

An Elementary Obstruction to Stability of Hessian Metrics

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Abstract

The property of admitting a positive definite Hessian metric is not open under small affine deformations of a compact manifold. On the fixed torus $\mathbb{T}^2$, the classical shear family with sole nonzero Christoffel coefficient $\Gamma_{22}^1 = t$ consists of complete, volume-preserving affine connections and converges smoothly to the Euclidean connection. For an arbitrary hypothetical Hessian metric, one component of the Codazzi equation integrates to $t \int_{\mathbb{T}^2} g_{11} = 0$, excluding every $t \neq 0$. Products give the same obstruction in all dimensions at least two. We also give an averaging criterion for translation-invariant affine connections on tori. The affine family is due to the classical theory described by Baues and Goldman; this note supplies a self-contained application to Shimizu’s stability question, without a novelty or priority claim.

1 The stability question

An *affine connection* in this note is a flat torsion-free connection D on the tangent bundle. A positive definite Riemannian metric g is *Hessian* for D if, near every point, $g = D(df)$ for a smooth local potential f . Equivalently,

$$(D_X g)(Y, Z) = (D_Y g)(X, Z) \tag{1}$$

for all vector fields. Necessity follows in affine coordinates by equality of third derivatives; sufficiency follows by applying the Poincaré lemma twice locally to the symmetric tensor coefficients. No global potential is required.

Shimizu’s question in [1, item 5(b), p. 126] asks whether every sufficiently small affine deformation of a compact Hessian manifold again admits a Hessian metric. The same source distinguishes the hyperbolic case: the affine universal cover is convex and contains no complete affine line. Koszul proves openness in that case [2, Theorem 1 and its corollary, pp. 103–104]. We consider the unrestricted question.

Theorem 1. *For every $n \geq 2$, the fixed torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ admits a real-analytic family D^t , $t \in \mathbb{R}$, of complete flat torsion-free connections preserving the standard volume, with the following properties:*

1. $D^t \rightarrow D^0$ in C^∞ as $t \rightarrow 0$;
2. D^0 admits a positive Hessian metric;
3. D^t admits no positive Hessian metric for any $t \neq 0$.

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The developing maps used below are exactly the classical polynomial shear maps displayed in Baues–Goldman [3, pp. 17–18 of the author manuscript]. Invariant averaging in Hessian geometry also appears in Yagi [4, p. 450]. Neither the affine family nor the averaging method is claimed as new. The purpose here is an elementary negative answer to the stated stability assertion, with its positivity and scope made explicit.

2 A shear family and its metric obstruction

Write $U = \partial_u$ and $V = \partial_v$ for the global coordinate vector fields on $\mathbb{T}^2$. Define

$$D_U^t U = D_U^t V = D_V^t U = 0, \quad D_V^t V = tU. \quad (2)$$

These coefficients are well defined on the quotient and symmetric in the lower indices. They therefore define torsion-free connections. In the frame (U, V) their connection matrices are

$$L_U = 0, \quad L_V = \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}.$$

The matrices are constant and commute, so the curvature vanishes. Both traces are zero, which gives $D^t(du \wedge dv) = 0$.

For completeness, the developing map is the global polynomial diffeomorphism

$$F_t(u, v) = (u + tv^2/2, v), \quad F_t^{-1} = F_{-t}. \quad (3)$$

Its pullback of the standard affine connection on $\mathbb{R}^2$ is (2). Conjugating the translation by $(m, n) \in \mathbb{Z}^2$ yields

$$(x, y) \mapsto (x + m + tny + tn^2/2, y + n). \quad (4)$$

These transformations are affine. Their action is free, proper and cocompact because it is conjugate to the usual lattice action. This identifies the family with the construction in [3]. Alternatively, the geodesic with initial point (u_0, v_0) and velocity (a_0, b_0) is

$$u(s) = u_0 + a_0 s - \frac{1}{2} t b_0^2 s^2, \quad v(s) = v_0 + b_0 s,$$

which exists for all real s . Thus D^t is complete for every t .

The difference tensor is

$$D^t - D^0 = t \, dv \otimes dv \otimes U.$$

This proves convergence in every C^k topology on the fixed compact torus. At $t = 0$, the metric $du^2 + dv^2$ is Hessian with local potential $(u^2 + v^2)/2$.

