

Central Symmetry and Seven Area Identities for Even N -Periodics in the Elliptic Billiard

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Abstract

Reznik, Garcia and Koiller listed many experimentally found invariants of the families of N -periodic billiard trajectories in an ellipse. Seven of their rows, $k_{406,a}$, k_{607} , k_{609} , k_{610} , k_{905} , k_{906} and $k_{907,b}$ (records AMR-050-0024, -0035, -0037, -0038, -0064, -0065 and -0067 of the UnsolvedMath collection), state for even N that a ratio of two signed areas, built from the trajectory, its outer or inner polygon and the foci, equals 1, or that two centroids lie at the centre. We prove them in the form in which they are true: for every trajectory of even least period whose caustic is a confocal ellipse, convex or self-intersecting, the two signed areas are equal at every member of the family (for k_{610} , at every member where the polygons are defined), and for $k_{406,a}$ the vertex centroid is the centre and the area centroid is the centre wherever the area is nonzero; so each ratio equals 1 wherever its denominator is nonzero. The argument is the one the source used for sibling rows: an even periodic trajectory is centrally symmetric, a classical fact printed by Stachel, and every construction involved commutes with the half-turn and with cyclic relabelling. We give a self-contained proof and record where the ratios are undefined. For k_{607} the common area vanishes along whole families, also inside the row's class $N \equiv 0 \pmod{4}$: for $N = 8, 12, 16$ at $b/a = 0.5394187523\dots$, $0.5618074057\dots$, $0.5688313760\dots$, for $N = 10$ at $b/a = 0.5543486017\dots$ and for $N = 6$ at $b/a = \frac{1}{2}$. The common areas of k_{905} , k_{610} and, in star families, k_{906} vanish at isolated members, the polygons of k_{610} degenerate exactly at the members with a vertex on a directrix of the caustic, and the area centroid of $k_{406,a}$ is undefined at isolated members. Apart from this degeneracy criterion these findings are numerical, and nonvanishing elsewhere is not proved. Row $k_{907,b}$ uses a symbol that the source does not define; the theorem holds for every reading that applies one isometry-natural recipe at both foci. This is an unrefereed note.

1 Introduction

1.1 The seven rows

Let $\mathcal{E}$ be the ellipse $x^2/a^2 + y^2/b^2 = 1$ with $a > b > 0$, centre O and foci $f_1 = (c, 0)$, $f_2 = (-c, 0)$, $c = \sqrt{a^2 - b^2}$. A billiard trajectory in $\mathcal{E}$ whose chords are all tangent to a confocal ellipse $\mathcal{C}$, its *caustic*, is called an N -periodic if it closes after N reflections and not earlier. By Poncelet's porism all trajectories with the same caustic are then N -periodic; they form a one-parameter family. In [RGK21] Reznik, Garcia and Koiller list quantities which they found experimentally to be constant along such families. Seven rows of their tables, marked as unproven there, are the subject of this note (Table 1). Each concerns the N -periodic $P = (P_1, \dots, P_N)$, its *outer polygon* P' , whose vertices are the intersections of the tangents to $\mathcal{E}$ at consecutive vertices of P , or its *inner polygon* P'' , whose vertices are the points of contact of the sides of P with $\mathcal{C}$, and polygons derived from these with

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respect to the centre or to the foci: pedal and antipedal polygons, inversions in unit circles, and polar and dual polygons. All areas are signed, as in [RGK21]. The precise definitions are in Section 2; we use those of the arXiv version [RGK21, v11], whose row codes are also used by the corpus [ula26], and the journal’s table numbering may differ.

record	row	quantity	value	N
AMR-050-0024	$k_{406,a}$	$C_0^{*'}, C_2^{*'}: \text{vertex and area centroids of the antipedal polygon of } P' \text{ with respect to } O$	O	even
AMR-050-0035	k_{607}	$\bar{A}_1^*/\bar{A}_2^*: \text{antipedal polygons of } P \text{ with respect to } f_1, f_2$	1	$\equiv 0 \pmod{4}$
AMR-050-0037	k_{609}	$\bar{A}_1''/\bar{A}_2'': \text{pedal polygons of } P'' \text{ with respect to } f_1, f_2$	1	even
AMR-050-0038	k_{610}	$\bar{A}_1^{''*}/\bar{A}_2^{''*}: \text{antipedal polygons of } P'' \text{ with respect to } f_1, f_2$	1	even
AMR-050-0064	k_{905}	$A_1^{\prime\prime\dagger}/A_2^{\prime\prime\dagger}: \text{inversions of } P'' \text{ in the unit circles at } f_1, f_2$	1	even
AMR-050-0065	k_{906}	$A_1^{\prime\dagger}/A_2^{\prime\dagger}: \text{inversions of } P' \text{ at the foci } f'_1, f'_2 \text{ of the ellipse of the outer vertices}$	1	even
AMR-050-0067	$k_{907,b}$	$A_{1,\text{dual}}^\dagger/A_{2,\text{dual}}^\dagger: \text{dual polygons constructed from } f_1, f_2 \text{ (symbol not defined in the source)}$	1	even

Table 1: The seven rows of [RGK21, v11] treated here: corpus record, row code, quantity (the ratios are ratios of the signed areas of the polygons named), value, and the N for which the source states it. The rows are in the source’s tables of antipedal polygons, of pairs of pedal polygons and of pairs of inversive objects.

The corpus records ask to prove that the quantity of the row, “equal to 1” (for $k_{406,a}$: the two centroids, “both equal O ”), is constant along the family. Since a ratio needs a nonzero denominator and an area centroid a nonzero area, the statement that is true without exception is the equality of the two signed areas.

1.2 Results

For every trajectory of *even least period* whose caustic is a confocal *ellipse*, convex or self-intersecting, we prove (Section 5, Theorems 1–7):

- (a) ($k_{406,a}$) the antipedal polygon R^* of P' with respect to O is defined at every member of the family, its vertex centroid is O , and its area centroid is O at every member at which its signed area is nonzero;
- (b) (the other six rows) at every member at which the polygon constructed from f_1 is defined, the polygon constructed from f_2 is defined too, and the two have equal signed areas; for k_{607} , k_{609} , k_{905} , k_{906} and $k_{907,b}$ both are defined at every member, and for k_{610} exactly at the members with no vertex on a directrix of the caustic.

