

Nonflat Metrics with Nonnegative Scalar Curvature That Are Flat at Infinity: A Counterexample to a Conjecture of Gromov

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Abstract

In his list of questions on scalar curvature, Gromov states that a complete Riemannian manifold X with $\text{Sc}(X) \geq 0$ which is isometric at infinity to a complete flat manifold X_{fl} is flat, provided there is a homomorphism $\pi_1(X) \rightarrow \pi_1(X_{\text{fl}})$ compatible with the isometry at infinity. He conjectures that this hypothesis is not needed when $\pi_1(X_{\text{fl}})$ acts on $\mathbb{R}^n$ by parallel translations, and remarks that X can then be simply connected. We show that this conjecture is false in every dimension $n \geq 3$. On $\mathbb{R}^2 \times S^{n-2}$ we write down a complete metric $dr^2 + f(r)^2 d\tau^2 + h(r)^2 g_{S^{n-2}}$, determined by one smooth step function, whose scalar curvature is nonnegative, and positive on an open set, and which outside a compact set is isometric to the complement of a compact set in the flat manifold $S^1 \times \mathbb{R}^{n-1} = \mathbb{R}^n/\mathbb{Z}$. For $n \geq 4$ these manifolds are simply connected and spin. The proof is elementary: the sign of the scalar curvature is read off from a one-line identity. Quotients of products with Euclidean spaces give such examples for all flat ends $T^k \times \mathbb{R}^N$ with $N \geq 2$, and a cut-and-paste gives examples that are isometric at infinity to $T^{m-1} \times \mathbb{R}$, for a class of flat tori T^{m-1} which includes the standard ones. The mechanism, that of Witten’s Kaluza–Klein bubble, is not new: the circle at infinity bounds a disc. Hao, Hu, Liu and Shi refuted the hyperbolic analogue of the conjecture by a gluing construction of this kind, and for $n = 4$ the existence of a metric as above also follows by combining published results: the Lohkamp-type compactification step in the positive mass theorem of Chen, Liu, Shi and Zhu, which does not use their incompressibility hypothesis, applied to the Euclidean Reissner–Nordström metric of negative mass on $\mathbb{R}^2 \times S^2$ (Brill and Horowitz). We claim as new only the explicit elementary construction, which is valid for all $n \geq 3$, including $n = 3$, the simply connected spin examples it gives for every $n \geq 4$, and the observation that Gromov’s conjecture is false. This is an unrefereed note.

1 Introduction

Throughout, manifolds are smooth and connected, and Sc denotes the scalar curvature, normalized so that the unit sphere S^n has $\text{Sc} = n(n-1)$. Two Riemannian manifolds X and Y are *isometric at infinity* if there are compact sets $K \subset X$ and $K' \subset Y$ and an isometry $X \setminus K \rightarrow Y \setminus K'$. This is how Gromov uses the phrase; see, for instance, the statement of Min-Oo’s hyperbolic rigidity theorem in [Gro23].

Section 8 of Gromov’s list of problems on scalar curvature [Gro17] contains the following proposition, under the heading “Semiperiodic metrics”. *Let X be a complete Riemannian manifold with $\text{Sc}(X) \geq 0$ which is isometric at infinity to a complete flat manifold X_{fl} . If there is a homomorphism $\eta: \pi_1(X) \rightarrow \pi_1(X_{\text{fl}})$ that is compatible with the isometry at infinity, then X is flat.* According to two footnotes, η may be taken to be induced by a continuous map $X \rightarrow X_{\text{fl}}$ which is isometric at infinity,

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and the sketched proof, a reduction to the non-existence of metrics of positive scalar curvature on closed manifolds that admit maps of non-zero degree to tori, needs X to be spin or $n \leq 8$.

Gromov then writes that the existence of η may in general be essential, but that if $\Gamma = \pi_1(X_{\text{fl}})$ acts on $\mathbb{R}^n$ by parallel translations, then, conjecturally [Gro17, item [?18]],

“all X with $\text{Sc}(X) \geq 0$ isometric to X_{fl} at infinity are flat.”

He adds that these X can, for instance, be simply connected. The question whether such X must be flat is record AMR-066-0018 of the UnsolvedMath dataset [ula26].

If Γ is a group of translations acting freely and properly discontinuously on $\mathbb{R}^n$, then $\Gamma \cong \mathbb{Z}^k$ is a lattice in a k -dimensional subspace, and $X_{\text{fl}} = \mathbb{R}^n/\Gamma$ is isometric to $T^k \times \mathbb{R}^{n-k}$ for a flat torus T^k . For $k = 0$ the trivial homomorphism is compatible, so this case is covered by Gromov’s proposition; it is the rigidity case of the positive mass theorem [SY79a, Wit81] for metrics that are Euclidean outside a compact set (see Remark 5.4). For $k = n$ the manifold X_{fl} is compact and the hypothesis is empty. So the conjecture concerns the range $1 \leq k \leq n - 1$. A complete simply connected flat manifold is isometric to $\mathbb{R}^n$, and $\mathbb{R}^n$ is not isometric at infinity to $T^k \times \mathbb{R}^{n-k}$ if $k \geq 1$, because only $\mathbb{R}^n$ contains isometric copies of Euclidean balls of every radius outside every compact set. Hence, for simply connected X , the conjecture says that no such X exists.

We show that the conjecture is false for every $n \geq 3$. Let $S^1(L) = \mathbb{R}/L\mathbb{Z}$ be the circle of length L , and let B_ρ and $\bar{B}_\rho$ be the open and the closed ball of radius ρ about the origin of a Euclidean space.

Theorem 1.1. *Let $n \geq 3$ and $m = n - 2$. The manifold $X = \mathbb{R}^2 \times S^m$ carries a smooth complete Riemannian metric g with the following properties, where $r: X \rightarrow [0, \infty)$ is the distance to the origin in the factor $\mathbb{R}^2$, and $a = 1/(4m)$, $b = 1/(2m)$, $\rho_0 = 1 + 1/(8m)$.*