Proof of the metric obstruction. Suppose g is a positive Hessian metric for D^t . Its lift to $\mathbb{R}^2$ has smooth periodic coefficients

$$g = a \, du^2 + 2b \, du \, dv + c \, dv^2, \quad a(u, v) = g(U, U) > 0.$$

Applying (1) to $X = V$, $Y = U$, $Z = V$ gives

$$(D_V^t g)(U, V) = \partial_v b - ta = (D_U^t g)(V, V) = \partial_u c.$$

Integrate over $[0, 1]^2$. Periodicity eliminates both derivative integrals, so

$$t \int_0^1 \int_0^1 a(u, v) \, du \, dv = 0. \quad (5)$$

The integral is strictly positive. Therefore $t = 0$, proving nonexistence for every nonzero parameter. Notice that the metric was not assumed invariant.

For $n > 2$, retain $\Gamma_{22}^1 = t$ and set all other coefficients to zero. The developing map (3) acts on the first two coordinates and leaves the rest unchanged. The same Codazzi component, integrated over $\mathbb{T}^n$, gives $t \int_{\mathbb{T}^n} g_{11} = 0$. This proves Theorem 1. $\square$

Remark 2. Positivity is essential. The indefinite tensor $2 du dv$ is Hessian for every t . In the affine coordinates $(x, y) = F_t(u, v)$ it equals

$$2 dx dy - 2ty dy^2 = \text{Hess}_{x,y}(xy - ty^3/3).$$

Thus the obstruction is not the absence of all nondegenerate Hessian tensors.

3 A criterion for translation-invariant connections

The integral argument has a finite-dimensional counterpart. Let D be a translation-invariant flat torsion-free connection on $\mathbb{R}^n/\Lambda$, with Λ a full lattice. For constant vector fields, write $C(x, y) = D_x y$ and $L_x y = C(x, y)$.

Proposition 3. *The connection D admits a positive Hessian metric if and only if there is a positive inner product h for which every L_x is self-adjoint. Equivalently, there is a real basis $e_1, \dots, e_n$ and real numbers λ_i such that*

$$C(e_i, e_j) = 0 \ (i \neq j), \quad C(e_i, e_i) = \lambda_i e_i. \quad (6)$$

If D is also geodesically complete, these conditions hold if and only if $C = 0$.

Proof. Translations preserve D . Average the pullbacks of a positive Hessian metric over the translation torus with normalized Haar measure. The resulting tensor h is constant and positive. Since the Codazzi equation is linear in the metric and translations commute with D , it still holds for h . This is the usual invariant averaging argument; compare [4, p. 450].

For constant h , expansion of the Codazzi equation and symmetry of C give

$$h(y, C(x, z)) = h(x, C(y, z)).$$

Permuting the variables shows that $h(C(x, y), z)$ is totally symmetric. This is equivalent to self-adjointness of every L_x . Conversely, that self-adjointness and symmetry of C imply the Codazzi equation for h .

Flatness gives $[L_x, L_y] = 0$. The commuting self-adjoint operators therefore have a common h -orthonormal eigenbasis. In this basis $C(e_i, e_j)$ is a multiple of e_j , and by commutativity also a multiple of e_i . For $i \neq j$ it must vanish, yielding (6). Conversely, a product of the form (6) has commuting multiplication operators, and the constant metric declaring this basis orthonormal satisfies the Codazzi equation. The basis need not be a lattice basis: a constant metric descends under every lattice of translations.

If some $\lambda_i \neq 0$, the geodesic starting with velocity e_i has velocity

$$\dot{\gamma}(s) = \frac{1}{1 + \lambda_i s} e_i.$$

It cannot extend through $s = -1/\lambda_i$, since its coefficient in the global translation frame diverges. Completeness thus forces every $\lambda_i = 0$. Conversely, $C = 0$ is the standard complete affine connection with a constant positive Hessian metric. $\square$

In the shear example, $L_V \neq 0$ is nilpotent for $t \neq 0$. A self-adjoint nilpotent operator on a positive inner-product space is zero, so Proposition 3 supplies a second obstruction. The direct integral proof above avoids any invariance reduction.

4 Scope and attribution

The developing domains in Theorem 1 are all $\mathbb{R}^n$ and contain complete affine lines. Hence the example lies outside Koszul’s hyperbolic stability theorem. It also changes the linear holonomy; no conclusion about deformations constrained to fixed linear holonomy or to an integral-affine class is asserted. The absence of a Hessian metric is preserved by affine isomorphism, so it is not an artifact of a marking. Smoothness of the family is proved directly for connections, without relying on smooth coordinates on a coarse moduli space.

The affine geometry and averaging background are classical. This elementary note does not assert a first solution, an exhaustive literature search, or priority over the work of Yagi or Baues–Goldman. The mathematical assertion is the explicit negative answer to the unrestricted question in [1], not a new classification of general compact Hessian manifolds. AI assistance was used in preparing the argument, checking formulas and drafting the text. The note is self-audited and unrefereed; the supplementary symbolic checks are not a formal proof or an independent review.

References

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