B_j}{(1-t)^2 B_j^2 \sqrt{1-tx}}.$$

By Lemma 5.10 the first bracket in the numerator equals $A_j + tB_j$, and $(1-tx)(A_j + tB_j) - tx(1-t)B_j = A_j(1-tx) + t(1-x)B_j$. With Lemma 5.9,

$$h'_x(b) = (2j-1)!! \Phi_b^{(j)}(E_x^c) \frac{b^{2j-2}}{(1-b^2)^2 B_j(b^2)^2}.$$

We have $h_x(0) = 0$ and $h_x(\sqrt{\tau}) = \tau^{j-1/2} \sqrt{1-\tau x} / (\mu B_j(\tau))$. Integrating from 0 to $\sqrt{\tau}$ gives the claim. $\square$

5.8 Proof of Theorem 5.1

Part (1) was proved above. The function $(b, E) \mapsto \Phi_b^{(j)}(E)$ is continuous on $[0, 1)$ times the bundle of tangent $2j$ -planes, because the numbers $e_r(\omega_1^2, \dots, \omega_j^2)$ are, up to sign, coefficients of the characteristic polynomial of the endomorphism Ω of Lemma 5.7. Hence $\Psi^{(j)}$ is continuous.

Part (2). By Lemmas 5.9 and 5.11, $\Psi^{(j)}(E) \leq \int \Phi_b^{(j)}(E_{x_E}^c) d\mathbf{m}_j = \sqrt{1-\tau x_E}$, with equality if E is of candidate type. Conversely, if equality holds, the continuous non-negative function $b \mapsto \Phi_b^{(j)}(E_{x_E}^c) - \Phi_b^{(j)}(E)$ has integral zero with respect to $\mathbf{m}_j$, so it vanishes for some $b > 0$, and E is of candidate type by Lemma 5.9. The total mass is computed below.

Part (3). Let $\mathbf{m} = \int_0^{\sqrt{\tau}} P_b d\mathbf{m}_j(b)$, a finite positive measure on $\mathcal{G}_{2j}$. The density of the $2j$ -tuple a is continuous in (b, a) , so $b \mapsto P_b(Z)$ is measurable for Borel sets Z ; and $\mathbf{m}(Z) = 0$ if and only if $\nu(Z) = 0$, since this is so for each P_b . By part (1) and Tonelli's theorem, $\int \#(N \cap W) d\mathbf{m}(W) = \frac{2}{\text{vol}(S^{2j})} \int_N \Psi^{(j)}(T_p N) d\mathcal{H}^{2j}$. So the hypotheses of Proposition 3.5 hold for $(\mathbf{m}, \Psi^{(j)}, V_0)$, by part (2) and Lemma 5.8(ii). Part (b) of that proposition is the inequality, with the value from Lemma 2.2(b), and part (a) gives $\mathbf{m}_j([0, \sqrt{\tau}]) = \mathbf{m}(\mathcal{G}_{2j}) = M_j(\mu)$.

Part (4). By Proposition 3.5(c), almost every complementary projective subspace meets N exactly once, so $N = \mathbb{RP}(V)$ with $\dim V = 2j+1$ by Lemma 3.7; and $\Psi^{(j)} = J_\mu$ almost everywhere, hence

everywhere, on the tangent planes of S_V . The operator K_V is skew-symmetric on the odd-dimensional space V , so there is a unit vector $p_0 \in V$ with $K_V p_0 = 0$, that is, $Jp_0 \perp V$. The tangent plane $E_0 = V \cap p_0^\perp$ of S_V at p_0 has $x_{E_0} = 0$ and is of candidate type by part (2); a plane of candidate type with $x = 0$ is $L \oplus \text{span}(Jh, h)$, which is J -invariant. Hence $V = E_0 \oplus \mathbb{R}p_0$ with E_0 complex, $p_0 \perp E_0$ and $Jp_0 \perp V$: V is of type $\mathbb{C}^j \oplus \mathbb{R}$. $\square$

5.9 Surfaces, hypersurfaces, and remarks

Remark 5.12 (surfaces, $j = 1$). For a tangent 2-plane E with orthonormal basis e_1, e_2 let $y_E = \omega_1(e_1, e_2)^2$. Then $\omega_1(E; b)^2 = y_E / (1 - b^2 x_E)$, $0 \leq y_E \leq 1 - x_E$, and E is of candidate type if and only if $y_E = 1 - x_E$. The formulas of this section become

$$\Phi_b^{(1)}(E) = \frac{1 - b^2 x_E + b^2 y_E}{\sqrt{1 - b^2 x_E}}, \quad \text{dm}_1(b) = \frac{\mu}{\sqrt{\tau}} \frac{db}{(1 - b^2)^2},$$

$$\Psi^{(1)}(E) = \sqrt{1 - \tau x_E} - (1 - x_E - y_E) \frac{\mu}{\sqrt{\tau}} \int_0^{\sqrt{\tau}} \frac{b^2 db}{\sqrt{1 - b^2 x_E} (1 - b^2)^2},$$

and the minimum in Theorem 5.1(3) is $2\pi M_1(\mu)$, where $M_1(\mu) = \frac{1}{2} + \mu \operatorname{artanh}(\sqrt{\tau}) / (2\sqrt{\tau})$. For $n = 1$ every tangent plane has $y_E = 1 - x_E$. For $j = 1$ the closed form of D_b only needs Lemma 5.4 with $P = \mathbb{C}$.

Remark 5.13 (hypersurfaces, $j = n$). Let $j = n$ and $0 < \mu < 1$. By Lemma 5.8(iv) and Theorem 5.1, $\Psi^{(n)} = J_\mu$ on all tangent hyperplanes, so

$$\text{Vol}_{g_\mu}(N) = \frac{\text{vol}(S^{2n})}{2} \int_0^{\sqrt{\tau}} \mathbb{E}_b \#(N \cap W) \text{dm}_n(b)$$

for every countably $2n$ -rectifiable Borel set $N \subset S$: an exact Crofton formula with great circles. The law P_b has a continuous density with respect to ν (Lemma 3.1), which by the $U(n+1)$ -invariance is a function $\varrho_b(\kappa(W))$. By the injectivity of the cosine transform on even continuous functions, quoted in Remark 4.6, the measure $\int P_b \text{dm}_n(b)$ coincides with the measure $\mu w_\mu(\kappa(W)) d\nu(W)$ of Theorem 4.2, and the same argument applied to each b , with Lemma 4.1 for the matrix $I - b^2 \xi \xi^T$, identifies the density: with $t = b^2$ and $k_t(\kappa) = (1 - t)(1 - t\kappa^2)^{-(n+1)}$,

$$\varrho_b = \frac{(A_n + tB_n)k_t + 2t(1 - t)B_n \partial_t k_t}{(2n - 1)!!}; \quad \text{for } n = 1: \quad \varrho_b(\kappa) = \frac{(1 - t)^2(1 + 3t\kappa^2)}{(1 - t\kappa^2)^3}.$$

The exact formula displayed above does not depend on the cited theorem; the identification of ϱ_b does. We do not use it.

Remark 5.14. (1) *The family of models.* A centred Gaussian law of a pair of functionals on $\mathbb{R}^{2n+2}$ which is invariant under $U(n+1)$ has covariances of the form $\mathbb{E}[a_i a_l^T] = S_{il}I + T_{il}J$ with S symmetric and T skew, because the matrices commuting with $U(n+1)$ are the combinations of I and J . Replacing the pair by a linear combination, which does not change the kernel, one gets $S = I$; so the models with parameter $b \in [0, 1]$ are all the invariant Gaussian models of pairs.

(2) *Where $0 < \mu < 1$ is used.* The measure $\mathbf{m}_j$ lives on $b^2 \in [0, 1 - \mu]$. For $\mu > 1$ the Berger density $\sqrt{1 + (\mu - 1)x}$ increases with x , while each $\Phi_b^{(j)}(E_x^c)$, $b > 0$, decreases with x (differentiate the formula of Lemma 5.9); so no positive mixture of the $\Phi_b^{(j)}$ equals the Berger density on these planes. See also Proposition 6.9.

(3) *The total mass.* The identity $\mathbf{m}_j([0, \sqrt{\tau}]) = M_j(\mu)$ was obtained above from the geometry. An independent verification run found a direct proof: $t^{1/2-j}(1 - t)B_j(t)$ is a solution of the hypergeometric equation satisfied by ${}_2F_1(-\frac{1}{2}, j; j + \frac{1}{2}; t)$, and $\mathbf{m}_j$ is the corresponding Wronskian measure. We omit it.

6 The first stretched case: proof of Theorem III

In this section $n = 2$, $S = S^5 \subset \mathbb{C}^3$, $k = 3$, $\mu > 1$ and $M = \mu - 1$. The horizontal space H_p has real dimension 4.