- (a) *$\text{Sc}(g) \geq 0$ on X , and $\text{Sc}(g) > 0$ on the open set $\{a < r < b\}$. In particular, g is not flat.*
- (b) *There are a number F with $a < F < 3/(8m)$ and an isometry $\Phi: X \setminus K \rightarrow X_{\text{fl}} \setminus K_{\text{fl}}$, where $X_{\text{fl}} = S^1(2\pi F) \times \mathbb{R}^{n-1}$, and $K = \{r \leq b\}$ and $K_{\text{fl}} = S^1(2\pi F) \times \bar{B}_{\rho_0}$ are compact. Here $X_{\text{fl}} = \mathbb{R}^n/\Gamma$ for the group $\Gamma = 2\pi F \mathbb{Z}e_1 \cong \mathbb{Z}$ of translations.*
- (c) *For $r_1 > b$ and $\omega_0 \in S^m$, the circle $\{r = r_1\} \times \{\omega_0\} \subset X \setminus K$ bounds a disc in X , while its image under Φ generates $\pi_1(X_{\text{fl}}) \cong \mathbb{Z}$. Hence no homomorphism $\pi_1(X) \rightarrow \pi_1(X_{\text{fl}})$ is compatible with Φ , and no continuous map $X \rightarrow X_{\text{fl}}$ agrees with Φ outside a compact set.*
- (d) *If $n \geq 4$, then X is simply connected and spin. If $n = 3$, then $X = \mathbb{R}^2 \times S^1$ is spin and $\pi_1(X) \cong \mathbb{Z}$.*

For every $\lambda > 0$ the metric $\lambda^2 g$ has the same properties, with $S^1(2\pi\lambda F)$ and $\bar{B}_{\lambda\rho_0}$ in (b).

The metric is $g = dr^2 + f(r)^2 d\tau^2 + h(r)^2 g_{S^m}$ in polar coordinates (r, τ) on $\mathbb{R}^2$, with profile functions f and h that are defined in Section 3 by two quadratures from one smooth step function. It is a product of a flat disc and the unit sphere for $r \leq a$, and it is flat for $r \geq b$. Products with Euclidean spaces, divided by suitable groups of isometries, give all flat tori at infinity (Proposition 5.1), and cutting and regluing such an example gives two ends (Proposition 5.2):

Corollary 1.2. *Let $n \geq 3$, and let $\Gamma \cong \mathbb{Z}^k$ be a discrete group of translations of $\mathbb{R}^n$. Suppose that $1 \leq k \leq n - 2$, or that $k = n - 1$ and the lattice of translation vectors has a basis $w_1, \dots, w_{n-1}$ such that the distance from w_{n-1} to the span of $w_1, \dots, w_{n-2}$ is at least $|w_1|$. Then there is a complete Riemannian n -manifold with $\text{Sc} \geq 0$ which is isometric at infinity to $\mathbb{R}^n/\Gamma$ and is not flat. In particular, the conjecture [Gro17, [?18]] is false in every dimension $n \geq 3$, and for $k = 1$ and $n \geq 4$ it fails for simply connected spin manifolds.*

The condition for $k = n - 1$ holds, for instance, for $\Gamma = \mathbb{Z}^{n-1}$. It does not hold for every lattice: for $n = 3$ it says that the lattice contains a primitive vector w_1 such that $|w_1|^2$ is at most the area of a fundamental parallelogram, and the hexagonal lattice contains no such vector. For arbitrary $\Gamma \cong \mathbb{Z}^{n-1}$ see Remark 5.3, which relies on a cited theorem. The conjecture is true for $n \leq 2$ (Remark 5.4), and for $k = 0$ as noted above. Gromov’s proposition is not affected; our examples show that its hypothesis on π_1 cannot be dropped for translation groups. This bears on the next item [Gro17, [?19]], which asks for the assumptions on $\pi_1(X)$ needed for each X_{fl} .

The mechanism and earlier work. In the manifold of Theorem 1.1 the circle at infinity is contractible. This is the mechanism of Witten’s Kaluza–Klein “bubble of nothing” [Wit82]. The time slices of Witten’s solution are diffeomorphic to $\mathbb{R}^2 \times S^2$, and the solution has zero total energy; so its time-symmetric slice is a complete, scalar-flat, non-flat metric which is asymptotic to the flat metric on $S^1 \times \mathbb{R}^3$ (see the introduction of [HL25]). Such initial data with negative energy were found by Brill and Pfister [BP89], Brill and Horowitz [BH91], and Horowitz and Lu [HL25]. In particular, the Euclidean Reissner–Nordström metrics on $\mathbb{R}^2 \times S^2$ are complete and scalar-flat, are asymptotic to $S^1 \times \mathbb{R}^3$, and have negative mass for suitable parameters [BH91]; this is recalled in [LSZ21, CLSZ25, KW25, AKK26]. These metrics are asymptotically flat, but not flat outside a compact set, and none of the recent papers just cited mentions Gromov’s conjecture.

Accordingly, a positive mass theorem for ends modelled on $S^1 \times \mathbb{R}^{n-1}$ needs an additional hypothesis. Known sufficient ones are: a matching condition for the spin structure at infinity [Min09, Theorem 2]; for $n \leq 7$, that the circle at infinity is not null-homotopic in X [LSZ21, Theorem 1.2], [CLSZ25, Theorem 1.8]; and for $4 \leq n \leq 7$, that a coordinate sphere S^{n-2} of the end is null-homologous in X [KW25, Theorem 1.2]. The theorems of [Min09, LSZ21, CLSZ25, KW25] include the statement that X is flat if the mass is zero; where they assume a rate of decay of the metric at infinity, the assumption holds trivially for ends that are exactly flat. Alaei, Khuri and Kunduri [AKK26] recall that the positive mass theorem fails in general for such ends, and they prove a sharp comparison theorem for the mass of toric 4-manifolds which explains how it fails. Our examples have mass zero and violate each of these hypotheses (Remark 5.6).

For $n = 4$ a counterexample to the conjecture can be derived from published results, although we did not find this conclusion stated anywhere. Chen, Liu, Shi and Zhu consider complete manifolds with $\text{Sc} \geq 0$ that are asymptotic to $\mathbb{R}^d \times T^k$ with $d \geq 3$. In the proof of their positive mass theorem they show that if the mass is negative, then the manifold carries another complete metric with $\text{Sc} \geq 0$, positive somewhere, which is equal to the flat product metric near infinity [CLSZ25, Proposition 4.11 and the first step of the proof of Theorem 1.8]. This is Lohkamp’s reduction [Loh99] of the positive mass theorem, carried over to these ends. It does not use the incompressibility hypothesis of [CLSZ25], which enters only afterwards, when the end is closed up. The same step, for ends $\mathbb{R}^d \times N$ with $d \geq 3$ and N closed and scalar-flat, is in the paper of Dai and Sun [DS23, Section 4]. Applied to a Euclidean Reissner–Nordström metric of negative mass, it gives a complete non-flat metric with $\text{Sc} \geq 0$ on $\mathbb{R}^2 \times S^2$ which is isometric at infinity to $S^1 \times \mathbb{R}^3$. The same reasoning applies for larger n whenever a complete metric with $\text{Sc} \geq 0$ and negative mass that is asymptotic to $S^1 \times \mathbb{R}^{n-1}$ is available, but not for $n = 3$, where $d = 2$.