Hence each ratio of Table 1 equals 1 wherever its denominator is nonzero. The restriction $N \equiv 0 \pmod{4}$ of k_{607} is not needed. The proof combines two lemmas. Lemma A (Section 3) is the central symmetry $P_{i+N/2} = -P_i$ of an N -periodic of even least period N , and of its outer and inner polygons. Lemma B (Section 4) says that every construction of Table 1 commutes with the half-turn about O and with cyclic relabelling, so that the construction at $f_2 = -f_1$ is the half-turn image of the construction at f_1 with relabelled vertices; signed areas are invariant under both operations.

The denominators do vanish (Section 6): for k_{607} along whole families, for $N = 6, 8, 10, 12, 16$ at one aspect ratio each (Table 2), three of them in the row’s class $N \equiv 0 \pmod{4}$; for k_{905} , k_{610} and, in star families, k_{906} , and for the area of R^* in $k_{406,a}$, at isolated members; and the antipedal polygons of k_{610} degenerate exactly when a vertex of P lies on a directrix of the caustic (Proposition 6.1). Apart

from Proposition 6.1 and one exact computation, these findings are numerical, and nonvanishing elsewhere is not proved. The rows also hold for repetitions of trajectories of even least period, and they fail, in all cases computed, for repetitions of trajectories of odd period (Section 7).

1.3 Credit and limits

The invariants, their names and the observation that they hold are due to Reznik, Garcia and Koiller [RGK21]. The central symmetry of even periodic trajectories is classical [Sta21, Corollary 4.2(i)], [BT22], and the source marks its analogous rows $k_{604,b}$, $k_{605,b}$, $k_{903,b}$, $k_{904,b}$ as proved by “symmetry”: opposite vertices of an even N -periodic “are reflections about the origin” [RGK21, v11]. Nothing here is deep: the note makes this one-line argument complete and checkable for the seven rows, with its exact hypotheses, and records where the ratios are undefined. Earlier unreferenced notes on sibling rows, by the author [Fer26a, Fer26b] and by A. Kriebel [Kri26], are related. No printed proof of any of the seven rows was found in the sources we searched (“Scope and priority” in Section 8).

2 Definitions and conventions

The billiard. We write $u \times v = u_1v_2 - u_2v_1$ for vectors of $\mathbb{R}^2$, $\sigma(x) = -x$ for the half-turn about O , and $A = \text{diag}(a^{-2}, b^{-2})$, so that $\mathcal{E} = \{x : \langle Ax, x \rangle = 1\}$. The caustic is the confocal ellipse

$$\mathcal{C} : \frac{x^2}{a''^2} + \frac{y^2}{b''^2} = 1, \quad a''^2 - b''^2 = c^2, \quad c < a'' < a.$$

It lies in the interior of $\mathcal{E}$, and f_1, f_2 and O are interior points of $\mathcal{C}$. A *billiard trajectory* is a sequence $(P_k)_{k \in \mathbb{Z}}$ of points of $\mathcal{E}$ with $P_{k+1} \neq P_k$ such that at every P_k the normal of $\mathcal{E}$ bisects the angle between the chords P_kP_{k-1} and P_kP_{k+1} (the reflection law); its caustic is $\mathcal{C}$ if every line P_kP_{k+1} is tangent to $\mathcal{C}$. It is *periodic* with period N if $P_{k+N} = P_k$ for all k ; the smallest such $N \geq 1$ is its *least period*. Following the source, an *N -periodic* is a trajectory with caustic $\mathcal{C}$ and least period N , viewed as the labelled polygon $P = (P_1, \dots, P_N)$ with indices taken modulo N . By Section 3, all trajectories with caustic $\mathcal{C}$ are then periodic with least period N , and either $\mathcal{C}$ lies to the left of every oriented chord $P_k \rightarrow P_{k+1}$ (counter-clockwise trajectories) or to the right of every one (clockwise trajectories). The *family* consists of all of them; we parametrize the counter-clockwise members by their first vertex $P_1 = (a \cos t, b \sin t)$, $t \in \mathbb{R}/2\pi\mathbb{Z}$, and the clockwise members are the same polygons with reversed labelling. The vertices of a member go τ times around O before closing (Section 3); τ is the turning number of [Sta21, Definition 4.1]. Convex N -periodics have $\tau = 1$; we call those with $\tau \geq 2$ star N -periodics.

Polygons. A *labelled polygon* is a tuple $R = (R_1, \dots, R_N)$ of points, indices modulo N ; it may self-intersect. Following [RGK21], its signed area, vertex centroid and area centroid are

$$S(R) = \frac{1}{2} \sum_{i=1}^N R_i \times R_{i+1}, \quad C_0(R) = \frac{1}{N} \sum_{i=1}^N R_i, \quad C_2(R) = \frac{1}{6S(R)} \sum_{i=1}^N (R_i \times R_{i+1})(R_i + R_{i+1}),$$

the last one defined only if $S(R) \neq 0$. For a point M and $r > 0$ we use four constructions, each producing a labelled polygon or being undefined. The *pedal polygon* $\text{Ped}(R; M)$ has as vertex i the foot of the perpendicular from M to the line R_iR_{i+1} (defined if all $R_i \neq R_{i+1}$). For the *antipedal polygon* $\text{Ant}(R; M)$ let ℓ_i be the line through R_i perpendicular to $R_i - M$; vertex i is $\ell_i \cap \ell_{i+1}$ (defined if all $R_i \neq M$ and $\ell_i \nparallel \ell_{i+1}$). The *inversion* $\text{Inv}_r(R; M)$ has the vertices $M + r^2(R_i - M)/|R_i - M|^2$ (defined if all $R_i \neq M$), and $\text{Inv} = \text{Inv}_1$. The *polar polygon* $\text{Pol}(R; M)$ has as vertex i the pole of

the line $R_i R_{i+1}$ with respect to the unit circle centred at M , that is, the inversion of the foot of the perpendicular from M to that line (defined if all $R_i \neq R_{i+1}$ and $M \notin R_i R_{i+1}$). Numbering the output vertices differently, for instance calling the foot on the side $R_i R_{i+1}$ vertex $i+1$, relabels the output cyclically and changes neither S nor C_0 nor C_2 ; so the index conventions of the source do not matter.