6.1 Statement

Notation. For a 2-dimensional subspace $F \subset T_p S$ with orthonormal basis f_1, f_2 let $y_F = \omega(f_1, f_2)^2$; then $0 \leq y_F \leq 1 - x_F$ (Remark 5.12). For a tangent 3-plane $E \subset T_p S$ let $P = E^\perp$ be its orthogonal complement in $T_p S$, a 2-plane; then $x_P = 1 - x_E$ and $0 \leq y_P \leq x_E$. We say that E is of *candidate type* if $y_P = 0$. For a 3-dimensional real subspace $W \subset \mathbb{R}^6$ the skew operator K_W has the singular values $\lambda_W, \lambda_W, 0$, and we put $z_W = \lambda_W^2 \in [0, 1]$.

For real $t < 1$ we define, for 2-planes F and 3-planes E in $T_p S$ and for $z \in [0, 1]$,

$$D_t(F) = \frac{1 - tx_F + ty_F}{\sqrt{1 - tx_F}}, \quad \Phi_t(E) = \text{mean of } D_t(E \cap \nu'^\perp) \text{ over the unit vectors } \nu' \in E,$$

$$f_t(z) = \frac{(1 - t)^3 (1 + tz)}{(1 - tz)^4}.$$

For $0 \leq t = b^2 < 1$, D_t is the function $\Phi_b^{(1)}$ of Remark 5.12; here it is used also for $t < 0$. Finally let

$$\sigma_\mu(\vartheta) = \frac{2\sqrt{\vartheta} (8\mu^2 - 4\mu(1 + \vartheta) - (1 + \vartheta)^2)}{3\pi M (1 + \vartheta)^3 \sqrt{M - \vartheta}} \quad (0 < \vartheta < M),$$

$$\Psi_\mu(E) = \int_0^M \Phi_{-\vartheta}(E) \sigma_\mu(\vartheta) d\vartheta, \quad \varphi_\mu(z) = \int_0^M f_{-\vartheta}(z) \sigma_\mu(\vartheta) d\vartheta.$$

Theorem 6.1. *Let $\mu > 1$.*

(1) *With $q = \sqrt{1 + (\mu - 1)z} \in [1, \sqrt{\mu}]$,*

$$\varphi_\mu(z) = \frac{\mu^2 (16q^4 + 48q^3 + 96q^2 + 90q + 30) - \mu (32q^4 + 36q^3 + 12q^2) - 6q^5 - 2q^4}{24q^7 (q + 1)^3}.$$

The function φ_μ is continuous and strictly positive on $[0, 1]$, and $\int_{\mathcal{G}_3} \varphi_\mu(z_W) d\nu(W) = m(\mu)$.

(2) *For every countably 3-rectifiable Borel set $N \subset S^5$,*

$$\int_{\mathcal{G}_3} \varphi_\mu(z_W) \#(N \cap W) d\nu(W) = \frac{2}{\text{vol}(S^3)} \int_N \Psi_\mu(T_p N) d\mathcal{H}^3(p).$$

(3) *For every tangent 3-plane E , $\Psi_\mu(E) \leq \sqrt{1 + (\mu - 1)x_E} = J_\mu(E)$, with equality if and only if E is of candidate type.*

(4) *Every 3-dimensional competitor $N \subset \mathbb{RP}^5$ satisfies $\text{Vol}_{g_\mu}(N) \geq \text{Vol}_{g_\mu}(\mathbb{RP}(V_0)) = v_3 m(\mu)$, where V_0 is of type $\mathbb{C} \oplus \mathbb{R}^2$.*

(5) *A closed C^1 submanifold of the non-zero class, of dimension 3, with $\text{Vol}_{g_\mu}(N) = v_3 m(\mu)$ is an $\mathbb{RP}(V)$ with V of type $\mathbb{C} \oplus \mathbb{R}^2$.*

Parts (4) and (5) are Theorem III. The values at the ends are $\varphi_\mu(0) = (35\mu^2 - 10\mu - 1)/24$ and $\varphi_\mu(1) = 2(\mu + \sqrt{\mu} + 1)/(3\mu^{3/2}(\sqrt{\mu} + 1))$.

6.2 Tangent 3-planes

Lemma 6.2. *Let E be a tangent 3-plane at p and $P = E^\perp$.*

(a) *There are unit vectors $e, e' \in H_p$ with $e' \perp e, Je$, and $g \in [0, \frac{\pi}{2}]$, such that*

$$p_1 = \sqrt{x_P} \xi + \sqrt{1 - x_P} e, \quad p_2 = \cos g Je + \sin g e'$$

is an orthonormal basis of P . Then $y_P = (1 - x_P) \cos^2 g = x_E \cos^2 g$. If $x_P < 1$, $\cos^2 g$ is determined by P ; if $x_P = 1$ it is not, and the formulas below do not depend on it.

(b) *The vectors $\epsilon_1 = \sqrt{1 - x_P} \xi - \sqrt{x_P} e$, $\epsilon_2 = -\sin g Je + \cos g e'$, $\epsilon_3 = Je'$ form an orthonormal basis of E , with $\eta(\epsilon_1) = \sqrt{x_E}$, $\eta(\epsilon_2) = \eta(\epsilon_3) = 0$ and*

$$\omega(\epsilon_2, \epsilon_3) = \cos g, \quad \omega(\epsilon_3, \epsilon_1) = 0, \quad \omega(\epsilon_1, \epsilon_2) = \sqrt{x_P} \sin g.$$

(c) *For every real $t < 1$,*

$$\Phi_t(E) = \int_0^1 \frac{1 - t x_E(1 - s^2) + t(s^2 \cos^2 g + \frac{1}{2} x_P \sin^2 g (1 - s^2))}{\sqrt{1 - t x_E(1 - s^2)}} ds. \quad (10)$$

In particular, if E is of candidate type and $x_E = x$, then $\Phi_t(E) = D(t, x)$, where

$$D(t, x) = \int_0^1 \frac{1 + \frac{t}{2}(1 - 3x)(1 - s^2)}{\sqrt{1 - t x(1 - s^2)}} ds, \quad (11)$$

and for every E , with $h(s) = s^2 - \frac{1}{2} x_P(1 - s^2)$ and $\varpi_t(s) = (1 - t x_E(1 - s^2))^{-1/2}$,

$$\Phi_t(E) - D(t, x_E) = t \cos^2 g \int_0^1 h(s) \varpi_t(s) ds. \quad (12)$$

Proof. (a) If $0 < x_P < 1$, let p_1 be the unit vector of P in the direction of $\text{pr}_P \xi$ and $p_2 \in P$ a unit vector orthogonal to p_1 . Then $\eta(p_1) = \sqrt{x_P}$, $\eta(p_2) = 0$, and $p_1 = \sqrt{x_P} \xi + \sqrt{1 - x_P} e$ with a unit vector $e \in H_p$. As $p_2 \perp p_1$ is horizontal, $\langle p_2, e \rangle = 0$, so p_2 lies in the sum of $\mathbb{R}Je$ and the complex line $H_p \cap \{e, Je\}^\perp$; changing the signs of p_2 and e' we get $0 \leq g \leq \frac{\pi}{2}$. If $x_P = 0$, take for $p_1 = e$ any unit vector of P . If $x_P = 1$, then $p_1 = \xi$, and we take $e' = p_2$, $g = \frac{\pi}{2}$ and any unit vector $e \in H_p \cap \{p_2, Jp_2\}^\perp$. By Lemma 2.1(a), $\omega(p_1, p_2) = \sqrt{1 - x_P} \langle Je, p_2 \rangle = \sqrt{1 - x_P} \cos g$.

(b) (ξ, e, Je, e', Je') is an orthonormal basis of $T_p S$, and the ϵ_i are orthonormal and orthogonal to p_1, p_2 . The values of η are clear. With $Je_2 = \sin g e + \cos g Je'$, $Je_3 = -e'$ and $\epsilon_1^h = -\sqrt{x_P} e$, Lemma 2.1(a) gives the three values of ω .

(c) Let $\nu' = \sum \nu_i \epsilon_i$ be a unit vector of E and $F' = E \cap \nu'^\perp$. Since $\text{pr}_E \xi = \sqrt{x_E} \epsilon_1$, $x_{F'} = x_E(1 - \nu_1^2)$. On the oriented 3-space E the 2-form ω is $\omega(f_1, f_2) = \langle v_\omega, f_1 \times f_2 \rangle$ with $v_\omega = (\cos g, 0, \sqrt{x_P} \sin g)$ in the basis (ϵ_i) , and $f_1 \times f_2 = \pm \nu'$ for an orthonormal basis of F' ; hence $y_{F'} = (\nu_1 \cos g + \nu_3 \sqrt{x_P} \sin g)^2$. For ν' uniformly distributed on the unit sphere of E , the coordinate $\nu_1 = s$ is uniformly distributed on $[-1, 1]$, and given $\nu_1 = s$ the mean of ν_3 is 0 and the mean of ν_3^2 is $\frac{1}{2}(1 - s^2)$. So the conditional mean of $y_{F'}$ is $s^2 \cos^2 g + \frac{1}{2} x_P \sin^2 g (1 - s^2)$, while $x_{F'}$ depends on s only; this gives (10). If $\cos g = 0$, the numerator in (10) is $1 + \frac{t}{2}(1 - s^2)(x_P - 2x_E)$, which is the numerator of (11); if $x_E = 0$, both (10) and (11) equal $1 + \frac{t}{3}$. As $y_P = x_E \cos^2 g$, these are the cases of candidate type. Subtracting the numerators gives (12). $\square$

