

Full-Dimensional One-Sided Trapping for Smooth Area-Preserving Plane Maps

Alper Ferudun*

17 October 2026

Abstract

A point of an invertible plane dynamical system is one-sided trapped if its forward semiorbit is bounded and its backward semiorbit is unbounded. We construct a single smooth area-preserving diffeomorphism of the plane whose trapped set has Hausdorff dimension two. The map has the explicit form $F(x, y) = (y, h(y) - x)$, with h given by a locally finite sum of smooth affine coding blocks. Its trapped set contains compact sets of dimensions $2 - 3/(j + 3)$, for every positive integer j , and their backward orbits escape geometrically through a common channel. Thus the discrete-time maximum is two in every C^m class, including C^∞ . In contrast, for every complete jointly smooth autonomous area-preserving flow on the plane, a nonempty trapped set has dimension one. We give a self-contained critical-value argument for this contrast. Finite-regularity continuous-time maxima are not determined.

1 The two smooth maxima

For a homeomorphism $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, define

$$T(F) = \{z : \{F^n z : n \geq 0\} \text{ is bounded and } \{F^{-n} z : n \geq 0\} \text{ is unbounded}\}.$$

For a complete flow g^t , define $T(g)$ by the same conditions with real $t \geq 0$. Boundedness is required for each forward semiorbit separately; a common compact region for all these orbits is not required.

Plakhov's Problem 1 in [1, p. 417] asks for the maximal Hausdorff dimension of such trapped sets for smooth area-preserving flows and cascades on the plane, and for its dependence on finite smoothness. We separate the discrete and continuous questions.

Theorem 1.1. *There is a C^∞ area-preserving diffeomorphism F of $\mathbb{R}^2$ with $\dim_{\text{H}} T(F) = 2$. More precisely, one such F has compact subsets $E_j \subseteq T(F)$ satisfying*

$$\dim_{\text{H}} E_j = 1 + \frac{j}{j+3} = 2 - \frac{3}{j+3}, \quad j \geq 1.$$

Consequently, the discrete-time maximum is two for every finite C^m class, $m \geq 1$, and for C^∞ .

Theorem 1.2. *Let $g : \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a jointly C^∞ , complete, autonomous area-preserving flow. Then $T(g)$ is empty or $\dim_{\text{H}} T(g) = 1$. The maximum one is attained by the linear saddle.*

The construction in Theorem 1.1 does not impose globally bounded derivatives: its affine slopes tend to infinity in blocks escaping to infinity. It does not give a compactly supported or an analytic example. Theorem 1.2 does not determine the finite-regularity continuous-time

*Mercury Software GmbH. Email: alper@mercurycodelab.com.

function asked for in the source. Accordingly the whole source record, AMR-051-0010 in the frozen `ulamai/UnsolvedMath` v1.6.0 corpus, is not declared closed.

Smooth cutoffs, the contraction principle, separated similarity sets, and the smooth rank-zero critical-value argument are classical. The point of the construction is the simultaneous affine coding, backward escape, and locally finite assembly in one global map. No absolute priority claim is made.

2 An explicit scalar function and global map

Let

$$e(t) = \begin{cases} \exp(-1/t), & t > 0, \\ 0, & t \leq 0, \end{cases} \quad \theta(t) = \frac{e(t)}{e(t) + e(1-t)}, \quad B(t) = 3t - 4t\theta(t).$$

Then θ is smooth, takes values in $[0, 1]$, and equals zero for $t \leq 0$ and one for $t \geq 1$. Thus $B(t) = 3t$ for $t \leq 0$, whereas $B(t) = -t$ for $t \geq 1$.

For each $j \geq 1$, put

$$a_j = 10j, \quad N_j = 2^j, \quad \rho_j = 8N_j, \quad r_j = \rho_j^{-1}, \quad \lambda_j = \rho_j + r_j. \quad (1)$$

Within a fixed block, omit the subscript j , and set

$$b_i = -\frac{1}{2} + \frac{i - \frac{1}{2}}{N} \quad (1 \leq i \leq N), \quad q = \frac{1}{4N}, \quad R = \frac{1}{3N}.$$

Define

$$\chi_i(u) = \theta\left(\frac{R^2 - (u - b_i)^2}{R^2 - q^2}\right).$$

This cutoff equals one if $|u - b_i| \leq q$ and vanishes if $|u - b_i| \geq R$. The supports are disjoint, since $2R < 1/N$, and are contained in $(-1/2, 1/2)$.