The seven quantities. For an N -periodic P , the outer polygon P' has the vertices $P'_i = T_{P_i} \cap T_{P_{i+1}}$, where T_X is the tangent of $\mathcal{E}$ at X , and the inner polygon P'' has as vertex P''_i the point of contact of the line $P_i P_{i+1}$ with $\mathcal{C}$. The outer vertices of the members of the family fill an ellipse $\mathcal{E}'$, concentric and axis-parallel with $\mathcal{E}$ (Theorem 6); f'_1, f'_2 denote its foci, with $f'_2 = -f'_1$, and $f'_1 = f'_2 = O$ if $\mathcal{E}'$ is a circle. For $j = 1, 2$ put $O_p^{(j)} = (1 + b^{-2})f_j$, and

$$\begin{aligned} k_{406,a} : R^* &= \text{Ant}(P'; O), \quad C_0^{*'} = C_0(R^*), \quad C_2^{*'} = C_2(R^*); & k_{607} : \bar{A}_j^* &= S(\text{Ant}(P; f_j)); \\ k_{609} : \bar{A}_j'' &= S(\text{Ped}(P''; f_j)); & k_{610} : \bar{A}_j^{''*} &= S(\text{Ant}(P''; f_j)); \\ k_{905} : A_j^{\prime\prime\dagger} &= S(\text{Inv}(P''; f_j)); & k_{906} : A_j^{\prime\dagger} &= S(\text{Inv}_r(P'; f'_j)); \\ k_{907,b} : \Pi_j &= \text{Pol}(P; f_j), \quad D_j = \text{Pol}(\Pi_j; O_p^{(j)}), \quad \tilde{D}_j = \text{Inv}(\Pi_j; O_p^{(j)}), & A_{j,\text{dual}} &= S(D_j). \end{aligned}$$

These are the definitions of [RGK21, v11]: the antipedal polygon is formed by the intersections of the lines through the P_i perpendicular to $P_i - M$; primes and double primes refer to P' and P'' ; $\dagger$ is the inversion in the unit circle centred at a focus, and $\ddagger$ the inversion of P' with respect to f'_j , for which no radius is given (we allow any $r > 0$, the same for $j = 1, 2$). The source's polar polygon, the “antipedal of the inversive”, is Π_j , and the centre it calls O_p is $O_p^{(j)}$ (Theorem 7). Its dual polygon, the “inversion of the polar with respect to O_p ”, is inscribed in a circle centred at O_p ; we read it as D_j , whose vertices lie on such a circle, and also treat the point-wise inversion $\tilde{D}_j$. The symbol $A_{j,\text{dual}}^\dagger$ of $k_{907,b}$ is not defined in the source (Remark 5.1).

3 Central symmetry

For a labelled polygon R and an integer k let $R^{[k]}$ be the cyclically relabelled polygon, $(R^{[k]})_i = R_{i+k}$, and $\sigma R = (-R_1, \dots, -R_N)$.

Lemma A (central symmetry). *Let $(P_k)_{k \in \mathbb{Z}}$ be a billiard trajectory in $\mathcal{E}$ with the confocal elliptic caustic $\mathcal{C}$ and even least period $m = 2h$. Then $P_{k+h} = -P_k$ for all k , and σ maps the line $P_k P_{k+1}$ onto the line $P_{k+h} P_{k+h+1}$. The outer and inner polygons are defined, and $P'_{k+h} = -P'_k$, $P''_{k+h} = -P''_k$. In particular an N -periodic P with $N = 2n$ satisfies $\sigma P = P^{[n]}$, $\sigma P' = P'^{[n]}$ and $\sigma P'' = P''^{[n]}$.*

For least period N this is [Sta21, Corollary 4.2(i)]: “For even N and odd τ , the billiard is centrally symmetric”, which Stachel derives from his theorem on Poncelet grids, [Sta21, Theorem 3.6]. Least period N implies that τ is odd (Step 5 below), but not conversely: $N = 12$, $\tau = 3$ is a 4-periodic traversed three times. We give a self-contained proof. Its two ingredients are classical: Joachimsthal's integral, in the form of Lemma 3.1, and the invariance of the area form under the billiard map.

Lemma 3.1. *Let $X \in \mathcal{E}$ and let v be a unit vector.*

- (a) *The line $X + \mathbb{R}v$ is tangent to $\mathcal{C}$ if and only if $|\langle AX, v \rangle| = J_{\mathcal{C}} := \sqrt{a^2 - a'^2}/(ab)$.*
- (b) *If $Y = X + \lambda v \in \mathcal{E}$ with $\lambda > 0$, and w is the reflection of v in the tangent of $\mathcal{E}$ at Y , then $\langle AY, w \rangle = \langle AX, v \rangle$.*

Proof. (a) The line is $\{z : \langle m, z \rangle = g\}$ with the unit normal $m = (-v_2, v_1)$ and $g = \langle m, X \rangle$. Such a line touches $\mathcal{C}$ if and only if $g^2 = a''^2 m_1^2 + b''^2 m_2^2$, and since $a''^2 = a^2 - (a^2 - a''^2)$, $b''^2 = b^2 - (a^2 - a''^2)$ and $m_1^2 + m_2^2 = 1$, this reads $a^2 - a''^2 = a^2 v_2^2 + b^2 v_1^2 - (X_1 v_2 - X_2 v_1)^2$. With $X = (a \cos \varphi, b \sin \varphi)$ the right-hand side equals $(av_2 \sin \varphi + bv_1 \cos \varphi)^2 = a^2 b^2 \langle AX, v \rangle^2$. (b) From $\langle AY, Y \rangle = \langle AX, X \rangle = 1$ we get $\lambda = -2\langle AX, v \rangle / \langle Av, v \rangle$, hence $\langle AY, v \rangle = \langle AX, v \rangle + \lambda \langle Av, v \rangle = -\langle AX, v \rangle$. With the unit normal $\nu = AY/|AY|$ we have $w = v - 2\langle v, \nu \rangle \nu$ and $\langle AY, w \rangle = \langle AY, v \rangle - 2\langle AY, v \rangle = \langle AX, v \rangle$. $\square$

The constant $J_{\mathcal{C}}$ is Stachel's expression for Joachimsthal's integral, quoted in [RGK21].