Lemma 6.3 (domination for $t \leq 0$). *For $t \leq 0$ and every tangent 3-plane E , $\Phi_t(E) \leq D(t, x_E)$, with equality if and only if $t = 0$ or E is of candidate type.*

Proof. For $t \leq 0$ both h and ϖ_t are non-decreasing on $[0, 1]$, so

$$\int_0^1 h \varpi_t - \int_0^1 h \int_0^1 \varpi_t = \frac{1}{2} \int_0^1 \int_0^1 (h(s) - h(s')) (\varpi_t(s) - \varpi_t(s')) ds ds' \geq 0.$$

Since $\int_0^1 h = \frac{1}{3} - \frac{1}{3}x_P = \frac{1}{3}x_E$ and $\varpi_t > 0$, we get $\int_0^1 h \varpi_t \geq \frac{1}{3}x_E \int_0^1 \varpi_t$, which is positive if $x_E > 0$. By (12), $\Phi_t(E) - D(t, x_E) = t \cos^2 g \int_0^1 h \varpi_t \leq 0$, with equality if and only if $t = 0$, or $\cos g = 0$, or $x_E = 0$; the last two cases are $y_P = 0$. $\square$

Lemma 6.4. *A tangent 3-plane E at p is of candidate type if and only if it is the tangent plane at p of a sphere S_V with V of type $\mathbb{C} \oplus \mathbb{R}^2$; then $V = \mathbb{R}p \oplus E$. If V is of type $\mathbb{C} \oplus \mathbb{R}^2$, all tangent planes of S_V are of candidate type.*

Proof. Let $V = \mathbb{C}c \oplus \mathbb{R}u_1 \oplus \mathbb{R}u_2$ be of type $\mathbb{C} \oplus \mathbb{R}^2$, with u_1, u_2 orthonormal; then $V^\perp$ is spanned by Ju_1, Ju_2 . For every $p \in S_V$ the normal plane of $T_p S_V$ in $T_p S$ is $P = V^\perp$, and $\omega(Ju_1, Ju_2) = -\langle u_1, Ju_2 \rangle = 0$ because $Ju_2 \perp V$. So $y_P = 0$. Conversely, let $y_P = 0$ and let n_1, n_2 be an orthonormal basis of P , so that $\langle Jn_1, n_2 \rangle = 0$. Put $V = \mathbb{R}p \oplus E$, the orthogonal complement of P in $\mathbb{R}^6$. The vectors Jn_1, Jn_2 are orthonormal and orthogonal to n_1, n_2 ; so $U = \text{span}(Jn_1, Jn_2) \subset V$ and $JU = P \perp V$. The orthogonal complement of U in V is the orthogonal complement in $\mathbb{R}^6$ of the complex plane spanned by n_1, n_2, Jn_1, Jn_2 , a complex line. Hence V is of type $\mathbb{C} \oplus \mathbb{R}^2$, and $T_p S_V = E$. $\square$

6.3 The Gaussian triple

Lemma 6.5. *Let $0 \leq t = b^2 < 1$, let (a_1, a_2) be a J -correlated pair with parameter b in $\mathbb{R}^6$ and a_3 an independent standard Gaussian vector, and $W = W(a_1, a_2, a_3)$.*

(a) *For every countably 3-rectifiable Borel set $N \subset S^5$, $\mathbb{E} \#(N \cap W) = \frac{2}{\text{vol}(S^3)} \int_N \Phi_t(T_p N) d\mathcal{H}^3(p)$.*

(b) *The law of W has the density $f_t(z_W)$ with respect to ν .*

Proof. (a) We use (1). The vector $(\langle a_i, p \rangle)_{i \leq 3}$ is standard Gaussian in $\mathbb{R}^3$, so its density at 0 is $(2\pi)^{-3/2}$. Conditionally on $\langle a_i, p \rangle = 0$, the pair is described by a circular family $\tilde{\ell}$ as in Lemma 5.2(c), and $\tilde{a}_3$ is an independent standard Gaussian vector of $T_p S$. Let $E = T_p N$, let $\tilde{a}_3^E$ be the projection of $\tilde{a}_3$ to E , a standard Gaussian vector of E , $\nu' = \tilde{a}_3^E / |\tilde{a}_3^E|$ and $F' = E \cap \nu'^\perp$. In an orthonormal basis (f_1, f_2, ν') of E the third row of the matrix $(\langle \tilde{a}_i, \cdot \rangle)$ is $(0, 0, |\tilde{a}_3^E|)$, so the absolute value of the determinant is $|\tilde{a}_3^E|$ times that of the 2×2 determinant of the pair on F' . The norm $|\tilde{a}_3^E|$ is a χ_3 variable independent of ν' , and by Lemma 5.3 and Proposition 5.6 with $j = 1$ the conditional mean of the 2×2 determinant given ν' is $D_t(F')$. Hence the mean is $\mathbb{E}\chi_3 \cdot \Phi_t(E)$, and $(2\pi)^{-3/2} \mathbb{E}\chi_3 = \pi^{-2} = 2/\text{vol}(S^3)$.

(b) *The span R of the pair.* By Lemma 3.1(b) with $m = 2$, $N = 6$, the law of R has a density proportional to $\int \gamma(g O_R) |\det g|^4 dg$, where γ is the density of (a_1, a_2) . By the proof of Lemma 5.2(b), $\gamma(a_1, a_2)$ is a constant times $(1 - b^2)^{-3} \exp(-\frac{1}{2}Q)$ with $Q = (|a_1|^2 + |a_2|^2 - 2b\langle Ja_1, a_2 \rangle)/(1 - b^2)$. If $a = g O_R$, where O_R has the orthonormal rows o_1, o_2 , then $|a_1|^2 + |a_2|^2 = |g|^2$ and $\langle Ja_1, a_2 \rangle = \kappa_0 \det g$ with $\kappa_0 = \langle J o_1, o_2 \rangle$. Write $g = \begin{pmatrix} \alpha + \gamma' & -\beta + \delta \\ \beta + \delta & \alpha - \gamma' \end{pmatrix}$, $r_+ = \alpha^2 + \beta^2$, $r_- = \gamma'^2 + \delta^2$. Then $|g|^2 = 2(r_+ + r_-)$, $\det g = r_+ - r_-$, dg is a constant times $dr_+ dr_-$ times the two angular measures, and $\frac{1}{2}Q = Ar_+ + Br_-$ with $A = (1 - b\kappa_0)/(1 - b^2)$, $B = (1 + b\kappa_0)/(1 - b^2)$. Now

$$\int_0^\infty \int_0^\infty (r_+ - r_-)^4 e^{-Ar_+ - Br_-} dr_+ dr_- = \sum_{l=0}^4 \binom{4}{l} (-1)^l \frac{(4-l)! l!}{A^{5-l} B^{l+1}} = \frac{24(A^5 + B^5)}{(A+B)(AB)^5}.$$

Normalising by the case $b = 0$, where the density is 1, we find that R has the density

$$f_t^{(2)}(\kappa_0) = \frac{A^5 + B^5}{(1 - b^2)^3 (A+B)(AB)^5} = \frac{(1-t)^3}{2} \left[(1 - b\kappa_0)^{-5} + (1 + b\kappa_0)^{-5} \right]$$

with respect to the invariant probability measure of the Grassmannian of 2-planes of $\mathbb{R}^6$.

From R to W . We have $W^\perp = R \oplus \mathbb{R}a_3$, and given R the unit vector of $W^\perp$ orthogonal to R is uniformly distributed on the unit sphere of $R^\perp$. The flags $(R \subset W')$, with $\dim R = 2$ and $\dim W' = 3$, form a homogeneous space of the compact group $O(6)$, which carries a unique invariant probability measure. It can be described in two ways: R is uniformly distributed and $W' = R \oplus \mathbb{R}u$ with u uniform in the unit sphere of $R^\perp$; or W' is uniformly distributed and $R = W' \cap v^\perp$ with v uniform in the unit sphere of W' . The law of the flag $(R, W^\perp)$ has the density $f_t^{(2)}(\kappa(R))$ with respect to this measure. Therefore $W^\perp$ has the density $W' \mapsto \mathbb{E}_v f_t^{(2)}(\kappa(W' \cap v^\perp))$, and, since $W \mapsto W^\perp$ preserves the invariant measures, W has the density given by the same expression evaluated at $W' = W^\perp$.