Restore the block subscripts and define

$$h(t) = B(t) + \sum_{j=1}^{\infty} \sum_{i=1}^{N_j} \chi_{j,i}(t - a_j)(2a_j + \lambda_j(t - a_j - b_{j,i}) - B(t)). \quad (2)$$

Only finitely many summands meet any compact interval. Hence $h \in C^\infty(\mathbb{R})$, with the exact properties

$$h(t) = 3t \quad (t \leq 0), \quad (3)$$

$$h(a + u) = 2a + \lambda(u - b_i) \quad (|u - b_i| \leq q), \quad (4)$$

$$h(a + u) = -(a + u) \quad (u \in J = [3/4, 7/8]). \quad (5)$$

For the last identity, $a + u > 1$ and no bump support meets $a + J$.

Now put

$$F(x, y) = (y, h(y) - x). \quad (6)$$

The inverse is $F^{-1}(x, y) = (h(x) - y, x)$, and

$$DF(x, y) = \begin{pmatrix} 0 & 1 \\ -1 & h'(y) \end{pmatrix}, \quad \det DF = 1.$$

Thus F is a global smooth diffeomorphism preserving Lebesgue area. Every positive and negative integer iterate is defined.

3 Coded forward orbits and backward escape

Fix one block and $u_0 \in J$. For an arbitrary infinite code $\sigma = (\sigma_1, \sigma_2, \dots) \in \{1, \dots, N\}^{\mathbb{N}}$, consider the recurrence

$$u_n = b_{\sigma_n} + \frac{u_{n-1} + u_{n+1}}{\lambda}, \quad n \geq 1, \quad (7)$$

with the prescribed value u_0 .

Lemma 3.1. *For every code and $u_0 \in J$, there is a unique solution of (7) in the closed unit ball of $\ell^\infty(\mathbb{N})$. It satisfies*

$$|u_n - b_{\sigma_n}| \leq \frac{2}{\lambda} < q \quad (n \geq 1)$$

and

$$F^n(a + u_0, a + u_1) = (a + u_n, a + u_{n+1}) \quad (n \geq 0).$$

In particular its forward orbit is bounded.

Proof. Use the right-hand side of (7) to define an operator on the closed unit ball, substituting the fixed u_0 when $n = 1$. Its norm is at most $1/2 + 2/\lambda < 1$, and its Lipschitz constant is at most $2/\lambda < 1$. The contraction theorem gives the fixed point. Since $\lambda = 8N + (8N)^{-1} > 8N$, the deviation from each branch center is smaller than $q = 1/(4N)$. Equation (4) and induction now give the iterate formula. Every point of this orbit lies in $[a - 1, a + 1]^2$. $\square$

Lemma 3.2. *Every point constructed in Lemma 3.1 has a backward orbit tending to infinity in norm.*

Proof. Write $x = a + u_0$, $y = a + u_1$. Both coordinates are positive. By (5), $F^{-1}(x, y) = (-x - y, x)$. Set $z_{-1} = y$, $z_0 = x$, and $z_1 = -x - y < 0$. Equation (3) gives

$$z_2 = 3z_1 - z_0 < 2z_1 < 0.$$

Inductively,

$$F^{-m}(x, y) = (z_m, z_{m-1}), \quad z_{m+1} = 3z_m - z_{m-1} < 2z_m < 0 \quad (m \geq 2).$$

Here $z_m < z_{m-1} < 0$ implies the next strict inequality. Thus $|z_m|$ grows at least geometrically. $\square$

The initial coordinate u_0 is outside every coding core. This causes no conflict: the first forward step evaluates h at $a + u_1$, whereas the first backward step evaluates it at $a + u_0$. This separation supplies the escape channel.