The hyperbolic counterpart of the conjecture is Gromov’s statement that parabolic quotients $\mathbb{H}^n/\Gamma$ of hyperbolic space admit no nontrivial compactly supported deformations with $\text{Sc} \geq -n(n-1)$, where a deformation may change the topology of a compact set; see [HHLS23, Statement 1.1]. Hao, Hu, Liu and Shi showed that it fails for $\mathbb{H}^n/\mathbb{Z}^{n-2}$ and for $\mathbb{H}^n/\mathbb{Z}^{n-1}$ [HHLS23, Theorem 1.2 and Section 2.4]. They remove a set $D^2 \times T^{n-2}$, glue in $S^1 \times D^2 \times T^{n-3}$ along a boundary where the mean curvatures satisfy a strict inequality, and smooth the corner. So a circle of the torus at infinity is filled by a disc, as in our examples. Theorem 1.1 can be seen as a flat counterpart of their construction, with

an explicit smooth metric in place of the gluing and with spheres in place of tori, and the proof of Proposition 5.2 uses their passage from $\mathbb{Z}^{n-2}$ to $\mathbb{Z}^{n-1}$. They also prove rigidity if the torus stays incompressible [HHLS23, Theorem 1.4]. Their paper does not discuss flat ends or [Gro17, [?18]].

In flat space the corresponding singular picture is the following. Let $l, R > 0$ and $m = n - 2$. Remove $S^1(2\pi l) \times B_R$ from $S^1(2\pi l) \times \mathbb{R}^{m+1}$, and glue in the flat product of a disc of radius l and a round sphere S^m of radius R , along the common boundary $S^1 \times S^m$. With respect to the normal pointing to infinity, this boundary has mean curvature $1/l$ in the new piece and m/R in the old one, so the inequality between the mean curvatures that such a gluing requires is $ml < R$. In the metric of Theorem 1.1 the function f plays the role of l and h that of R , with $h = 1$ at the centre, and the condition $m f h^{m-1} < 1$ used in Section 2 is the smooth form of this inequality. Liu, Shi and Zhu remark that fill-ins of the form $\Sigma_0 \times D^2(l)$ have negative Brown–York mass when l is small [LSZ21, Remark 1.8]; this is the same picture.

Finally, Dahl and Kröncke [DK24] point out that $\mathbb{R}^{n-1} \times S^1$ is scalar curvature rigid: by Minerbe’s theorem [Min09], its flat metric cannot be changed on a compact subset so that $\text{Sc} \geq 0$ everywhere and $\text{Sc} > 0$ somewhere. Our metrics are not deformations of this kind: for $n \geq 4$ the manifold X is not homeomorphic to X_{fl} , and for $n = 3$ the isometry at infinity does not extend to a homeomorphism $X \rightarrow X_{\text{fl}}$, by Theorem 1.1(c). So the change of topology is essential.

What is new. We do not claim the phenomenon as new. We claim as new only the following: the explicit and elementary construction, which is uniform in $n \geq 3$ and includes $n = 3$; the explicit simply connected spin examples that it gives for every $n \geq 4$, which is the situation of Gromov’s remark; and the explicit observation that [Gro17, [?18]] is false. Experts on mass theorems for such ends may well regard the counterexample as folklore.

2 The scalar curvature of a doubly warped product

Lemma 2.1. *Let $m \geq 1$, let $I \subset \mathbb{R}$ be an open interval, and let $f, h: I \rightarrow (0, \infty)$ be smooth. On $M = I \times S^1 \times S^m$, with coordinates $r \in I$ and $\tau \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$, consider the metric*

$$g = dr^2 + f(r)^2 d\tau^2 + h(r)^2 g_{S^m}, \quad (1)$$

where g_{S^m} is the round metric of curvature 1. Let $e_0 = \partial_r$, $e_1 = f^{-1}\partial_\tau$ and $e_i = h^{-1}\bar{e}_i$ for $2 \leq i \leq m+1$, where $(\bar{e}_i)$ is a local orthonormal frame of S^m . Then the planes spanned by e_a and e_b , $a \neq b$, have sectional curvatures

$$K_{01} = -\frac{f''}{f}, \quad K_{0i} = -\frac{h''}{h}, \quad K_{1i} = -\frac{f'h'}{fh}, \quad K_{ij} = \frac{1-h'^2}{h^2}, \quad (2)$$

for $2 \leq i, j \leq m+1$, $i \neq j$, and all components R_{abcd} of the curvature tensor with $\{a, b\} \neq \{c, d\}$ vanish. (For $m = 1$ there is no K_{ij} .) Consequently

$$\text{Sc}(g) = -\frac{2f''}{f} - \frac{2mh''}{h} - \frac{2mf'h'}{fh} + \frac{m(m-1)(1-h'^2)}{h^2}, \quad (3)$$

$$\frac{fh^m}{2} \text{Sc}(g) = -(h^m f')' + fh^{m-2} \left[\frac{m(m-1)}{2} (1-h'^2) - m h h'' \right]. \quad (4)$$

In particular, if $h' = \varphi$ and $h^m f' = 1 - \varphi$ for a smooth function φ on I , then

$$\frac{fh^m}{2} \text{Sc}(g) = (1 - m f h^{m-1}) \varphi' + \frac{m(m-1)}{2} f h^{m-2} (1 - \varphi^2). \quad (5)$$

Proof. Let $(\bar{\theta}^i)$ be the coframe dual to $(\bar{e}_i)$, and let $\bar{\omega}_j^i = -\bar{\omega}_i^j$ be the connection forms of S^m , so that $d\bar{\theta}^i = -\sum_j \bar{\omega}_j^i \wedge \bar{\theta}^j$. Since S^m has curvature 1, we have $d\bar{\omega}_j^i + \sum_l \bar{\omega}_l^i \wedge \bar{\omega}_j^l = \bar{\theta}^i \wedge \bar{\theta}^j$. (For $m = 1$ there are no forms $\bar{\omega}_j^i$.) The coframe $\theta^0 = dr$, $\theta^1 = f d\tau$, $\theta^i = h \bar{\theta}^i$ is orthonormal for g . Put

$$\omega_0^1 = f' d\tau, \quad \omega_0^i = h' \bar{\theta}^i, \quad \omega_1^i = 0, \quad \omega_j^i = \bar{\omega}_j^i,$$