The mean over v . For $w \in W$ write $Jw = K_W w + L_W w$ with $L_W = P_{W^\perp} J|_W$. Then $K_W^T K_W + L_W^T L_W$ is the identity of W , the same holds for $W^\perp$, and $L_{W^\perp} = -L_W^T$. As $\dim W = \dim W^\perp$, the operators $L_W^T L_W$ and $L_{W^\perp} L_{W^\perp}^T$ have the same eigenvalues, so K_W and $K_{W^\perp}$ have the same singular values $\lambda, \lambda, 0$. A skew endomorphism of an oriented Euclidean 3-space with these singular values is $u \mapsto \lambda \mathbf{n} \times u$ for a unit vector $\mathbf{n}$. So for $R = W^\perp \cap v^\perp$ with orthonormal basis o_1, o_2 , $\langle J o_1, o_2 \rangle = \langle K_{W^\perp} o_1, o_2 \rangle = \pm \lambda \langle \mathbf{n}, v \rangle$, and $\langle \mathbf{n}, v \rangle$ is uniformly distributed on $[-1, 1]$. With $\beta = b\lambda$ the density of W is

$$\frac{(1-t)^3}{2} \int_{-1}^1 \frac{ds}{(1-\beta s)^5} = (1-t)^3 \frac{(1-\beta)^{-4} - (1+\beta)^{-4}}{8\beta} = \frac{(1-t)^3(1+\beta^2)}{(1-\beta^2)^4} = f_t(z_W),$$

because $(1+\beta)^4 - (1-\beta)^4 = 8\beta(1+\beta^2)$ and $\beta^2 = t z_W$. $\square$

For a bounded Borel function ϕ on $[0, 1]$, of any sign, and a tangent 3-plane E at p let

$$\mathcal{A}\phi(E) = \frac{\mathbb{E}_F[\phi(z_F) |\langle E^\perp, F \rangle|]}{\mathbb{E}_F[|\langle E^\perp, F \rangle|]}, \quad z_F := x_F + y_F,$$

where F is a uniformly distributed 2-plane of $T_p S$. If $W \in \mathcal{G}_3$ and $p \in S \cap W$, then $z_W = z_F$ for $F = W \cap p^\perp$: in the orthonormal basis p, f_1, f_2 of W , $z_W = \frac{1}{2}|K_W|^2 = \langle Jp, f_1 \rangle^2 + \langle Jp, f_2 \rangle^2 + \langle Jf_1, f_2 \rangle^2$. So Lemma 3.4 with $G(p, F) = \phi(z_F)$ reads, for $\phi \geq 0$,

$$\int_{\mathcal{G}_3} \phi(z_W) \#(N \cap W) d\nu(W) = \frac{2}{\text{vol}(S^3)} \int_N \mathcal{A}\phi(T_p N) d\mathcal{H}^3(p). \quad (13)$$

Lemma 6.6 (identification and continuation). *For every real $t < 1$ and every tangent 3-plane E , $\mathcal{A}f_t(E) = \Phi_t(E)$.*

Proof. Let first $0 \leq t < 1$. By Lemma 6.5(b) and (13) with $\phi = f_t \geq 0$, and by Lemma 6.5(a),

$$\int_N \mathcal{A}f_t(T_p N) d\mathcal{H}^3 = \frac{\text{vol}(S^3)}{2} \mathbb{E} \#(N \cap W) = \int_N \Phi_t(T_p N) d\mathcal{H}^3$$

for every countably 3-rectifiable Borel set N . Both $\mathcal{A}f_t$ and Φ_t are continuous functions on the bundle of tangent 3-planes. Every tangent 3-plane E at p is the tangent plane at p of the great sphere $S_{\mathbb{R}p \oplus E}$; applying the identity to geodesic balls of this sphere around p and letting the radius tend to 0 gives $\mathcal{A}f_t(E) = \Phi_t(E)$.

Now fix E . Both sides are holomorphic functions of t on $\mathbb{C} \setminus [1, \infty)$. For the left side: $f_t(z)$ is rational in t with its only pole at $t = 1/z \geq 1$, and it is bounded for t in a compact subset of $\mathbb{C} \setminus [1, \infty)$, uniformly in $z \in [0, 1]$. For the right side: $1 - t x_{F'}$ does not lie in $(-\infty, 0]$ for such t , because $x_{F'} \in [0, 1]$, and we use the principal branch of the square root, which is the positive root for real $t < 1$. The two sides agree on $[0, 1)$, hence for all real $t < 1$. $\square$

For $t < 0$ the measure $f_t(z_W) d\nu(W)$ is not the law of the kernel of a Gaussian triple, and for $t < -1$ it is not even positive, since $f_t(1) = (1+t)/(1-t) < 0$.

6.4 The mixing measure and the function φ_μ

Lemma 6.7. *The function σ_μ is positive and integrable on $(0, M)$, and for every $x \in [0, 1]$*

$$\int_0^M D(-\vartheta, x) \sigma_\mu(\vartheta) d\vartheta = \sqrt{1 + (\mu - 1)x}.$$

Proof. For $0 < \vartheta < M$ we have $1 + \vartheta < \mu$, so $8\mu^2 - 4\mu(1 + \vartheta) - (1 + \vartheta)^2 > 3\mu^2$, and σ_μ is positive; it is bounded near 0 and of order $(M - \vartheta)^{-1/2}$ near M . Put $\tilde{\Phi}(t', x) = (1 + \frac{t'}{2}(1 - 3x))/\sqrt{1 - t'x}$, so that $D(t, x) = \int_0^1 \tilde{\Phi}(t(1 - s^2), x) ds$ by (11). The substitution $r = \vartheta(1 - s^2)$ gives $D(-\vartheta, x) = \int_0^\vartheta \tilde{\Phi}(-r, x) dr / (2\sqrt{\vartheta}\sqrt{\vartheta - r})$. As $|\tilde{\Phi}(-r, x)| \leq 1 + r$, Fubini's theorem applies:

$$\int_0^M D(-\vartheta, x) \sigma_\mu(\vartheta) d\vartheta = \int_0^M \tilde{\Phi}(-r, x) \left[\int_r^M \frac{\sigma_\mu(\vartheta) d\vartheta}{2\sqrt{\vartheta}\sqrt{\vartheta - r}} \right] dr.$$

The inner integral equals $\frac{\mu^{3/2}}{M}(1 + r)^{-5/2}$. With $a = 1 + \vartheta$ and $a_0 = 1 + r$ it is

$$\frac{1}{3\pi M} \int_{a_0}^\mu \frac{8\mu^2 - 4\mu a - a^2}{a^3 \sqrt{(\mu - a)(a - a_0)}} da = \frac{8\mu^2 I_3 - 4\mu I_2 - I_1}{3\pi M}, \quad I_l := \int_{a_0}^\mu \frac{da}{a^l \sqrt{(\mu - a)(a - a_0)}}.$$

The substitution $a = 1/u$ turns I_l into $\pi(a_0\mu)^{-1/2}$ times the moment of order $l - 1$ of the arcsine law on $[1/\mu, 1/a_0]$, whose mean is $\frac{1}{2}(\frac{1}{a_0} + \frac{1}{\mu})$ and whose variance is $\frac{1}{8}(\frac{1}{a_0} - \frac{1}{\mu})^2$. Hence

$$I_1 = \frac{\pi}{\sqrt{a_0\mu}}, \quad I_2 = \frac{\pi(a_0 + \mu)}{2(a_0\mu)^{3/2}}, \quad I_3 = \frac{\pi(3a_0^2 + 2a_0\mu + 3\mu^2)}{8(a_0\mu)^{5/2}},$$

and $8\mu^2 I_3 - 4\mu I_2 - I_1 = 3\pi\mu^{3/2}a_0^{-5/2}$.

A derivative. One checks directly that $\frac{d}{dt'} [t'(1 - t')^{-3/2}\sqrt{1 - t'x}] = (1 - t')^{-5/2} \tilde{\Phi}(t', x)$. Therefore

$$\int_0^M \tilde{\Phi}(-r, x) (1 + r)^{-5/2} dr = \left[t'(1 - t')^{-3/2}\sqrt{1 - t'x} \right]_{t'=-M}^0 = M\mu^{-3/2}\sqrt{1 + Mx},$$

and the lemma follows. $\square$

Lemma 6.8. *The formula of Theorem 6.1(1) holds, and $\varphi_\mu > 0$ on $[0, 1]$.*

Proof. Since $f_{-\vartheta}(z) = (1 + \vartheta)^3(1 - \vartheta z)(1 + \vartheta z)^{-4}$,

$$f_{-\vartheta}(z) \sigma_\mu(\vartheta) = \frac{2}{3\pi M} \sqrt{\frac{\vartheta}{M - \vartheta}} \frac{(P_0 - P_1\vartheta - \vartheta^2)(1 - \vartheta z)}{(1 + \vartheta z)^4}, \quad P_0 = 8\mu^2 - 4\mu - 1, \quad P_1 = 4\mu + 2.$$

This is bounded by a constant times $(M - \vartheta)^{-1/2}$, so φ_μ is continuous on $[0, 1]$, and it suffices to prove the formula for $z > 0$. With $y = 1 + \vartheta z$,

$$\frac{(P_0 - P_1\vartheta - \vartheta^2)(1 - \vartheta z)}{(1 + \vartheta z)^4} = P_0 \left(\frac{2}{y^4} - \frac{1}{y^3} \right) + \frac{P_1}{z} \left(\frac{1}{y^2} - \frac{3}{y^3} + \frac{2}{y^4} \right) + \frac{1}{z^2} \left(\frac{1}{y} - \frac{4}{y^2} + \frac{5}{y^3} - \frac{2}{y^4} \right).$$