4 The exact dimension calculation

Since $\lambda = \rho + \rho^{-1}$, multiplying (7) by ρ^{-n} and telescoping yields

$$\lambda \sum_{n=1}^M \rho^{-n} b_{\sigma_n} = u_1 - \rho^{-1} u_0 + \rho^{-(M+1)} u_M - \rho^{-M} u_{M+1}.$$

The solution is bounded, so

$$u_1 = r u_0 + \lambda \sum_{n=1}^{\infty} \rho^{-n} b_{\sigma_n}. \quad (8)$$

Let K_N consist of the series values after $r u_0$ is removed. It is the invariant compact set of the similarities

$$S_i(v) = r v + \lambda r b_i, \quad 1 \leq i \leq N.$$

Its convex hull is $[-A, A]$, where

$$A = \frac{\lambda(N-1)}{2N(\rho-1)}.$$

The gap between consecutive first-level hull intervals is

$$\frac{\lambda r}{N} - 2rA = \frac{\lambda r(\rho - N)}{N(\rho - 1)} > 0. \quad (9)$$

Thus strong separation holds, and the classical separated-similarity formula [2, Theorem 5.3(1)] gives

$$\dim_{\text{H}} K_N = s := \frac{\log N}{\log \rho}. \quad (10)$$

Here is the elementary argument in this particular case. At level k , the N^k hull intervals of length $2Ar^k$ give the upper bound in (10). Assign mass N^{-k} to each cylinder. Distinct level- k hulls have gaps at least cr^{k-1} , where $c > 0$ is the first-level gap. If $r^k \leq \ell < r^{k-1}$, an interval of length ℓ meets at most a constant number of those hulls, independently of k . Its mass is therefore at most $CN^{-k} = C(r^k)^s \leq C\ell^s$. The mass distribution principle gives the lower bound.

By Lemma 3.1, every infinite code has a bounded solution for every $u_0 \in J$. Hence (8) describes the whole compact initial set

$$E_j = \{(a+u, a+ru+v) : u \in J, v \in K_N\}.$$

It is an invertible affine image of $J \times K_N$. The product of interval measure and the preceding Cantor measure gives the lower dimension bound $1+s$. For the upper bound, cover each of the N^k cylinders by $O(r^{-k})$ squares of side $O(r^k)$. Thus

$$\dim_{\text{H}} E_j = 1 + s = 1 + \frac{j}{j+3}.$$

All constants may depend on the fixed block j . Lemmas 3.1 and 3.2 show that every E_j belongs to the trapped set of the *same* map F . Countable stability of Hausdorff dimension now gives

$$2 \geq \dim_{\text{H}} T(F) \geq \dim_{\text{H}} \left(\bigcup_{j \geq 1} E_j \right) = \sup_{j \geq 1} \left(1 + \frac{j}{j+3} \right) = 2.$$

This proves Theorem 1.1, including attainment rather than merely a supremum over different maps.

Proposition 4.1. *For any area-preserving plane diffeomorphism, $T(F)$ has Lebesgue area zero.*

Proof. For an integer $R \geq 1$, let $K_R = \bigcap_{n \geq 0} F^{-n}(\overline{B(0, R)})$. It is compact, $F(K_R) \subseteq K_R$, and the two sets have the same finite area. Hence $K_R \setminus F(K_R)$ is null. If $z \in K_R$ has unbounded backward orbit, take its first backward exit from K_R . The previous point lies in $K_R \setminus F(K_R)$, so $z \in F^m(K_R \setminus F(K_R))$ for some $m \geq 0$. Every trapped point belongs to some K_R . A countable union over m, R is null and contains $T(F)$. $\square$

Thus the full dimension in Theorem 1.1 does not describe a positive-area basin.

5 The continuous-time contrast

We first recall, with a proof, the smooth rank-zero critical-value fact needed here.

Lemma 5.1. *If f is smooth on an open subset of $\mathbb{R}^d$, with $d = 1$ or $d = 2$, then*

$$\dim_{\mathbb{H}} f(\{df = 0\}) = 0$$

whenever the critical-value set is nonempty.

Proof. Let C_k be the set where all derivatives of orders one through k vanish, and let $C_{\infty} = \bigcap_{k \geq 1} C_k$.