and $\omega_a^b = -\omega_b^a$. Then $d\theta^a = -\sum_b \omega_b^a \wedge \theta^b$ for all a . Indeed, $d\theta^1 = f' dr \wedge d\tau$ and $d\theta^i = h' dr \wedge \bar{\theta}^i - h \sum_j \bar{\omega}_j^i \wedge \bar{\theta}^j$, and for $a = 0$ both sides vanish, because $\sum_b \omega_b^0 \wedge \theta^b = -f f' d\tau \wedge d\tau - h h' \sum_i \bar{\theta}^i \wedge \bar{\theta}^i = 0$. So the ω_b^a are the Levi-Civita connection forms of g . The curvature forms $\Omega_b^a = d\omega_b^a + \sum_c \omega_c^a \wedge \omega_b^c$ are

$$\begin{aligned} \Omega_0^1 &= f'' dr \wedge d\tau = -\frac{f''}{f} \theta^1 \wedge \theta^0, & \Omega_1^i &= \omega_0^i \wedge \omega_1^0 = -\frac{f' h'}{f h} \theta^i \wedge \theta^1, \\ \Omega_0^i &= h'' dr \wedge \bar{\theta}^i = -\frac{h''}{h} \theta^i \wedge \theta^0, & \Omega_j^i &= \bar{\theta}^i \wedge \bar{\theta}^j + \omega_0^i \wedge \omega_j^0 = \frac{1 - h'^2}{h^2} \theta^i \wedge \theta^j. \end{aligned}$$

In the formula for Ω_0^i we used that the remaining terms are $h'(d\bar{\theta}^i + \sum_j \bar{\omega}_j^i \wedge \bar{\theta}^j) = 0$. Thus $\Omega_b^a = K_{ab} \theta^a \wedge \theta^b$ for all $a \neq b$, with K_{ab} as in (2). With the convention just used, in which the unit sphere has curvature forms $\bar{\theta}^i \wedge \bar{\theta}^j$, this says that K_{ab} is the sectional curvature of the plane spanned by e_a and e_b , and that $R_{abcd} = 0$ unless $\{a, b\} = \{c, d\}$. Hence $\text{Sc}(g) = \sum_{a \neq b} K_{ab} = 2K_{01} + 2m K_{0i} + 2m K_{1i} + m(m-1)K_{ij}$, which is (3). Multiplying (3) by $f h^m/2$ and using $(h^m f')' = h^m f'' + m h^{m-1} h' f'$ gives (4). Inserting $h'' = \varphi'$ and $(h^m f')' = -\varphi'$ into (4) gives (5). $\square$

The construction rests on the following observation: if $0 \leq \varphi \leq 1$ and $\varphi' \geq 0$, the right-hand side of (5) is nonnegative as soon as $m f h^{m-1} \leq 1$.

3 The profile functions

Let $\psi(t) = e^{-1/t}$ for $t > 0$ and $\psi(t) = 0$ for $t \leq 0$, and let $\chi(t) = \psi(t)/(\psi(t) + \psi(1-t))$ for $t \in \mathbb{R}$.

Lemma 3.1. *The function χ is smooth, with $0 \leq \chi \leq 1$. Moreover $\chi(t) = 0$ if and only if $t \leq 0$, and $\chi(t) = 1$ if and only if $t \geq 1$; $\chi' > 0$ on $(0, 1)$; and $\chi(t) + \chi(1-t) = 1$ for all t . In particular $\int_0^1 \chi(t) dt = \frac{1}{2}$.*

Proof. The function ψ is smooth, and $\psi(t) + \psi(1-t) > 0$ for all t , since $t > 0$ or $1-t > 0$. So χ is smooth, and the statements about the values 0 and 1 and the identity $\chi(t) + \chi(1-t) = 1$ are clear. For $0 < t < 1$ we have $\chi(t) = (1 + e^{u(t)})^{-1}$ with $u(t) = \frac{1}{t} - \frac{1}{1-t}$, and $u' < 0$; so $\chi' > 0$. Finally $\int_0^1 \chi(t) dt = \int_0^1 \chi(1-t) dt$, and the two integrals add up to 1. $\square$

Fix $m \geq 1$ and numbers $0 < a < b$. For $r \geq 0$ put

$$\varphi(r) = \chi\left(\frac{r-a}{b-a}\right), \quad h(r) = 1 + \int_0^r \varphi(s) ds, \quad f(r) = \int_0^r \frac{1 - \varphi(s)}{h(s)^m} ds. \quad (6)$$

Lemma 3.2. *The functions φ , h and f are smooth on $[0, \infty)$, and $h' = \varphi$, $h^m f' = 1 - \varphi$. Moreover:*

- (i) $\varphi = 0$ on $[0, a]$, $\varphi = 1$ on $[b, \infty)$, $0 < \varphi < 1$ and $\varphi' > 0$ on (a, b) .
- (ii) $h = 1$ on $[0, a]$, h is non-decreasing, $h(b) = 1 + \frac{b-a}{2}$, and $h(r) = r - c$ for $r \geq b$, where $c = \frac{a+b}{2} - 1$.
- (iii) $f(r) = r$ on $[0, a]$, $0 \leq f' \leq 1$, $f' > 0$ on $[0, b)$, and $f = F$ on $[b, \infty)$, where $F = f(b)$ satisfies $a < F < \frac{a+b}{2}$. In particular $0 < f(r) \leq r$ for all $r > 0$.

(iv) $m f h^{m-1} \leq m b \left(1 + \frac{b-a}{2}\right)^{m-1}$ on $[0, b]$. For $a = \frac{1}{4m}$ and $b = \frac{1}{2m}$ the right-hand side equals $\frac{1}{2} \left(1 + \frac{1}{8m}\right)^{m-1}$, and it is less than 0.567.

Proof. (i) follows from Lemma 3.1. So h is smooth with $h' = \varphi \geq 0$ and $h \geq 1$, and then f is smooth with $f' = (1 - \varphi)h^{-m}$.

(ii) Clearly $h = 1$ on $[0, a]$. By Lemma 3.1, $\int_a^b \varphi = (b - a) \int_0^1 \chi = \frac{b-a}{2}$, which gives $h(b)$. For $r \geq b$ we get $h(r) = h(b) + (r - b) = r - c$.

(iii) Since $h \geq 1$ and $0 \leq 1 - \varphi \leq 1$, we have $0 \leq f' \leq 1$; also $f' = 1$ on $[0, a]$, $f' > 0$ on $(0, b)$ because $\varphi < 1$ there, and $f' = 0$ on $[b, \infty)$. Hence $f(r) = r$ on $[0, a]$, $f(r) \leq r$, $f > 0$ on $(0, \infty)$, $f = F$ on $[b, \infty)$ and $F > f(a) = a$. Since $h > 1$ on $(a, b]$, we get $F < a + \int_a^b (1 - \varphi) = \frac{a+b}{2}$.