Let $K_l = \int_0^M \sqrt{\vartheta/(M - \vartheta)} (1 + \vartheta z)^{-l} d\vartheta$. From $\int_0^1 \sqrt{u/(1 - u)} (c + \zeta u)^{-1} du = \frac{\pi}{\zeta} (1 - \sqrt{c/(c + \zeta)})$ (substitute $u = \sin^2 \theta$) and its derivatives with respect to c at $c = 1$, $\zeta = Mz$, we get, with $q = \sqrt{1 + Mz}$,

$$K_1 = \frac{\pi}{z} \left(1 - \frac{1}{q} \right), \quad K_2 = \frac{\pi M}{2q^3}, \quad K_3 = \frac{\pi M(q^2 + 3)}{8q^5}, \quad K_4 = \frac{\pi M(q^4 + 2q^2 + 5)}{16q^7}.$$

Using $z = (q^2 - 1)/M$ one finds

$$\begin{aligned} 2K_4 - K_3 &= \frac{\pi M}{8q^7}(5 - q^2), & \frac{1}{z}(K_2 - 3K_3 + 2K_4) &= \frac{\pi M^2}{8q^7}(2q^2 - 5), \\ \frac{1}{z^2}(K_1 - 4K_2 + 5K_3 - 2K_4) &= \frac{\pi M^3}{8q^7} \cdot \frac{8q^4 + 4q^3 - 12q^2 - 15q - 5}{(q+1)^3}; \end{aligned}$$

for the last one note that $8q^6 + (q+1)(-12q^4 + 13q^2 - 5) = (q-1)^2(8q^4 + 4q^3 - 12q^2 - 15q - 5)$. Since $P_0(5 - q^2) + MP_1(2q^2 - 5) = 20\mu^2 - 10\mu + 5 - 3q^2$, we obtain

$$\varphi_\mu(z) = \frac{1}{12q^7} \left[20\mu^2 - 10\mu + 5 - 3q^2 + (\mu - 1)^2 \frac{8q^4 + 4q^3 - 12q^2 - 15q - 5}{(q+1)^3} \right],$$

and putting this over the denominator $24q^7(q+1)^3$ and collecting the powers of μ gives the formula of the theorem.

Positivity. Write the numerator of that formula as $\text{Num}(\mu, q) = \mu^2 X - \mu Y - Z$ with $X = 16q^4 + 48q^3 + 96q^2 + 90q + 30$, $Y = 32q^4 + 36q^3 + 12q^2$, $Z = 6q^5 + 2q^4$, and regard μ and q as independent variables. For $q \geq 1$ and $\mu \geq 1$, $\partial_\mu \text{Num} = 2\mu X - Y \geq 2X - Y = 60(q+1)^3 > 0$. The range $0 \leq z \leq 1$ is $1 \leq q \leq \sqrt{\mu}$, that is, $\mu \geq q^2$. Hence

$$\text{Num}(\mu, q) \geq \text{Num}(q^2, q) = q^4 X - q^2 Y - Z = 16q^4(q+1)^2(q^2 + q + 1) > 0.$$

The values at $z = 0$ ($q = 1$) and $z = 1$ ($q = \sqrt{\mu}$) follow by substitution. $\square$

6.5 Proof of Theorem 6.1

Part (2). On $[0, M] \times [0, 1]$ we have $|f_{-\vartheta}(z)| \leq \mu^4$, and σ_μ is integrable; so φ_μ is bounded, and by Fubini's theorem and Lemma 6.6

$$\mathcal{A}\varphi_\mu(E) = \int_0^M \mathcal{A}f_{-\vartheta}(E) \sigma_\mu(\vartheta) d\vartheta = \int_0^M \Phi_{-\vartheta}(E) \sigma_\mu(\vartheta) d\vartheta = \Psi_\mu(E).$$

Since $\varphi_\mu \geq 0$ (Lemma 6.8), (13) with $\phi = \varphi_\mu$ is part (2).

Part (3). By Lemma 6.3 with $t = -\vartheta < 0$, the positivity of σ_μ and Lemma 6.7, $\Psi_\mu(E) \leq \int_0^M D(-\vartheta, x_E) \sigma_\mu(\vartheta) d\vartheta = \sqrt{1 + (\mu - 1)x_E}$, with equality if and only if E is of candidate type.

Parts (1) and (4). The measure $\mathbf{m} = \varphi_\mu(z_W) d\nu(W)$ and the function Ψ_μ satisfy the hypotheses of Proposition 3.5 for V_0 of type $\mathbb{C} \oplus \mathbb{R}^2$: (i) is part (2), and (ii), (iii) follow from part (3) and Lemma 6.4. By Lemma 6.8, φ_μ is continuous and positive on $[0, 1]$, so $\mathbf{m}$ and ν have the same null sets. Proposition 3.5(a), (b) and Lemma 2.2(b) give $\int \varphi_\mu(z_W) d\nu = m(\mu)$ and part (4).

Part (5). By Proposition 3.5(c) and Lemma 3.7 (with $m = 5$, $k = 3$), $N = \mathbb{RP}(V)$ with $\dim V = 4$, and $\Psi_\mu = J_\mu$ almost everywhere on the tangent planes of S_V . Their normal planes are all equal to $V^\perp$, so $V^\perp$ has $y = 0$, and by Lemma 6.4 V is of type $\mathbb{C} \oplus \mathbb{R}^2$. $\square$

6.6 Why Gaussian mixtures cannot work, and how the measure was found

As in Remark 5.14(1), a $U(3)$ -invariant centred Gaussian law of a triple of functionals with a density is, after a linear change of the triple, the law of a J -correlated pair with some parameter $b \in [0, 1]$ and an independent standard functional (a real skew 3×3 matrix is conjugate under $O(3)$ to a matrix with a single 2×2 block). By Lemma 6.5 the Crofton densities of these models are the functions Φ_t , $0 \leq t < 1$.

Proposition 6.9. *Let $\mu > 1$. There is no finite positive measure ς on $[0, 1)$ such that $\Psi = \int \Phi_t d\varsigma(t)$ satisfies $\Psi \leq J_\mu$ on all tangent 3-planes with equality on those of candidate type.*

Proof. By (11), for $0 \leq t < 1$ we have $D(t, 0) = 1 + \frac{t}{3} \geq 1$ and $D(t, 1) = \int_0^1 \sqrt{1 - t(1 - s^2)} ds \leq 1$. On planes of candidate type with $x_E = 0$ the Berger density is 1, so equality there gives $\varsigma([0, 1)) \leq \int D(t, 0) d\varsigma = 1$. On planes of candidate type with $x_E = 1$ we then get $\Psi = \int D(t, 1) d\varsigma \leq 1 < \sqrt{\mu} = J_\mu$. $\square$

Remark 6.10. For $0 \leq t < 1$ the inequality of Lemma 6.3 is reversed: $\Phi_t(E) \geq D(t, x_E) + \frac{1}{6}t^2 y_P$. For $t = 0$ both sides are 1. Let $0 < t < 1$ and put $c = tx_E$, so that $\varpi_t(s) = (1 - c + cs^2)^{-1/2}$ depends on c only. For fixed c the integral $\int_0^1 h\varpi_t$ decreases when x_P increases, and $x_P = 1 - c/t \leq 1 - c$. Hence, with $A = \int_0^1 s^2 \varpi_t$ and $B = \int_0^1 \varpi_t$, $\int_0^1 h\varpi_t \geq (1 + \frac{1-c}{2})A - \frac{1-c}{2}B$. Integrating $\frac{d}{ds} [s\sqrt{1 - c + cs^2}] = (1 - c + 2cs^2)\varpi_t$ gives $(1 - c)B + 2cA = 1$, and $A \geq \frac{1}{3}$ because $\varpi_t \geq 1$. So $\int_0^1 h\varpi_t \geq \frac{3+c}{2}A - \frac{1}{2} \geq \frac{c}{6}$, and (12) gives the claim. Thus among the planes with a given x_E the Gaussian models give the *smallest* density to the planes of candidate type.

The measure of Theorem 6.1 was found as follows. By (13), a $U(3)$ -invariant measure $\phi(z_W) d\nu(W)$ has the Crofton density $\mathcal{A}\phi$, and the conditions “ $\mathcal{A}\phi \leq J_\mu$, with equality on the planes of candidate type” are linear in ϕ . A discretised version of this linear programme, with a Monte Carlo approximation of the operator $\mathcal{A}$, had a solution whose total mass agreed with the target $m(\mu)$ within $2 \cdot 10^{-5}$ for $\mu \leq 4$; the numerical solution was smooth and positive. To first order in $\mu - 1$ it was $1 + (\mu - 1)(\frac{5}{2} - \frac{15}{4}z)$, which is the first-order expansion of $(1 + \frac{\mu-1}{4})f_t(z)$ with $t = -\frac{3}{4}(\mu - 1) < 0$. This pointed to the functions f_t with negative t , that is, to the analytic continuation of the Gaussian model of Lemma 6.5. The equation of Lemma 6.7 for the mixing density is an Abel integral equation, and solving it gave σ_μ . The closed form φ_μ agrees with the numerical solution of the linear programme. These computations are described in Section 8; the proof does not use them.

7 Further remarks

7.1 An exact weighted Crofton formula for every dimension

The formula of Theorem 4.2(a) has an analogue for every k , in which the weight depends on the point of the slice and not only on the slice.