For $d = 1$, a point of $C_k \setminus C_{k+1}$ is locally an isolated zero of $f^{(k)}$, since its derivative is nonzero. Each such stratum is countable, so its image has dimension zero. On a compact portion of C_{∞} , Taylor's theorem at a flat point bounds the difference between function values at flat points in an interval of length δ by $O(\delta^M)$, for every fixed positive integer M . There are $O(\delta^{-1})$ grid intervals. For any $s > 1/M$, the sum of the s -powers of these image diameters tends to zero. The image dimension is at most $1/M$ for every M , hence zero. A countable compact exhaustion handles the domain.

For $d = 2$, near a point of $C_k \setminus C_{k+1}$, some derivative $D^{\alpha} f$, of order k , has nonzero gradient. Its zero set is a smooth curve containing C_k in that neighborhood. Every point of C_k is critical for the restriction of f to the curve. The one-dimensional case, in a smooth parametrization, gives image dimension zero. Countably many such charts cover all finite strata. For the flat set, the preceding grid argument has $O(\delta^{-2})$ cells and image diameter $O(\delta^M)$ in each. It gives dimension at most $2/M$ for every M . Combining the strata proves the lemma. $\square$

Proof of Theorem 1.2. The smooth generator $X = (P, Q)$ is divergence-free because the flow preserves area. The one-form $-Q dx + P dy$ is closed and, on the simply connected plane, is dH for a smooth real function H . Thus $X = (H_y, -H_x)$, and H is constant on every orbit. The equilibria are exactly $C = \{dH = 0\}$. By Lemma 5.1, the critical-value set $V = H(C)$ has dimension zero if nonempty.

Suppose a trapped point had energy outside V . Its connected level component is a closed embedded one-manifold without boundary. A nonzero complete orbit is open in this component, and the orbits partition it into open sets; connectedness implies that one orbit covers it. If the component is a circle, both semiorbits are bounded. If it is a line, its closed embedding is proper. A bounded positive semiorbit would lie in a compact part of that line. In a line coordinate, the continuous nonzero vector field has constant sign and speed bounded away from zero on that compact interval, which is impossible for all positive time. Thus every trapped point has energy in V .

A trapped point is not stationary, so it lies in the regular set of H . Cover that set by countably many relatively compact submersion charts. On compact subcharts the coordinate map and its inverse are Lipschitz, and H is one coordinate. Locally the trapped set lies in $I \times V$ for a bounded interval I . This set has dimension at most one: for every $s > 0$, cover the relevant part of V by intervals of lengths δ_i with arbitrarily small $\sum_i \delta_i^s$, then cover each $I \times I_i$ by $O(\delta_i^{-1})$ squares of side δ_i . The sum of their $(1+s)$ -powers tends to zero. The countable chart cover proves $\dim_{\mathbb{H}} T(g) \leq 1$.

If $T(g)$ is nonempty, it contains a nonconstant smooth orbit segment, since time translation preserves trapping. This gives the matching lower bound. Finally

$$g^t(x, y) = (e^{-t}x, e^ty)$$

is a complete smooth area-preserving flow with $T(g) = \{(x, 0) : x \neq 0\}$, attaining dimension one. $\square$

6 Scope and reproducibility

The supplied standard-library Python checker uses exact rational arithmetic. It compares the coding series with independent finite linear solves for periodic codes, checks finite telescoping

identities and cylinder gaps, and replays the forward and backward branches. These finite tests supplement the written universal arguments; they are not a formal verification of smoothness, Hausdorff dimension, or every infinite orbit. In particular, the flow theorem uses all orders of smoothness, and makes no assertion for the remaining finite- C^m flow problem.

This is an unrefereed preprint. AI assistance was used in developing the construction, checking algebra, searching the literature, and preparing the manuscript and reproducibility code; no independent expert review or proof-assistant certification is claimed.

References

- [1] M. Bialy, C. Fierobe, A. Glutsyuk, M. Levi, A. Plakhov and S. Tabachnikov, *Open problems on billiards and geometric optics*, Arnold Mathematical Journal **8** (2022), 411–422. arXiv:2110.10750.
- [2] J. E. Hutchinson, *Fractals and self similarity*, Indiana University Mathematics Journal **30** (1981), 713–747. Author’s retyped version.