(iv) On $[0, b]$ we have $f \leq b$ and $h \leq h(b)$ by (ii) and (iii). For $a = \frac{1}{4m}$ and $b = \frac{1}{2m}$ this gives $m f h^{m-1} \leq \frac{1}{2} \left(1 + \frac{1}{8m}\right)^{m-1} \leq \frac{1}{2} e^{(m-1)/(8m)} < \frac{1}{2} e^{1/8}$. Finally $e^{1/8} < 1.134$, because $1.134^2 > 1.2859$, $1.2859^2 > 1.6535$ and $1.6535^2 > 2.734 > e$. $\square$

Remark 3.3. From now on $a = \frac{1}{4m}$ and $b = \frac{1}{2m}$. Then $h(b) = 1 + \frac{1}{8m} = \rho_0$, $c = \frac{3}{8m} - 1 < 0$ and $\frac{1}{4m} < F < \frac{3}{8m}$. The proof of Theorem 1.1 only uses that $m b \left(1 + \frac{b-a}{2}\right)^{m-1} < 1$; any $0 < a < b$ with this property would do, for instance any $0 < a < b < 1$ if $m = 1$. The number F has no closed form; numerically $F = 0.374182 \dots$ for $m = 1$ and $F = 0.187088 \dots$ for $m = 2$.

4 Proof of Theorem 1.1

Let $m = n - 2 \geq 1$ and $X = \mathbb{R}^2 \times S^m$, and let f, h, φ, F, c be as in Section 3, with $a = \frac{1}{4m}$ and $b = \frac{1}{2m}$. Write $r = |p|$ for $(p, \omega) \in X$.

Step 1: the metric. On $U_1 = \{r > 0\}$ we use polar coordinates $p = r e^{i\tau}$ in $\mathbb{R}^2$, so that $U_1 = (0, \infty) \times S^1 \times S^m$, and define g by (1). This is a smooth Riemannian metric on U_1 , since f and h are smooth and positive on $(0, \infty)$. On $U_0 = \{r < a\}$ let $g = g_{\mathbb{R}^2} + g_{S^m}$ be the product of the Euclidean metric and the round metric. On $U_0 \cap U_1$ the two definitions agree, because $f(r) = r$ and $h(r) = 1$ for $r \leq a$ and $g_{\mathbb{R}^2} = dr^2 + r^2 d\tau^2$. So g is a smooth Riemannian metric on $X = U_0 \cup U_1$.

Step 2: completeness. On U_1 the function r is smooth with $|dr|_g = 1$, and $r \geq 0$ vanishes on $X \setminus U_1$. Hence r is 1-Lipschitz for the distance d_g . Indeed, let γ be a piecewise smooth curve from x to y , where $r(y) > r(x)$, and let t_0 be the last parameter with $r(\gamma(t_0)) = r(x)$. After t_0 the curve stays in U_1 , where $|(r \circ \gamma)'| \leq |\gamma'|_g$; so its length is at least $r(y) - r(x)$. Therefore every d_g -bounded set is contained in a compact set $\{r \leq R\}$, and g is complete by the Hopf-Rinow theorem.

Step 3: the scalar curvature. On U_0 the metric is a product of a flat metric and the unit sphere, so $\text{Sc}(g) = m(m-1) \geq 0$. On U_1 we have $h' = \varphi$ and $h^m f' = 1 - \varphi$ by Lemma 3.2, so (5) holds. For $a \leq r \leq b$ we have $1 - m f h^{m-1} > 1 - 0.567 > 0.43$ by Lemma 3.2(iv), and $\varphi' \geq 0$, $0 \leq \varphi \leq 1$, $f > 0$, $h > 0$. So $\text{Sc}(g) \geq 0$ on $\{a \leq r \leq b\}$, and $\text{Sc}(g) > 0$ on $\{a < r < b\}$, where $\varphi' > 0$. For $r \geq b$ we have $\varphi = 1$ and $\varphi' = 0$, so $\text{Sc}(g) = 0$. This proves (a).

Step 4: the end. On $\{r > b\}$ we have $f = F$ and $h = r - c$, so $g = dr^2 + F^2 d\tau^2 + (r - c)^2 g_{S^m}$ there. Let $L = 2\pi F$, and regard S^m as the unit sphere of $\mathbb{R}^{m+1} = \mathbb{R}^{n-1}$. Define $\Phi: \{r > b\} \rightarrow S^1(L) \times \mathbb{R}^{m+1}$ by $\Phi(r, \tau, \omega) = (F\tau \bmod L, (r - c)\omega)$. The map $\tau \mapsto F\tau$ is a diffeomorphism $\mathbb{R}/2\pi\mathbb{Z} \rightarrow \mathbb{R}/L\mathbb{Z}$, and $(r, \omega) \mapsto (r - c)\omega$ is a diffeomorphism of $(b, \infty) \times S^m$ onto $\{y \in \mathbb{R}^{m+1} : |y| > b - c\}$, where $b - c = h(b) = \rho_0$. So Φ is a diffeomorphism of $X \setminus K$ onto $X_{\text{fl}} \setminus K_{\text{fl}}$. In polar coordinates $y = \rho\omega$ the Euclidean metric of $\mathbb{R}^{m+1} \setminus \{0\}$ is $d\rho^2 + \rho^2 g_{S^m}$. Hence the flat metric $d\sigma^2 + |dy|^2$ of X_{fl} pulls back

under Φ , where $\sigma = F\tau$ and $\rho = r - c$, to $F^2 d\tau^2 + dr^2 + (r - c)^2 g_{S^m} = g$. So Φ is an isometry. The sets K and K_{fl} are compact, and the bound on F is Lemma 3.2(iii). This proves (b).

Step 5: topology. Let $r_1 > b$ and $\omega_0 \in S^m$. The loop $\gamma(\tau) = (r_1, \tau, \omega_0)$ bounds the disc $\{r \leq r_1\} \times \{\omega_0\}$ in X , so it is null-homotopic in X . Its image $\Phi \circ \gamma$ is the loop $\tau \mapsto (F\tau, (r_1 - c)\omega_0)$, which winds once around the factor $S^1(L)$ and so generates $\pi_1(X_{\text{fl}}) \cong \mathbb{Z}$. A homomorphism $\eta: \pi_1(X) \rightarrow \pi_1(X_{\text{fl}})$ compatible with Φ would have to send the class of γ , which is trivial, to the class of $\Phi \circ \gamma$, which is not. Likewise, if a continuous map $\Psi: X \rightarrow X_{\text{fl}}$ agreed with Φ outside a compact set, then for large r_1 the loop $\Psi \circ \gamma = \Phi \circ \gamma$ would be null-homotopic in X_{fl} . This proves (c).