Proposition 7.1. *Let $1 \leq k \leq 2n$ and $\mu > 0$. For every countably k -rectifiable Borel set $N \subset S^{2n+1}$,*

$$\text{Vol}_{g_\mu}(N) = \mu^{(2n+2-k)/2} \frac{\text{vol}(S^k)}{2} \int_{\mathcal{G}_k} \sum_{p \in N \cap W} (1 + (\mu - 1)|K_W p|^2)^{-(n+1)} d\nu(W).$$

For $k = 2n$ and $p \in S \cap W$ we have $|K_W p| = \kappa(W)$, and the formula is (4). The proof rests on the following extension of Lemma 4.1, which is the case $m = 1$.

Lemma 7.2. *Let $1 \leq m < d$, let P be an m -dimensional subspace of $\mathbb{R}^d$ and B a positive definite symmetric $d \times d$ matrix. For a uniformly distributed m -dimensional subspace F ,*

$$\mathbb{E}_F[|\langle P, F \rangle| \det(B|_F)^{-(d+1)/2}] = \mathbb{E}_F[|\langle P, F \rangle| \cdot (\det B)^{-m/2} \det(B^{-1}|_P)^{1/2}],$$

where $B|_F$ and $B^{-1}|_P$ denote the restrictions of the quadratic forms.

Proof. Let $\mathbf{p}$ be a $d \times m$ matrix whose columns are an orthonormal basis of P , and compute $I = \int_{\mathbb{R}^{d \times m}} |\det(\mathbf{p}^T a)| \exp(-\frac{1}{2} \text{tr}(a^T B a)) da$ in two ways. With $a = B^{-1/2}y$ we have $da = (\det B)^{-m/2} dy$

and $\mathbf{p}^T B^{-1/2} = \mathbf{s} \mathbf{o}$ with $\mathbf{s} = (\mathbf{p}^T B^{-1} \mathbf{p})^{1/2}$ and $\mathbf{o}$ a matrix with orthonormal rows; as $\mathbf{o} \mathbf{y}$ is a standard Gaussian $m \times m$ matrix, $I = (\det B)^{-m/2} \det(B^{-1}|_P)^{1/2} C_1$ with a constant C_1 depending only on d, m . On the other hand, by Lemma 3.1 (applied to a^T), $a = \mathbf{f} h$ with $\mathbf{f}$ a matrix whose columns are an orthonormal basis of F , $h \in GL(m, \mathbb{R})$ and $da = c |\det h|^{d-m} dh d\nu_m(F)$. Then $|\det(\mathbf{p}^T a)| = |\langle P, F \rangle| |\det h|$ and $\text{tr}(a^T B a) = \text{tr}(h^T B_F h)$ with $B_F = \mathbf{f}^T B \mathbf{f}$, and the substitution $h = B_F^{-1/2} h'$ gives $\int |\det h|^{d-m+1} \exp(-\frac{1}{2} \text{tr}(h^T B_F h)) dh = \det(B_F)^{-(d+1)/2} C_2$. Hence $I = c C_2 \mathbb{E}_F[|\langle P, F \rangle| \det(B|_F)^{-(d+1)/2}]$. Comparing with the case $B = I$ determines the constants. $\square$

Proof of Proposition 7.1. Apply Lemma 3.4 with $m = d - k$ and $G(p, F) = \det(B_p|_F)^{-(n+1)} = (1 + (\mu - 1)x_F)^{-(n+1)}$, where $B_p = I + (\mu - 1)\xi\xi^T$ on $T_p S$. For $W = \mathbb{R}p \oplus F$ we have $x_F = |\text{pr}_F \xi|^2 = |K_W p|^2$, because $\xi \perp p$. By Lemma 7.2, $\det B_p = \mu$, and $\det(B_p^{-1}|_{E^\perp}) = \det(B_p|_E) / \det B_p = J_\mu(E)^2 / \mu$ (complementary minors of the inverse matrix), $\mathcal{A}G(p, E) = \mu^{-(m+1)/2} J_\mu(E)$, and $m + 1 = 2n + 2 - k$. $\square$

A competitor of dimension k meets almost every slice, so Proposition 7.1 bounds its volume from below by an integral of the minimum of the weight over the slice. For $k < 2n$ this bound is not sharp; for $n = 2$, $k = 3$, $\mu = 2$ it is about $0.83 v_3$, while the minimum is $m(2) v_3 \approx 1.22 v_3$. The measures of Theorems II and III are of a different kind: their Crofton densities are strictly smaller than the Berger density away from the candidate planes.

7.2 The complex endpoint and the Poincaré formulas of Tasaki

For $b \rightarrow 1$ the pair $(a, bJa + \sqrt{1 - b^2}a')$ degenerates to a complex linear functional, the slices W become complex subspaces of complex codimension j , the form G_b becomes the horizontal metric, and for a tangent plane E with $x_E < 1$ the numbers $\omega_a(E; 1)$ are the cosines of the multiple Kähler angles θ_a of the image of E under the Hopf projection to $\mathbb{CP}^n$. The limit of the density of Theorem 5.1(1) is

$$\sqrt{1 - x_E} \frac{1}{(2j - 1)!!} \sum_{r=0}^j (2r - 1)!! (2j - 2r - 1)!! e_r(\cos^2 \theta_1, \dots, \cos^2 \theta_j),$$

where $\sqrt{1 - x_E}$ is the Jacobian of the Hopf projection on E . We have not proved the Kac–Rice formula at $b = 1$, where the Gaussian law is degenerate, and we do not use it. The limit should be the density of the Poincaré formula for a real submanifold of dimension $2j$ and a complex submanifold of complex codimension j in $\mathbb{CP}^n$. Such formulas are due to Tasaki [Tas03] (for surfaces, Kang and Tasaki [KT01]; see also [Kan04]); in the form given by Bernig and Fu [BF11, Theorems 1.2 and 3.9], the mean number of intersection points of submanifolds A, B of complementary dimensions $2j$ and $2n - 2j$, $2j \leq n$, is $\sum_{i,l} (T_{2j}^n)_{il} \int_A \sigma_i \int_B \sigma_l$, where σ_i is the i -th elementary symmetric function of the squared cosines of the multiple Kähler angles of the tangent plane, and T_{2j}^n is a symmetric matrix computed in [BF11, Corollary 5.14]. For a complex submanifold B all angles vanish and $\sigma_l = \binom{j}{l}$, so the density on A is $\sum_i (\sum_l (T_{2j}^n)_{il} \binom{j}{l}) \sigma_i$. Our closed form predicts that $\sum_l (T_{2j}^n)_{il} \binom{j}{l}$ is proportional to $(2i - 1)!! (2j - 2i - 1)!!$. For $j = 1$ and $j = 2$ this is immediate from the matrices T_2^n and T_4^n displayed in [BF11]; for $j \leq 8$ and $2j \leq n \leq 2j + 5$ it was checked by exact rational arithmetic (Section 8). We have not looked for a proof for all j , and we do not know whether the coefficients $(2r - 1)!! (2j - 2r - 1)!!$ appear in the literature in this form. The fact that the density is of degree one in each $\cos^2 \theta_a$, which at $b = 1$ is part of Tasaki’s theorem, persists for all b (Lemma 5.5).

So the family of Theorem 5.1 interpolates between the round Crofton formula ($b = 0$) and the complex Poincaré formula ($b = 1$), and the Berger metric with parameter $\mu < 1$ is matched by a mixture over $b^2 \in [0, 1 - \mu]$.

7.3 Limits

$\mu \rightarrow 1$. All three measures tend to the invariant measure ν : $w_\mu \rightarrow 1$; $\mathbf{m}_j$ concentrates at $b = 0$, where $\Phi_0^{(j)} = 1$, with mass $M_j(\mu) \rightarrow 1$; and $\varphi_\mu \rightarrow 1$ uniformly, by the closed form.

$\mu \rightarrow 0$. By Gauss' theorem $M_j(\mu) \rightarrow {}_2F_1(-\frac{1}{2}, j; j + \frac{1}{2}; 1) = (2j-1)!!/(2^j j!)$, so the least volume in Theorems I and II tends to $v_{2j}(2j-1)!!/(2^j j!) = \pi^j/j!$, the volume of $\mathbb{CP}^j$. This is the volume of the image of $\mathbb{RP}(V_0)$ under the Hopf map $\mathbb{RP}^{2n+1} \rightarrow \mathbb{CP}^n$, which does not increase g_μ -volumes and maps $\mathbb{RP}(V_0)$ onto a complex projective subspace of dimension j , injectively outside a null set. Correspondingly $\mathbf{m}_j$ concentrates at $b = 1$.

$\mu \rightarrow \infty$. Here $m(\mu)/\sqrt{\mu} \rightarrow \frac{2}{3}$, and for fixed $z > 0$ the closed form gives $\varphi_\mu(z) \sim 2/(3\mu z^3)$: after division by $\sqrt{\mu}$ the measure of Theorem 6.1 concentrates on the slices with $z_W = 0$, that is, on the totally real 3-dimensional subspaces. Similarly the measure of Theorem 4.2 concentrates on the great circles with $\kappa = 0$, which are the horizontal ones.