Corollary 3.2. *Let $X \in \mathcal{E}$. There are exactly two lines through X tangent to $\mathcal{C}$. The reflection ρ_X in the normal line ν_X of $\mathcal{E}$ at X exchanges them, and neither of them is ν_X . Hence if a chord of $\mathcal{E}$ ending at X is tangent to $\mathcal{C}$, the chord leaving X by the reflection law lies on the other tangent line from X .*

Proof. X lies outside $\mathcal{C}$, so there are two tangent lines $\ell \neq \tilde{\ell}$ from X . Let ℓ meet $\mathcal{E}$ again at Z and let v be the unit vector from Z to X . By Lemma 3.1, the reflection w of v in the tangent T_X of $\mathcal{E}$ at X satisfies $|\langle AX, w \rangle| = J_{\mathcal{C}}$, so the line $X + \mathbb{R}w$, which is $\rho_X(\ell)$, touches $\mathcal{C}$. Thus the involution ρ_X maps $\{\ell, \tilde{\ell}\}$ to itself. The only lines through X fixed by ρ_X are ν_X and T_X , and T_X does not meet the interior of $\mathcal{E}$, which contains $\mathcal{C}$. If ρ_X fixed ℓ and $\tilde{\ell}$, both would be ν_X ; so ρ_X exchanges them, and then it fixes neither. $\square$

Proof of Lemma A. Phase space. Let $\mathcal{M}$ be the set of pairs (X, v) with $X \in \mathcal{E}$ and v a unit vector pointing into $\mathcal{E}$, $\langle AX, v \rangle < 0$. Let s be the counter-clockwise arc length on $\mathcal{E}$, L the perimeter, and $\theta \in (0, \pi)$ the angle from the counter-clockwise unit tangent at X to v ; then (s, θ) identifies $\mathcal{M}$ with $(\mathbb{R}/L\mathbb{Z}) \times (0, \pi)$. The billiard map $T(X, v) = (Y, w)$, with Y the second point of $\mathcal{E}$ on the ray $X + \mathbb{R}_{>0}v$ and w the reflection of v in T_Y , is a diffeomorphism of $\mathcal{M}$; its inverse is the billiard map of the reversed motion. It preserves $\Omega = \sin \theta ds \wedge d\theta$ (Birkhoff; see [Tab05]). Indeed, $(X, v) \mapsto (X, Y)$ is a diffeomorphism of $\mathcal{M}$ onto the set of pairs of distinct points of $\mathcal{E}$, with inverse $(X, Y) \mapsto (X, (Y - X)/|Y - X|)$, so $s_1 = s(X)$, $s_2 = s(Y)$ are coordinates on $\mathcal{M}$. With $\chi(s_1, s_2) = |Y - X|$ we have $\partial \chi / \partial s_1 = -\cos \theta(X, v)$ and, since the reflection law preserves the tangential component, $\partial \chi / \partial s_2 = \cos \theta(Y, w)$; and $d(\chi) = 0$ gives $\sin \theta(X, v) ds_1 \wedge d\theta(X, v) = \sin \theta(Y, w) ds_2 \wedge d\theta(Y, w)$.

Put $J(X, v) = -\langle AX, v \rangle = |AX| \sin \theta > 0$ and $G(X, v) = X \times v$. By Lemma 3.1(b), $J \circ T = J$, and by Lemma 3.1(a) the chord of (X, v) is tangent to $\mathcal{C}$ if and only if $J(X, v) = J_{\mathcal{C}}$. For such a chord, $|G|$ is the distance of O from its line, which is positive because O is an interior point of $\mathcal{C}$, and $G > 0$ if and only if $\mathcal{C}$ lies to the left of the oriented line $X + \mathbb{R}v$. Let

$$\Gamma = \{J = J_{\mathcal{C}}, G > 0\}, \quad \Gamma^- = \{J = J_{\mathcal{C}}, G < 0\}.$$

A trajectory with caustic $\mathcal{C}$ is an orbit (P_k, v_k) of T in $\Gamma \cup \Gamma^-$, where v_k is the unit vector from P_k to P_{k+1} .

Step 1: Γ is a smooth closed curve over $\mathcal{E}$. Let $X \in \mathcal{E}$. By Corollary 3.2 each of the two tangent lines from X to $\mathcal{C}$ carries one inward unit vector at X , say v' and v'' , pointing to the points of contact. The convex set $\mathcal{C}$ lies in the angle $\{X + \alpha v' + \beta v'' : \alpha, \beta \geq 0\}$; if $v' \times v'' > 0$, every point Z of this angle satisfies $v' \times (Z - X) \geq 0$ and $v'' \times (Z - X) \leq 0$, so $\mathcal{C}$ lies to the left of (X, v') and to the right of (X, v'') , and if $v' \times v'' < 0$ the roles are exchanged. Hence Γ and Γ^- contain exactly one point over each $X \in \mathcal{E}$. By Corollary 3.2, $\theta \neq \pi/2$ on $\Gamma \cup \Gamma^-$, so $\partial J / \partial \theta = |AX| \cos \theta \neq 0$ there, and by the implicit function theorem Γ is the graph of a smooth function $\theta = \theta_{\Gamma}(s)$, a smooth closed curve on which $dJ \neq 0$; likewise Γ^- .

Step 2: $T(\Gamma) = \Gamma$. Let $(X, v) \in \Gamma$ and $T(X, v) = (Y, w)$. Then $J(Y, w) = J_{\mathcal{C}}$. The point of contact Q of the line $X + \mathbb{R}v$ with $\mathcal{C}$ lies on the open chord between X and Y , since $\mathcal{C}$ lies in the interior of $\mathcal{E}$; so the ray from Y through Q is $(Y, -v)$, and $\mathcal{C}$ lies to its right because it lies to the left of (X, v) . By

Corollary 3.2 (this is where the reflected chord is shown to be the *other* tangent), $Y + \mathbb{R}w$ is the second tangent line from Y , and w points to its point of contact. So $(Y, -v)$ and (Y, w) are the two tangent rays from Y , and by Step 1 $\mathcal{C}$ lies to the left of (Y, w) : $T(X, v) = (Y, w) \in \Gamma$. Likewise $T(\Gamma^-) \subseteq \Gamma^-$. As T is a bijection with $J \circ T = J = J \circ T^{-1}$, it maps $\Gamma \cup \Gamma^-$ onto itself, so $T(\Gamma) = \Gamma$ and $T(\Gamma^-) = \Gamma^-$. In particular every trajectory with caustic $\mathcal{C}$ is counter-clockwise (in Γ) or clockwise (in Γ^-).