For $m \geq 2$ the manifold $\mathbb{R}^2 \times S^m$ is simply connected. For all $m \geq 1$ its tangent bundle is the sum of the pull-back of TS^m and a trivial plane bundle; it is trivial, because the sum of TS^m and a trivial line bundle is trivial. So X is parallelizable, hence spin. This proves (d).

Finally, for $\lambda > 0$ the metric $\lambda^2 g$ is complete with scalar curvature $\lambda^{-2} \text{Sc}(g)$, and Φ followed by $(\sigma, y) \mapsto (\lambda\sigma, \lambda y)$ is an isometry of $(X \setminus K, \lambda^2 g)$ onto $S^1(\lambda L) \times (\mathbb{R}^{n-1} \setminus \bar{B}_{\lambda\rho_0})$. This proves Theorem 1.1.

5 Other flat ends, and remarks

For a lattice $\Lambda \subset \mathbb{R}^k$ let $T_\Lambda = \mathbb{R}^k / \Lambda$ be the corresponding flat torus.

Proposition 5.1. *Let $k \geq 1$ and $N \geq 2$ be integers, $n = k + N$, and let $\Lambda \subset \mathbb{R}^k$ be a lattice. There is a complete Riemannian n -manifold (X_Λ, g_Λ) with $\text{Sc} \geq 0$, and with $\text{Sc} > 0$ on a nonempty open set, which is isometric at infinity to $T_\Lambda \times \mathbb{R}^N$.*

Proof. Let (X, g) be the manifold of Theorem 1.1 for $m = N - 1$, so that $\dim X = N + 1$. Choose a basis $w_1, \dots, w_k$ of Λ , and rescale g so that the circle at infinity has length $L = |w_1|$; we keep the notation g, Φ, K and write $\rho_1 = \rho_0 L / (2\pi F)$ for the radius of the removed ball. Let $V = w_1^\perp \subset \mathbb{R}^k$, and write $w_i = t_i w_1 + u_i$ with $t_i \in \mathbb{R}$ and $u_i \in V$, for $2 \leq i \leq k$. The vectors $u_2, \dots, u_k$ are linearly independent, so they span a lattice Λ_V in V .

For $\alpha \in \mathbb{R}$ let R_α be the rotation $\tau \mapsto \tau + \alpha$ of the factor $\mathbb{R}^2$ of X . It is an isometry of g , because f and h depend only on r . On $X \times V$ with the product metric $g + g_V$, let the i -th generator of $\mathbb{Z}^{k-1}$ act by $(x, v) \mapsto (R_{2\pi t_i} x, v + u_i)$, for $2 \leq i \leq k$. These isometries commute. The action of $\mathbb{Z}^{k-1}$ is free and properly discontinuous, because it covers the action of Λ_V on V by translations. So $X_\Lambda = (X \times V) / \mathbb{Z}^{k-1}$ is a manifold with a complete metric g_Λ , which is locally isometric to $g + g_V$. Hence $\text{Sc}(g_\Lambda) \geq 0$, with strict inequality on the image of $\{a < r < b\} \times V$. The image of $K \times V$ in X_Λ is compact, since it is the image of the compact set $K \times P$, where P is a closed fundamental parallelepiped of Λ_V .

It remains to describe the complement of this compact set. The map $\Phi \times \text{id}_V$ is an isometry of $(X \setminus K) \times V$ onto $S^1(L) \times (\mathbb{R}^N \setminus \bar{B}_{\rho_1}) \times V$. Under Φ , the rotation R_α corresponds to the translation $\sigma \mapsto \sigma + L\alpha / 2\pi$ of $S^1(L)$. Identify $S^1(L) \times V$ with $\mathbb{R}^k / \mathbb{Z}w_1$ by $(\sigma, v) \mapsto \sigma w_1 / L + v$. Then the i -th generator acts on $\mathbb{R}^k / \mathbb{Z}w_1$ as the translation by $t_i w_1 + u_i = w_i$, and trivially on $\mathbb{R}^N \setminus \bar{B}_{\rho_1}$. Therefore the complement is isometric to $(\mathbb{R}^k / \Lambda) \times (\mathbb{R}^N \setminus \bar{B}_{\rho_1})$, which is the complement of a compact set in $T_\Lambda \times \mathbb{R}^N$. $\square$

If $\Lambda = \mathbb{Z}w_1 \oplus \Lambda'$ is an orthogonal sum and $w_2, \dots, w_k \in \Lambda'$, then all $t_i = 0$ and $X_\Lambda = X \times T_{\Lambda'}$. The next proposition treats two ends; its proof is the flat version of [HHLS23, Section 2.4.1].

Proposition 5.2. *Let $n \geq 3$, and let $\tilde{\Lambda} \subset \mathbb{R}^{n-1}$ be a lattice with a basis $w_1, \dots, w_{n-1}$ such that the distance from w_{n-1} to the span H of $w_1, \dots, w_{n-2}$ is at least $|w_1|$. Then there is a complete Riemannian n -manifold with $\text{Sc} \geq 0$, and with $\text{Sc} > 0$ on a nonempty open set, which is isometric at infinity to $T_{\tilde{\Lambda}} \times \mathbb{R}$.*

Proof. Let $\Lambda \subset H \cong \mathbb{R}^{n-2}$ be the lattice spanned by $w_1, \dots, w_{n-2}$, and let $Z = X_\Lambda$ be the manifold constructed in the proof of Proposition 5.1 for $N = 2$ and this basis. So there are a compact set $K_Z \subset Z$ and an isometry Ψ of $Z \setminus K_Z$ onto $T_\Lambda \times (\mathbb{R}^2 \setminus \bar{B}_{\rho_1})$. Here $m = 1$, $\rho_0 = \frac{9}{8}$ and $\rho_1 = \rho_0|w_1|/(2\pi F)$. Since $h \leq \frac{9}{8}$ on $[a, b]$, we have $F \geq a + \frac{8}{9} \int_a^b (1 - \varphi) = \frac{13}{36}$, and so $\rho_1 \leq \frac{81}{52\pi}|w_1| < \frac{1}{2}|w_1|$. Write $w_{n-1} = (s, 2A) \in H \times \mathbb{R}$ with $A > 0$; then $A \geq \frac{1}{2}|w_1| > \rho_1$.