7.4 The open cases: a numerical probe

Nothing in this subsection is proved. Let $\mu > 1$, $n \geq 3$ and $n < k < 2n$. The conjectured minimisers are of type $\mathbb{C}^{k-n} \oplus \mathbb{R}^{2n+1-k}$ [Whe26]. In Theorems II and III the slices are kernels of c pairs and $k-2c$ independent functionals, where c is the complex dimension of the candidate, with $t = b^2$ between 0 and $1-\mu$. For $k = n+1$, where the candidate is of type $\mathbb{C} \oplus \mathbb{R}^n$, the analogous family is “one pair with $t \in [1-\mu, 0]$ and $k-2$ independent functionals”; the first case is $k = 4$ in $\mathbb{RP}^7$. The Crofton density of this family on the planes of candidate type was computed, and a non-negative least squares fit on a grid was used to see whether a positive mixture over t reproduces the Berger density there (program `e14_next_case_probe.py`; the formulas behind it were derived at the working stage and have not been verified independently). For $k = 3$ the fit is exact to rounding, as it must be by Lemma 6.7. For $k = 4$ the best positive mixture misses the Berger density by $5 \cdot 10^{-3}$ for $\mu = 1.2$, by $1.5 \cdot 10^{-5}$ for $\mu = 2$ and by $7 \cdot 10^{-7}$ for $\mu = 5$; for $k = 5$ the errors are between $3 \cdot 10^{-4}$ and $5 \cdot 10^{-2}$. Signed mixtures fit exactly. So the proof of Theorem III does not extend to these cases as it stands. For $n \geq 3$ the slices have several Kähler angles, the invariant measures on $\mathcal{G}_k$ are given by densities in several variables, and the continued Gaussian models form only a small family among them. The linear programme of Section 6.6 over all invariant densities, which led to Theorem III, has not been carried out for $\mathbb{RP}^7$.

7.5 Open problems

Problem 7.3. *Let $\mu > 1$, $n \geq 3$ and $n < k < 2n$. Do the projective subspaces of type $\mathbb{C}^{k-n} \oplus \mathbb{R}^{2n+1-k}$ minimise the g_μ -volume in the non-zero class of $H_k(\mathbb{RP}^{2n+1}; \mathbb{Z}_2)$? This is the remaining part of [Whe26, Conjecture 7.1]; the first cases are $k = 4, 5$ in $\mathbb{RP}^7$. Is there a positive $U(n+1)$ -invariant measure on $\mathcal{G}_k$ that proves it in the sense of Proposition 3.5?*

Problem 7.4. *Prove uniqueness among all mass-minimising flat cycles mod 2 in Theorems II and III, that is, carry out Remark 3.10 or find an argument that avoids regularity theory, as Proposition 4.4 does in codimension one.*

Problem 7.5. *Explain the measures of Theorems 5.1 and 6.1 conceptually, for instance within Hermitian integral geometry [BF11]. In particular, is there a direct description of the positive density φ_μ which does not pass through the signed densities f_t , $t < -1$?*

8 Computations and reproducibility

No proof in this note depends on a computation; in particular none depends on floating-point arithmetic. The algebraic identities used in the proofs (Lemma 5.10, the derivatives in Lemmas 5.11

and 6.7, the integrals I_l and K_l , the closed form and the positivity of φ_μ in Lemma 6.8) are proved in the text. The computations described here served to find the statements and to test them. The programs and their outputs are in the folder `reproducibility` of the release package; its README lists the commands. They are written in Python and use numpy, scipy, sympy and mpmath. In the programs the theorems carry the names of the working notes; the README contains the dictionary. Monte Carlo deviations are given in units of the estimated standard error (s.e.).

Identities. `e1_exact_identities.py` checks, exactly or with 25 to 40 digits: Lemma 3.3; the inversion in Proposition 5.6 and Lemma 5.10 for $j \leq 120$; the derivative identity of Lemma 5.11 symbolically for $j \leq 6$; the mixture identity for $j \leq 8$ and the total mass for $j \leq 10$; the formulas of Remark 5.13; and the limits. `e12_phi_closed_form.py` checks symbolically the partial fractions, the closed form of φ_μ , its positivity (by another certificate: after the substitution $q = 1 + p$, $\mu = q^2 + r$ the numerator has non-negative coefficients) and the identities of Lemma 6.7, and it checks the integrals K_l by quadrature; the two certificates used in the proof of Lemma 6.8 are checked by `manuscript_checks.py`; `e11_stretched_exact.py` (part M) checks Lemma 6.7 with 25 digits. `e7_tasaki_check.py` is the computation with the matrices of [BF11, Corollary 5.14] quoted in Section 7; it first reproduces the matrices T_2^n , T_4^n printed there. `manuscript_checks.py` tests the formulas as they are printed in this text (Lemmas 4.1 and 4.3, (7), Lemmas 5.8, 5.10, 5.11, 6.7 and 6.8 and Remarks 5.13 and 6.10, and the inequality of Theorem 6.1(3) on a grid). The results (W1) and (W2) of [Whe26] were checked as well: `verify_cone_exact.py` verifies the mean curvature formula in exact arithmetic for 22 subspaces, through the metric cone, and `verify_volumes.py` the volume formula and the comparison among linear subspaces for $n \leq 4$.

Monte Carlo tests of the densities. `e2_mc_structure.py` tests the closed form of F_j for $j = 2, 3, 4$ (18 parameter vectors, 10^7 samples each, worst deviation 2.2 s.e.), Lemma 5.4 (36 cases) and the formula of Lemma 5.3 with the conditional law computed numerically from the covariance of the unconditioned model (64 cases, worst 2.0 s.e.). `e10_psi_bruteforce.py` computes $\Psi^{(j)}$ from the Gaussian model without any closed form. `e11_stretched_exact.py` tests Lemma 6.5 (law of z_W and density of the triple) and, with $2.4 \cdot 10^9$ uniformly distributed slices, Lemma 6.6 for $t = 0.5, -0.3, -0.8, -1.5, -3$. `e9_stretched_checks.py` tests the pointwise identity behind Proposition 7.1 (48 cases).

The Crofton formulas on non-linear submanifolds. `verify_crofton.py` and `verify_crofton_generic.py` compare both sides of (5) for small spheres, Clifford hypersurfaces and a hypersurface without symmetry in S^3 and S^5 . The latter program was run twice: with $4 \cdot 10^6$ great circles, eight of the ten comparisons are within 1.8 s.e. and the other two, both in S^5 , deviate by 2.8 and 2.2 s.e.; with another seed (hence another hypersurface) and $2 \cdot 10^7$ great circles, all ten comparisons are within 1.3 s.e., with a relative error of at most $3 \cdot 10^{-4}$. `verify_theoremD.py` does the same for Theorem 5.1(1) with $j = 1$ on a torus in S^5 and S^7 , and `e5_endtoend.py` for $j = 2, 3$ on products of spheres in S^5 to S^{11} (in general position, except for the one in S^5), with exact intersection counts (24 comparisons, worst 2.0 s.e.; the dependence on b , up to 24%, is resolved). `e13_theoremF_endtoend.py` (part X) compares both sides of Theorem 6.1(2) on $S^1 \times S^2$ in two general positions in S^5 , for $\mu = 1.3, 2, 5, 20$ ($2 \cdot 10^7$ slices; eight comparisons within 1.0 s.e.).

The pointwise inequalities. `e3_pointwise.py` evaluates $J_\mu - \Psi^{(j)}$ on 4000 random planes for nine pairs (j, n) and seven values of μ : the gap is positive when $n > j$, and zero to rounding when $n = j$ and on planes of candidate type; numerical minimisation of the gap over the Grassmannian ends on planes of candidate type. `e15_theoremF_pointwise.py` evaluates $\Psi_\mu/J_\mu - 1$ on 40401 plane

types for six values of μ between 1.01 and 100: it vanishes to rounding on the candidate types and is negative elsewhere.

Searches for competitors. `verify_perturbation.py`, `e6_perturbation.py` and `e13_theoremF_endtoend.py` (part V) compute the volumes of images of the candidate under random odd polynomial perturbations. No competitor with smaller volume was found in the cases of Theorems I to III; in control runs with the wrong sign of $\mu - 1$ (where the statements are false) competitors with smaller volume were found at once (44 of 120, and 60 of 180).

The linear programme and the probe. `e8_stretched_lp.py` and `e8b_closed_form_search.py` are the linear programmes of Section 6.6 (kernel from $2.4 \cdot 10^9$ slices; the optimal total mass agrees with $m(\mu)$ within $2 \cdot 10^{-5}$ for $\mu \leq 4$; for $\mu = 2$ the numerical density at $z = 0, 0.1, 0.2$ is 4.970, 3.551, 2.613, and φ_μ gives 4.958, 3.547, 2.612). `e14_next_case_probe.py` is the probe of Section 7.4.