Step 3: $T|_\Gamma$ is a rotation. Orient Γ by the counter-clockwise orientation of $\mathcal{E}$ (Step 1). $T|_\Gamma$ is a homeomorphism without fixed points ($Y \neq X$). An orientation-reversing homeomorphism of a circle $\mathbb{R}/L\mathbb{Z}$ has a fixed point: a lift F satisfies $F(t + L) = F(t) - L$, so $g(t) = F(t) - t$ satisfies $g(t + L) = g(t) - 2L$ and takes a value in $L\mathbb{Z}$ on $[t, t + L]$. Hence $T|_\Gamma$ preserves the orientation. Next we construct an invariant measure. For $x \in \Gamma$ choose $\zeta \in T_x\mathcal{M}$ with $dJ(\zeta) = 1$ and put $\eta_x(\xi) = \Omega(\zeta, \xi)$ for $\xi \in T_x\Gamma = \ker dJ_x$. This does not depend on ζ , since two choices differ by a vector of the line $T_x\Gamma$, on which Ω vanishes; and $\eta_x \neq 0$, since otherwise $\Omega(\cdot, \xi)$ would vanish on $T_x\mathcal{M} = \text{span}(\zeta, \xi)$ for some $\xi \neq 0$, whereas Ω is nondegenerate. (With $\zeta = (|AX| \cos \theta)^{-1} \partial_\theta$ one finds $\eta = -(\tan \theta_\Gamma(s)/|AX(s)|) ds$.) If F is a diffeomorphism between neighbourhoods of Γ with $F^*\Omega = \Omega$, $J \circ F = J$ and $F(\Gamma) = \Gamma$, then $dF(\zeta)$ is admissible at $F(x)$, because $dJ(dF\zeta) = d(J \circ F)(\zeta) = 1$, so $\eta_{F(x)}(dF\xi) = \Omega(dF\zeta, dF\xi) = \Omega(\zeta, \xi) = \eta_x(\xi)$. Thus $\mu = |\eta|$ is a smooth positive measure on Γ , of total mass U say, invariant under T . Let $u(x) \in \mathbb{R}/U\mathbb{Z}$ be the μ -measure of the counter-clockwise arc of Γ from a fixed point to x ; u is a homeomorphism. If F is an orientation-preserving homeomorphism of Γ preserving μ , and $\tilde{F}$ an increasing lift in the coordinate u , then for $r < t < r + U$ the arcs from $u^{-1}(r)$ to $u^{-1}(t)$ and from $F(u^{-1}(r))$ to $F(u^{-1}(t))$ have the measures $t - r$ and $\tilde{F}(t) - \tilde{F}(r)$, which are equal; so $\tilde{F}(t) - t$ is constant, and F is a rotation. In particular $T|_\Gamma$ is the rotation $u \mapsto u + \delta$ for some $\delta \in (0, U)$.

Step 4: the half-turn. The half-turn acts on $\mathcal{M}$ by $\hat{\sigma}(X, v) = (-X, -v)$, which in the coordinates (s, θ) is $(s, \theta) \mapsto (s + L/2, \theta)$; so $\hat{\sigma}^*\Omega = \Omega$. It commutes with T and preserves J and G , hence Γ . By Step 3, $\hat{\sigma}|_\Gamma$ preserves μ , has no fixed point ($-X \neq X$), preserves the orientation, and is a rotation by some ε . Since $\hat{\sigma}^2 = \text{id}$, $2\varepsilon \in U\mathbb{Z}$, and since $\hat{\sigma}$ has no fixed point, $\varepsilon \equiv U/2 \pmod{U}$.

Step 5: the period. Let (P_k) be counter-clockwise with least period m , and $x = (P_1, v_1) \in \Gamma$. Since Γ has one point over each vertex, $T^k x = x$ if and only if $P_{k+1} = P_1$; so x has least period m under the rotation by δ . Hence $\delta = \tau U/m$ with integers $0 < \tau < m$, $\gcd(\tau, m) = 1$. If $m = 2h$, then τ is odd and T^h is the rotation by $\tau U/2 \equiv U/2$, that is, $T^h = \hat{\sigma}$ on Γ . Therefore $(P_{k+h}, v_{k+h}) = (-P_k, -v_k)$ for all k , which gives $P_{k+h} = -P_k$ and $\sigma(P_k + \mathbb{R}v_k) = P_{k+h} + \mathbb{R}v_{k+h}$. If (P_k) is clockwise, the reversed sequence (P_{-k}) is a counter-clockwise trajectory (the reflection law is symmetric in the two chords at a vertex, and reversing the chords puts $\mathcal{C}$ on their left) with the same least period, and the statement for it is the statement for (P_k) . (Since $P_k \times P_{k+1} = |P_{k+1} - P_k| P_k \times v_k > 0$, the vertex P_{k+1} lies on the open counter-clockwise arc of $\mathcal{E}$ from P_k to $-P_k$; so $\delta < U/2$, and the vertices go τ times around O before closing, $0 < \tau < m/2$. Every orbit of the rotation has least period m , which is Poncelet's porism.)