Let (θ, y_1, y_2) be the coordinates on $T_\Lambda \times \mathbb{R}^2$, and fix ε with $0 < \varepsilon < A - \rho_1$. The set $\{|y_1| > A - \varepsilon\}$ lies in $T_\Lambda \times \{|y| > \rho_1\}$, the image of Ψ , where the metric is the flat product metric. Let $Z_A \subset Z$ be the complement of $\Psi^{-1}\{|y_1| > A\}$. It is a connected manifold with the two boundary components $\Psi^{-1}\{y_1 = \pm A\}$. Let M be obtained from Z_A by identifying $\Psi^{-1}(\theta, -A, y_2)$ with $\Psi^{-1}(\theta + s, A, y_2)$. The translation $(\theta, y_1, y_2) \mapsto (\theta + s, y_1 + 2A, y_2)$ is an isometry of the collar $\{|y_1 + A| < \varepsilon\}$ of the first boundary component onto the collar $\{|y_1 - A| < \varepsilon\}$ of the second, and it maps the part of the first collar inside Z_A onto the part of the second collar outside Z_A . So M is a smooth Riemannian manifold which is locally isometric to Z ; hence $\text{Sc} \geq 0$ on M , with $\text{Sc} > 0$ on a nonempty open set.

The part $E = \{|y_2| > \rho_1\}$ of M is isometric to the product of $\mathbb{R} \setminus [-\rho_1, \rho_1]$ with the quotient of $T_\Lambda \times \mathbb{R}$ by the translation $(\theta, y_1) \mapsto (\theta + s, y_1 + 2A)$. This quotient is $T_{\tilde{\Lambda}}$, because Λ and $w_{n-1} = (s, 2A)$ generate $\tilde{\Lambda}$. The complement of E in M is the image of a closed subset of $Z \setminus \Psi^{-1}(T_\Lambda \times \{|y| > \rho_2\})$, where $\rho_2 = (A^2 + \rho_1^2)^{1/2}$; this set is compact, as in the proof of Proposition 5.1. On E the function $|y_2|$ is 1-Lipschitz, so every bounded subset of M is relatively compact, as in Step 2 of Section 4. Therefore M is complete, and it is isometric at infinity to $T_{\tilde{\Lambda}} \times \mathbb{R}$. $\square$

Proof of Corollary 1.2. Let $W \subset \mathbb{R}^n$ be the span of the translation vectors of Γ . They form a lattice Λ in $W \cong \mathbb{R}^k$, and $\mathbb{R}^n/\Gamma$ is isometric to $T_\Lambda \times W^\perp = T_\Lambda \times \mathbb{R}^{n-k}$. If $n - k \geq 2$, Proposition 5.1 applies; for $k = 1$ this is Theorem 1.1, and then X is simply connected and spin if $n \geq 4$. If $n - k = 1$, Proposition 5.2 applies. $\square$

Remark 5.3 (Arbitrary tori for $k = n - 1$). Let $\Lambda \subset \mathbb{R}^{n-1}$ be any lattice, $n \geq 3$. The surface $C = (\mathbb{R}^2, dr^2 + f_1(r)^2 d\tau^2)$ with $f_1(r) = \int_0^r (1 - \varphi)$ has Gauss curvature $\varphi'/f_1 \geq 0$ and is a flat half-cylinder for $r \geq b$. The construction of Proposition 5.1, with C in place of X , gives a complete n -manifold Y with $\text{Sc} \geq 0$, with $\text{Sc} > 0$ on an open set, and with one end, which is isometric to $T_\Lambda \times (0, \infty)$. By the connected sum theorem of Gromov–Lawson [GL80] and Schoen–Yau [SY79b], in the local form in which the metric is changed only in small balls, the connected sum of two copies of Y at points with $\text{Sc} > 0$ carries a complete non-flat metric with $\text{Sc} \geq 0$ which is isometric at infinity to $T_\Lambda \times \mathbb{R}$. By the same theorem, surgery on circles in the region where $\text{Sc} > 0$ makes the manifolds X_Λ of Proposition 5.1 simply connected if $n \geq 4$. We have not written out these gluings, so these two statements rest on the cited theorem.

Remark 5.4 (Where the conjecture holds). For $k = 0$, that is, $X_\Lambda = \mathbb{R}^n$, the statement is the rigidity case of the positive mass theorem for metrics that are Euclidean outside a compact set. It holds if X is spin, by Witten’s argument [Wit81]. Without the spin condition, closing up a large cube around the compact set gives a metric with $\text{Sc} \geq 0$ on a connected sum $T^n \# N$ with N closed. Such a manifold carries no metric of positive scalar curvature, for $n \leq 7$ by Schoen and Yau [SY79b] (see [SY79a] for the positive mass theorem in dimension three) and in all dimensions by [SY19]; by a standard argument, its metrics with $\text{Sc} \geq 0$ are then flat.

For $n = 2$, a complete surface X with Gauss curvature $K \geq 0$ that is isometric at infinity to $S^1(L) \times \mathbb{R}$ or to $\mathbb{R}^2$ is flat. Indeed, cut off the ends along the closed geodesics $S^1(L) \times \{\pm R\}$, respectively along a large round circle. For the remaining compact surface W the Gauss–Bonnet formula gives $\int_W K = 2\pi\chi(W)$, respectively $\int_W K = 2\pi(\chi(W) - 1)$, and both are ≤ 0 .

Remark 5.5 (Dimension three). For $n = 3$ and $k \in \{1, 2\}$, a complete manifold X that is isometric at infinity to $T^k \times \mathbb{R}^{3-k}$ is not simply connected. Indeed, X is homotopy equivalent to a compact

3-manifold W whose boundary consists of one or two tori. If W is orientable and $H_1(W; \mathbb{Q}) = 0$, then $H_2(W, \partial W; \mathbb{Q}) \cong H^1(W; \mathbb{Q}) = 0$ by Lefschetz duality, and the exact sequence of the pair gives $H_1(\partial W; \mathbb{Q}) = 0$, which is absurd. So Gromov's remark on simply connected X concerns $n \geq 4$.

Remark 5.6 (Positive mass theorems). Let (X, g) be as in Theorem 1.1. Its end is isometric to an end of $S^1(L) \times \mathbb{R}^{n-1}$, so for $n \geq 4$ its mass vanishes for each of the definitions used in [Min09, LSZ21, CLSZ25, KW25]; but g is not flat. This is consistent with the theorems listed in the introduction. The circle at infinity is null-homotopic in X . The coordinate sphere $\{p\} \times S^{n-2}$, $|p| > b$, generates $H_{n-2}(X; \mathbb{Z}) \cong \mathbb{Z}$. Finally, every spin structure of X restricts to the disc $\{r \leq r_1\} \times \{\omega_0\}$, and so induces on its boundary circle the spin structure that bounds. The parallel frame of X_{fl} , pulled back by Φ , rotates once along this circle relative to a frame that extends over the disc. So, with respect to the parallel frame, spinors on the end are antiperiodic along the circle (compare [Wit82, HL25]). The flat model has no nonzero parallel spinors for this spin structure, so Witten's argument, on which [Min09] relies, does not apply.