The programs of the fifth verification run. They were written from the formulas printed in this text, without the programs above. `r5_exact_I_II.py` and `r5_exact_III.py` check, exactly or with 45 digits: the inversion in Proposition 5.6 and Lemma 5.10 for $j \leq 150$; the hypergeometric equation of Remark 5.14(3) for $j \leq 150$; Lemma 5.11; the constants of (7); the identity of Lemma 6.7 for thirty pairs (μ, x) with μ between 1.001 and 1000 (largest deviation $4 \cdot 10^{-46}$); the closed form of φ_μ , which is derived symbolically from the integral $\int_0^1 \sqrt{u/(1-u)} (c + \zeta u)^{-1} du$ and compared with a quadrature of the defining integral for 48 pairs (μ, z) (relative deviation at most $2 \cdot 10^{-45}$); the two positivity certificates; and $\int \varphi_\mu(z_W) d\nu = m(\mu)$, computed with the Beta law with parameters $(\frac{3}{2}, 1)$ of z_W under ν , which the next program tests. `r5_mc.py` tests the Crofton densities of Theorem I, of Theorem II for $j = 2$ and of Theorem III by Monte Carlo, with $4 \cdot 10^6$ samples per case: pointwise, from Lemma 3.2 alone, where the conditional Gaussian law is computed from the covariance of the unconditioned model; and end to end, on small spheres in S^3 , S^5 and S^7 which are not totally geodesic. In 115 comparisons the largest deviation is 2.7 s.e.

Verification. The proofs are written out completely in the text. They were checked as follows.

1. The statements and proofs were first written as working notes, together with the programs of the working stage described above. For this version the sixteen quick programs listed in `rerun_quick.py` were run again, the last time from the extracted archive of the release package. All checks pass, and the outputs agree with the saved ones up to running times, one warning message and, for one program whose saved output had been produced with another seed, the random planes. The long Monte Carlo programs were not run again.
2. Four independent verification runs, all AI-assisted, examined the working notes before the present text was written. Each run re-derived its part line by line, wrote its own programs, and tried to refute the statements.
 - The first run checked Lemma 3.7 and Theorem I, in the form (5) derived from Santaló's formula. It confirmed both, tested (5) by Monte Carlo on hyperplanes, geodesic spheres, quadrics and graphs of cubics for $n = 1, 2$, computed the second variation of the hyperplanes, and searched for competitors for μ between 10^{-3} and 10^3 . It required the words “without boundary” in the statements, a proof or a restriction of the statement for general cycles, and the presentation of Theorem I as an application of the classical mechanism; and it supplied the argument with sets of finite perimeter on which Proposition 4.4 is based.

- The second run checked the case $j = 1$ of Theorem II and, by an independent exact computation, the mean curvature formula of [Whe26]. It confirmed both, proved Proposition 6.9, and proposed the extension to all even dimensions, including the domination lemma and the mixing measure; a proof of Lemma 5.10 for all j and the remaining steps were then added.
- The third run checked Theorem II and its proof, step by step, and Proposition 3.8. It confirmed them, required corrections of wording and of citations in Remarks 3.9, 3.10 and 5.13, read the statements quoted from [Sim83] in the scan, and found the proof mentioned in Remark 5.14(3).
- The fourth run checked Theorem III and Proposition 7.1. It confirmed them, supplied the proof of Remark 6.10 and direct proofs of the mass identity, and required clarifications in Lemmas 6.2, 6.5 and 6.6, which are incorporated.

No run found an error in a statement. All required changes were made. The programs and outputs of the four runs are in the release package.

3. The present text was written after these runs. The following parts were written at that stage and had not been seen by the four runs: the common framework of Lemmas 3.1 and 3.4 and Proposition 3.5; the derivation of Theorem 4.2(a) from Lemma 3.4 instead of Santaló's formula (the formula itself is the one confirmed by the first run, and it is the case $k = 2n$ of Proposition 7.1, which the fourth run confirmed); Lemma 4.3 and Proposition 4.4 in their present form, including the equality case; the converse statement in Lemma 5.8(ii); and the derivation of the closed form of φ_μ in the proof of Lemma 6.8, which replaces a simplification by computer algebra. The program `manuscript_checks.py` tests the formulas of these parts.

A fifth independent verification run, AI-assisted, then examined the final text, with these parts first. It re-derived the proofs line by line; compared the question, the quoted results of [Whe26] and the statements cited from [GM16, Amb26, AMN25, ÁPB06, TU22, BF11, Sim83, Hat02] with the sources; re-derived the closed form of φ_μ symbolically and confirmed it by quadrature; wrote the programs described above; ran the quick programs of the package from the extracted archive; and repeated the literature search. It found no error in a statement or in a proof, and it confirmed that no proof depends on a computation. It required the following changes, which are made in this text: the second half of part (a) and part (d) of Lemma 3.1, with their proofs, which make explicit a measurability step used in Lemma 3.4 and Propositions 3.5 and 3.8; two more exact sentences in the abstract; three corrections in the description of the computations above (both runs of `verify_crofton_generic.py` are reported; the ranges of `e1_exact_identities.py`; what `e12_phi_closed_form.py` checks); and corrections of wording and of notation. The programs and outputs of this run are in the release package as well.

Scope and priority. This note is a partial answer to the second question of Gil-Medrano in [MP21], record AMR-067-0008 of [ula26]. Theorems I to III determine the minimisers in the regimes of Table 1 within the classes of competitors of Definition 1.1 and Corollary 1.2, with uniqueness among closed C^1 submanifolds of the non-zero class, and for hypersurfaces also among sets of finite perimeter (Proposition 4.4). The statements about arbitrary flat chains mod 2 (Remarks 3.9, 3.10 and 4.5) rely on cited theorems and are not claimed as theorems; Remark 4.7 relies on the cited theorem of [AMN25]. The regime $\mu > 1$, $n < k < 2n$, $n \geq 3$ is open. The first question, and the determination of the minimisers among projective subspaces, are due to Wheeler [Whe26]; that preprint is unrefereed, and our proofs do not depend on it.

Sources. We read the section of [MP21] in the $\text{\TeX}$ source on the web page of P. Pansu; [Whe26] and [GM16] completely; in [Amb26] the classification theorem, the example which identifies a Berger

metric, and the remark on $\mathbb{RP}^3$ and on [ÁPF07]; the correspondence theorem of [AMN25, Section 9.4]; Section 5 of [ÁPB06]; Lemma 1 and Propositions 2 and 5 of [TU22]; Theorems 1.2, 3.9, 3.10 and Corollary 5.14 of [BF11]; the statements of [Sim83] and [Hat02] that we cite; and the abstracts of [Wan26, Mar26] (a verification run read the first part of [Wan26]). For [Amb26, AMN25, ÁPB06, TU22, BF11] we used the arXiv versions (version 9 of [BF11]), whose numbering we follow. We could not read [GS94] and [ÁPF07]: we found no openly accessible copy and have no subscription (the publisher’s page of [ÁPF07] asks for a login). What we say about them is taken from [ÁPB06] and [Amb26]. According to [Amb26], Theorem 8.5 of [ÁPF07] classifies the Finsler metrics on $\mathbb{RP}^n$ whose hyperplanes minimise the Holmes–Thompson area in their homology class by positive measures on the Grassmannian of 2-planes. We therefore cannot exclude that the inequality of Theorem I follows directly from a statement in one of these two papers; Theorem I is presented here as an application of the classical Crofton mechanism (for a Riemannian metric the pointwise density is forced by the injectivity of the cosine transform, and it is positive because ellipsoids are zonoids) with an explicit positive density, and no novelty is claimed for that mechanism. We also could not read [Tas03, KT01, Kan04] (the publishers’ sites refused automated access, and we did not pursue it); what we say about them is taken from [BF11] and from their reviews in zbMATH, which were consulted in a verification run. The works [Ber72, Fom72, BGM10, BBEN10, L  93, How93, San04, Sch14, EK95, Ste22, Fed69, Fle66, All72, AM20] are cited for attribution or for standard facts and were not consulted for this note; in particular the paragraph numbers quoted from [Fed69] and the section quoted from [Sch14] were not checked against the books. The bibliographic data were checked with Crossref and arXiv; Crossref lists no page numbers for [KT01, Kan04], and those of [KT01] are taken from zbMATH.

Priority. On 8 October 2026 we searched arXiv (Berger spheres and projective spaces together with systoles, volume minimisation, Crofton formulas and K  hler angles; homological systoles; the name Gil-Medrano), OpenAlex (the works citing [MP21, GM16, AMN25,   PF07, Whe26]), zbMATH and the web; the first four verification runs made their own searches on the same day, the arXiv queries were repeated when this text was written (8 October 2026, 22:05 UTC), and the arXiv, OpenAlex and zbMATH queries and a web search were repeated in the fifth verification run (9 October 2026, 00:20 UTC). We found no statement of Theorems I to III. The only paper on the question that we found is [Whe26]; only its first version exists, it lists the three cases as open, and OpenAlex lists no work citing it. For Theorem I the mechanism is classical, as explained in Remark 4.6; the result may be known to specialists, or contained in [  PF07]. For Theorems II and III the tools are classical as well (Kac–Rice formulas, Poincar   formulas in complex space forms, analytic continuation), and the route was proposed in [Whe26]. We did not find the family of J -correlated Gaussian slices, the closed form of its Crofton density, or the continued measures of Section 6 in the literature, but our reading of Hermitian integral geometry was limited to [BF11]. A search that finds nothing is not a proof of priority, and we claim none.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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