Step 6: the outer and inner polygons. The line $P_k P_{k+1}$ touches $\mathcal{C}$ at P_k'' , and σ maps $\mathcal{C}$ onto itself and this line onto $P_{k+h} P_{k+h+1}$; so $P_{k+h}'' = -P_k''$. The tangents T_{P_k} and $T_{P_{k+1}}$ are not parallel, for otherwise $P_{k+1} = -P_k$ and the chord $P_k P_{k+1}$, which touches $\mathcal{C}$, would pass through O , an interior point of $\mathcal{C}$. So P_k' is defined, and since $\sigma(T_{P_k}) = T_{-P_k} = T_{P_{k+h}}$, $P_{k+h}' = -P_k'$. $\square$

Remark 3.3. (a) The proof also shows: a trajectory with caustic $\mathcal{C}$ that closes after N reflections, $N = 2n$, satisfies $P_{k+n} = -P_k$ for all k if and only if $N\delta/U$ is odd, and $P_{k+n} = P_k$ otherwise; and a trajectory of odd least period has no two antipodal vertices, because $T^k \neq \hat{\sigma}$ on Γ for every k . (b) By Corollary 3.2, consecutive side lines of a trajectory are the two distinct tangents from their common vertex; so $P_{k+2} \neq P_k$, and the points of contact $P_k'' \neq P_{k+1}''$.

4 Naturality

Definition 4.1. A construction Φ assigns to a labelled polygon R and a tuple $M = (M_1, \dots, M_q)$ of points either a labelled polygon $\Phi(R; M)$ with the same number of vertices, or “undefined”. It is *natural* if for all R, M and k

$$\Phi(\sigma R; \sigma M) = \sigma \Phi(R; M) \quad \text{and} \quad \Phi(R^{[k]}; M) = \Phi(R; M)^{[k]},$$

where in each identity one side is defined if and only if the other is.

The four constructions Ped, Ant, Inv_r (with a fixed r) and Pol of Section 2 are natural: vertex i of the output is obtained from R_i, R_{i+1} and M by feet of perpendiculars, perpendicular lines and their intersections, and inversions in circles of a fixed radius, all of which commute with every isometry of the plane; for instance $\text{Inv}_r(\sigma X; \sigma M) = \sigma M + r^2(\sigma X - \sigma M)/|\sigma X - \sigma M|^2 = \sigma \text{Inv}_r(X; M)$. A composition of natural constructions, in which each step may use its own points among $M_1, \dots, M_q$, is natural.

Lemma B (naturality). *Let Φ be natural, and let R be a labelled polygon with $\sigma R = R^{[k]}$ for some k . Then for every tuple M , $\Phi(R; \sigma M)$ is defined if and only if $\Phi(R; M)$ is, and then*

$$\Phi(R; \sigma M) = \sigma(\Phi(R; M)^{[k]}), \quad S(\Phi(R; \sigma M)) = S(\Phi(R; M)).$$

If moreover $\sigma M = M$, then $X = \Phi(R; M)$ satisfies $\sigma X = X^{[k]}$.

Proof. Using $\sigma^2 = \text{id}$, the two naturality identities and the hypothesis, $\Phi(R; \sigma M) = \Phi(\sigma(\sigma R); \sigma M) = \sigma \Phi(\sigma R; M) = \sigma \Phi(R^{[k]}; M) = \sigma(\Phi(R; M)^{[k]})$, where at each step one side is defined if and only if the other is. The signed area is invariant under σ , since $(-X_i) \times (-X_{i+1}) = X_i \times X_{i+1}$, and under cyclic relabelling. If $\sigma M = M$, the identity reads $X = \sigma(X^{[k]})$, that is, $\sigma X = X^{[k]}$. $\square$

Lemma 4.2. *If $\sigma X = X^{[k]}$ for some k , then $C_0(X) = O$ and $\sum_i (X_i \times X_{i+1})(X_i + X_{i+1}) = 0$; hence $C_2(X) = O$ if $S(X) \neq 0$.*

Proof. $\sum_i X_i = \sum_i X_{i+k} = -\sum_i X_i$. The terms $w_i = (X_i \times X_{i+1})(X_i + X_{i+1})$ satisfy $w_{i+k} = -w_i$, so $\sum_i w_i = \sum_i w_{i+k} = -\sum_i w_i$. $\square$

Only the half-turn σ is used; the external points come in σ -related pairs, $f_2 = \sigma f_1$, $f'_2 = \sigma f'_1$, $O_p^{(2)} = \sigma O_p^{(1)}$, $O = \sigma O$.

5 The seven identities

Throughout this section (P_k) is a billiard trajectory in $\mathcal{E}$ with the confocal elliptic caustic $\mathcal{C}$ and even least period $m = 2h$, listed as the labelled polygon $P = (P_1, \dots, P_N)$, where N is a multiple of m ($N = m$ for an N -periodic); it may be convex or a star polygon, counter-clockwise or clockwise. By Lemma A, $\sigma P = P^{[h]}$, $\sigma P' = P'^{[h]}$ and $\sigma P'' = P''^{[h]}$, so Lemma B applies with $k = h$.

Theorem 1 ($k_{406,a}$, AMR-050-0024). *The antipedal polygon $R^* = \text{Ant}(P'; O)$ is defined. Its vertex centroid $C_0^{*'} is O , and $\sum_i (R_i^* \times R_{i+1}^*)(R_i^* + R_{i+1}^*) = 0$. Hence its area centroid $C_2^{*'} is O wherever $S(R^*) \neq 0$; where $S(R^*) = 0$ it is undefined.$$*

Proof. $P'_i \neq O$, since P'_i lies outside $\mathcal{E}$, and the line ℓ_i through P'_i perpendicular to P'_i is $\{z : \langle z, P'_i \rangle = |P'_i|^2\}$. Two such lines are parallel if and only if O, P'_i, P'_{i+1} are collinear. Both P'_i and P'_{i+1} lie on the tangent $T_{P_{i+1}}$, and they are distinct: otherwise the tangents at the three distinct points P_i, P_{i+1}, P_{i+2} (Remark 3.3(b)) would pass through one point outside $\mathcal{E}$, from which there are only two tangents. So the line $P'_i P'_{i+1}$ is $T_{P_{i+1}}$, which does not contain O , and R^* is defined. By Lemma B with $\Phi = \text{Ant}$ and $M = O = \sigma O$, $\sigma R^* = R^{*[h]}$, and Lemma 4.2 gives the rest. $\square$

Theorem 2 (k_{607} , AMR-050-0035). *The antipedal polygons $\text{Ant}(P; f_1)$ and $\text{Ant}(P; f_2)$ are defined and $\bar{A}_1^* = \bar{A}_2^*$. Hence $\bar{A}_1^*/\bar{A}_2^* = 1$ wherever $\bar{A}_2^* \neq 0$. No condition on N modulo 4 is needed.*