Conversely, the examples show that these hypotheses cannot be dropped from the rigidity statements. For $n = 3$ the end is $S^1(L) \times (\mathbb{R}^2 \setminus \bar{B}_{\rho_0})$, which is asymptotically conical with fibre S^1 and angle 2π at infinity in the sense of [CLSZ25]. By [LSZ21, Theorem 1.2] and [CLSZ25, Theorem 1.10], a complete 3-manifold with $\text{Sc} \geq 0$ and such an end is flat if the fibre circle is not null-homotopic. The three-dimensional example of Theorem 1.1 shows that this hypothesis on π_1 is necessary. The manifolds X_Λ of Proposition 5.1 with $N = 2$ and $n \leq 7$ show the same for the fibres T^{n-2} in [CLSZ25, Theorem 1.10]: the class of w_1 in $\pi_1(T_\Lambda)$ dies in $\pi_1(X_\Lambda)$.

Verification. The proofs above do not depend on computation. The formulas and inequalities were also checked by computer; all programs and their outputs are in the source files.

1. A program by the author, `lead_checks.py`, checks the numerical inequalities in the proofs of Lemma 3.2(iv) and Proposition 5.2 in exact rational arithmetic. For $m \leq 12$ and $m = 25, 100$ it computes h and f by quadrature and checks Lemma 3.2 and that $\text{Sc}(g)$, evaluated from (3), is positive at 399 grid points of (a, b) and agrees there with (5); likewise for 40 random parameters as in Remark 3.3. As a negative control, three parameter choices violating that condition give $\text{Sc} < 0$ somewhere. It also checks symbolically, for $m \leq 3$, that Φ^* of the flat metric is g .
2. Two programs written when the construction was found compare (3) symbolically with the scalar curvature computed from the Christoffel symbols for $m \leq 5$, check (4) and (5) for symbolic m and the flatness of the end metric for $m \leq 3$, and check Lemma 3.2 and $\text{Sc} \geq 0$ numerically for seven values of $m \leq 20$.
3. A first independent verification run, AI-assisted, re-derived Lemma 2.1 and the inequalities, and compared the conjecture with a fresh copy of [Gro17]. Its own programs, with a general curvature routine and S^m in a graph chart, confirm (3) symbolically for $m \leq 3$ and at a rational point for $m = 4$, (5) for symbolic m and for $m \leq 8$, and the flatness of the end for $m \leq 3$, with a negative control. A computation with 25 digits for seven values of $m \leq 50$ finds $\text{Sc}(g)$, evaluated from (3), positive at all 399 grid points. The run reproduced the outputs of the two earlier programs and found no mathematical error. It asked for the account of earlier work in the introduction, including the derivation from [CLSZ25] and [BH91] for $n = 4$, and for Remark 5.3 to be marked as resting on a cited theorem; this has been done. This run examined Theorem 1.1, Proposition 5.1 and the remarks in the form of a working write-up; Proposition 5.2 and the comparison with [HHLS23, DS23] were written later.
4. A second independent verification run, AI-assisted, checked the present text line by line, including Proposition 5.2, and read [Gro17] and the arXiv versions of [HHLS23, DS23, CLSZ25, LSZ21,

KW25, AKK26, DK24, HL25, Min09]. Its own programs use a curvature routine for arbitrary metrics, with S^m in stereographic coordinates. For $m \leq 3$ they confirm (2), the vanishing of all other components of the curvature tensor, (3), (5) with the scalar curvature computed from the Christoffel symbols, the flatness of the end, and that Φ pulls the flat metric back to g ; for $m \leq 2$ they obtain the same scalar curvature in a non-diagonal chart. Interval arithmetic gives rigorous enclosures of h and f : for eight values of $m \leq 100$ the supremum of $m f h^{m-1}$ over $[a, b]$ is at most 0.424, and for $m = 1$ one has $0.37417 < F < 0.37419$, so that $\rho_1 < 0.4786 |w_1|$ in the proof of Proposition 5.2. The programs check the lattice computations of Propositions 5.1 and 5.2 in exact arithmetic, and that the condition of Corollary 1.2 fails for the hexagonal lattice. The run re-ran all earlier programs and found no mathematical error. It found that neither [HHLS23] nor [DS23] contains a nonflat metric with $\text{Sc} \geq 0$ that is flat at infinity, or mentions [Gro17, [?18]]. It asked for a more accurate title, for the paragraph on the singular picture and on [LSZ21, Remark 1.8], for the statement on the case $k = 0$ in Remark 5.4, for two additions to the proof of Proposition 5.2, and for corrections to the record of sources and searches; this has been done.

Scope and priority. Theorem 1.1 and Corollary 1.2 refute [Gro17, [?18]] as stated, for every $n \geq 3$ and every translation group of rank $1 \leq k \leq n-2$, and for the lattices of rank $n-1$ in Proposition 5.2. Arbitrary lattices of rank $n-1$ are treated only in Remark 5.3, which relies on [GL80, SY79b]. The conjecture holds for $n \leq 2$; for $k = 0$ it is the rigidity case of the positive mass theorem (Remark 5.4). The mechanism is not new, the hyperbolic analogue was refuted in [HHLS23], and for $n = 4$ a counterexample follows from [CLSZ25] or [DS23] together with [BH91]; our claims of novelty are limited to those listed in the introduction.

We read [Gro17] in the file cited, and [Gro23, Min09, LSZ21, CLSZ25, DS23, HHLS23, DK24, KW25, AKK26, HL25] in their arXiv versions, to which the theorem numbers refer. The theorem of [Min09] that we cite is the second theorem of its introduction; we follow [LSZ21, KW25, AKK26] in calling it Theorem 2. The published version of [CLSZ25] could not be accessed; only its bibliographic data and its list of references were checked. We did not read [Wit82, BP89, BH91, Loh99] themselves, and rely on their abstracts and on the papers just listed; [GL80, SY79b, SY79a, SY19, Wit81] are quoted in their standard form. We searched arXiv, Crossref, zbMATH, INSPIRE, OpenCitations and Semantic Scholar, and Gromov’s later texts on scalar curvature, for the conjecture, for the phrases “isometric at infinity” and “flat at infinity”, for compactly supported deformations, for Kaluza–Klein bubbles in connection with scalar curvature, and for the works citing [CLSZ25, DS23, HHLS23], most recently in October 2026; we also ran general web searches. We found no source that states that [Gro17, [?18]] is false, or that discusses it, and no explicit example of the kind constructed here. A list of the works citing [Gro17] was not available to us: OpenAlex could not be queried, and MathSciNet and Google Scholar were not searched. This negative search is not a proof of priority.

AI-assisted tools were used to search the problem corpus and the literature, to find the construction and the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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