Proof. $P_i \neq f_j$, and the lines through P_i, P_{i+1} perpendicular to $P_i - f_j, P_{i+1} - f_j$ are parallel if and only if f_j lies on the line $P_i P_{i+1}$; but that line touches $\mathcal{C}$ and f_j is an interior point of $\mathcal{C}$. Now apply Lemma B with $\Phi = \text{Ant}$, $M = f_1$, $\sigma M = f_2$. $\square$

Theorem 3 (k_{609} , AMR-050-0037). *The pedal polygons $\text{Ped}(P''; f_1)$ and $\text{Ped}(P''; f_2)$ are defined and $\bar{A}_1'' = \bar{A}_2''$. Hence $\bar{A}_1''/\bar{A}_2'' = 1$ wherever $\bar{A}_2'' \neq 0$.*

Proof. $P_i'' \neq P_{i+1}''$ by Remark 3.3(b); apply Lemma B with $\Phi = \text{Ped}$. $\square$

Theorem 4 (k_{610} , AMR-050-0038). *$\text{Ant}(P''; f_1)$ is defined if and only if no vertex of P lies on the line $x = a''^2/c$, and $\text{Ant}(P''; f_2)$ if and only if no vertex lies on $x = -a''^2/c$; these two conditions are equivalent, and where they hold, $\bar{A}_1''^* = \bar{A}_2''^*$. Hence $\bar{A}_1''^*/\bar{A}_2''^* = 1$ wherever both are defined and $\bar{A}_2''^* \neq 0$. Members at which the two polygons are undefined exist if and only if $a''^2 \leq ac$.*

Proof. $P_i'' \neq f_j$, since $P_i'' \in \mathcal{C}$. The lines through P_i'' and P_{i+1}'' perpendicular to $P_i'' - f_j$ and $P_{i+1}'' - f_j$ are parallel if and only if f_j lies on the line $P_i'' P_{i+1}''$. The sides i and $i+1$ of P are the two tangents from P_{i+1} to $\mathcal{C}$ (Corollary 3.2), so this line is the chord of contact, that is, the polar of P_{i+1} with respect to $\mathcal{C}$. By the symmetry of polarity, $f_1 = (c, 0)$ lies on the polar of P_{i+1} if and only if P_{i+1} lies on the polar of f_1 , the line $x = a''^2/c$; likewise for f_2 and $x = -a''^2/c$. The vertex set of P is symmetric under σ , so the two conditions are equivalent, and Lemma B with $\Phi = \text{Ant}$ gives the equality. The line $x = a''^2/c$ meets $\mathcal{E}$ if and only if $a''^2/c \leq a$, and every point of $\mathcal{E}$ is a vertex of some member of the family (Steps 1 and 2 in the proof of Lemma A). $\square$

Theorem 5 (k_{905} , AMR-050-0064). *The polygons $\text{Inv}(P''; f_1)$ and $\text{Inv}(P''; f_2)$ are defined and $A_1''^\dagger = A_2''^\dagger$. Hence $A_1''^\dagger/A_2''^\dagger = 1$ wherever $A_2''^\dagger \neq 0$.*

Proof. $P_i'' \in \mathcal{C}$ and f_j is interior to $\mathcal{C}$; apply Lemma B with $\Phi = \text{Inv}$. $\square$

Theorem 6 (k_{906} , AMR-050-0065). *Put $A' = a^2/a''$ and $B' = b^2/b''$.*

- (a) *The outer vertices of all members of the family are exactly the points of the ellipse $\mathcal{E}' : x^2/A'^2 + y^2/B'^2 = 1$.*
- (b) *$A' > B'$ if and only if $a'' > a^2/\sqrt{a^2 + b^2}$, and $A' = B'$ if and only if $a'' = a^2/\sqrt{a^2 + b^2}$, the caustic of the 4-periodics; then $\mathcal{E}'$ is the circle of radius $\sqrt{a^2 + b^2}$. Thus $f'_{1,2} = (\pm\sqrt{A'^2 - B'^2}, 0)$ if $a'' > a^2/\sqrt{a^2 + b^2}$, $f'_{1,2} = (0, \pm\sqrt{B'^2 - A'^2})$ if $a'' < a^2/\sqrt{a^2 + b^2}$, and $f'_1 = f'_2 = O$ for the 4-periodics; in all cases $f'_2 = -f'_1$.*
- (c) *For every $r > 0$ the polygons $\text{Inv}_r(P'; f'_1)$ and $\text{Inv}_r(P'; f'_2)$ are defined and $A_1'^\dagger = A_2'^\dagger$; the common value is r^4 times its value for $r = 1$. Hence $A_1'^\dagger/A_2'^\dagger = 1$ wherever $A_2'^\dagger \neq 0$, and the ratio does not depend on r .*

Proof. (a) The side $P_i P_{i+1}$ touches $\mathcal{C}$ at a point $(a'' \cos \varphi, b'' \sin \varphi)$, so its line is $x \cos \varphi/a'' + y \sin \varphi/b'' = 1$. The intersection P'_i of the tangents of $\mathcal{E}$ at the ends of this chord is its pole with respect to $\mathcal{E}$, and the pole of the line $ux + vy = 1$ is $(a^2 u, b^2 v)$; so $P'_i = (a^2 \cos \varphi/a'', b^2 \sin \varphi/b'') \in \mathcal{E}'$. Conversely, every tangent line of $\mathcal{C}$ is a side line of some member. (b) Since $b''^2 = a''^2 - c^2$ and $a^4 - b^4 = c^2(a^2 + b^2)$, $A' > B' \iff a^4 b''^2 > b^4 a''^2 \iff a''^2(a^4 - b^4) > a^4 c^2 \iff a''^2(a^2 + b^2) > a^4$, and likewise with $=$. The rhombus with vertices $(\pm a, 0), (0, \pm b)$ is a 4-periodic (the reflection law holds by symmetry), and its side $x/a + y/b = 1$ touches the confocal ellipse with $a''^2/a^2 + b''^2/b^2 = 1$, that is, $a''^2 = a^4/(a^2 + b^2)$;

then $A'^2 = B'^2 = a^2 + b^2$. Every family of 4-periodics has this caustic: its member through $(a, 0)$ has $P_3 = (-a, 0)$ and $P_4 = -P_2$ (Lemma A), and by the reflection law at $(a, 0)$ the vertex P_4 is the mirror image of P_2 in the x -axis; so $P_2 = (0, \pm b)$, and this member is the rhombus. (c) $P'_i \in \mathcal{E}'$ and f'_j is an interior point of $\mathcal{E}'$, so $P'_i \neq f'_j$; apply Lemma B with $\Phi = \text{Inv}_r$, $M = f'_1$, $\sigma M = f'_2$. Finally $\text{Inv}_r(X; M) = M + r^2(\text{Inv}(X; M) - M)$ is a homothety of ratio r^2 about M applied to $\text{Inv}(X; M)$, which multiplies signed areas by r^4 . $\square$

Since the rotation number τ/N of the family decreases strictly as the caustic grows, a standard fact that we do not prove here, and equals $\frac{1}{4}$ for the 4-periodics, the foci f'_j lie on the x -axis if and only if $\tau/N < \frac{1}{4}$, for instance for every convex N -periodic with $N \geq 5$ and every $a > b$, and on the y -axis if and only if $\tau/N > \frac{1}{4}$, as for $N = 3$ or the star family $N = 8$, $\tau = 3$.

Theorem 7 ($k_{907,b}$, AMR-050-0067). *For $j = 1, 2$:*

- (a) $\Pi_j = \text{Pol}(P; f_j)$ is defined and equals $\text{Ant}(\text{Inv}(P; f_j); f_j)$. Its vertices lie on the circle with centre $(1+b''^{-2})f_j$ and radius a''/b''^2 , and its side lines touch the circle K_j with centre $O_p^{(j)} = (1+b^{-2})f_j$ and radius a/b^2 .
- (b) $D_j = \text{Pol}(\Pi_j; O_p^{(j)})$ and $\tilde{D}_j = \text{Inv}(\Pi_j; O_p^{(j)})$ are defined, and the vertices of D_j lie on the circle with centre $O_p^{(j)}$ and radius b^2/a .
- (c) $S(D_1) = S(D_2)$ and $S(\tilde{D}_1) = S(\tilde{D}_2)$. More generally, for every natural construction Φ , $S(\Phi(P; f_1, O_p^{(1)})) = S(\Phi(P; f_2, O_p^{(2)}))$ wherever the left-hand side is defined, and the right-hand side is then defined.

Hence $A_{1,\text{dual}}/A_{2,\text{dual}} = 1$ wherever $A_{2,\text{dual}} \neq 0$, under either reading of the dual polygon.

Proof. (a) The side lines of P touch $\mathcal{C}$ and so do not pass through f_j , and consecutive side lines are distinct; so Π_j is defined. The polar of a point $Z \neq f_j$ with respect to the unit circle centred at f_j is the line through $\text{Inv}(Z; f_j)$ perpendicular to $Z - f_j$; so vertex i of $\text{Ant}(\text{Inv}(P; f_j); f_j)$ is the intersection of the polars of P_i and P_{i+1} , which is the pole of the line $P_i P_{i+1}$. If a tangent of $\mathcal{C}$ is $\langle m, z \rangle = g$ with $|m| = 1$, then $g^2 = a''^2 m_1^2 + b''^2 m_2^2$, and the foot $F = f_j + (g - \langle m, f_j \rangle)m$ of the perpendicular from the focus f_j satisfies $|F|^2 = c^2 + g^2 - \langle m, f_j \rangle^2 = a''^2$: the feet lie on the circle $|z| = a''$. Inversion in the unit circle centred at f_j maps a circle $|z| = \rho$ with $\rho > c$ onto the circle with centre $(1 + (\rho^2 - c^2)^{-1})f_j$ and radius $\rho/(\rho^2 - c^2)$; for $\rho = a''$ this is the stated circle. Side i of Π_j joins the poles of two lines through P_{i+1} , so it is the polar $\{z : \langle z - f_j, P_{i+1} - f_j \rangle = 1\}$ of P_{i+1} . For $f_j = (\pm c, 0)$ and $P_{i+1} = (x, y) \in \mathcal{E}$ we have $|P_{i+1} - f_j| = (a^2 \mp cx)/a$ and $\langle f_j, P_{i+1} - f_j \rangle = \pm cx - c^2$, so the distance of $O_p^{(j)}$ from this line is $|\langle b^{-2}f_j, P_{i+1} - f_j \rangle - 1|/|P_{i+1} - f_j| = ((a^2 \mp cx)/b^2)/((a^2 \mp cx)/a) = a/b^2$. (b) The side lines of Π_j are at distance $a/b^2 > 0$ from $O_p^{(j)}$, so their poles with respect to the unit circle centred at $O_p^{(j)}$ exist and lie at distance b^2/a from $O_p^{(j)}$; consecutive side lines of Π_j are the polars of distinct points and so are distinct. The point $O_p^{(j)}$ lies inside the circle of the vertices of Π_j : $c(b''^{-2} - b^{-2}) < a''b''^{-2}$ is equivalent, by $b^2 - b''^2 = a^2 - a''^2$ and $b^2 = a^2 - c^2$, to $(a'' - c)(a^2 + ca'') > 0$. So no vertex of Π_j is $O_p^{(j)}$, and $\tilde{D}_j$ is defined. (c) Π_j , D_j and $\tilde{D}_j$ are compositions of Pol and Inv with the external points f_j and $O_p^{(j)}$, and σ maps $(f_1, O_p^{(1)})$ to $(f_2, O_p^{(2)})$; apply Lemma B. $\square$

Remark 5.1 (the symbol $A_{j,\text{dual}}^\dagger$). The symbol table of [RGK21, v11] defines $A_{j,\text{dual}}$, the area of the dual polygon with respect to f_j , but not $A_{j,\text{dual}}^\dagger$ (rows $k_{907,a}$, $k_{907,b}$); the dagger, elsewhere the inversion in the unit circle at a focus, is also used undefined in rows k_{815} , k_{816} , k_{908} . Theorem 7(c) proves the row for every reading in which the polygon for f_j comes from one and the same natural recipe applied

to P , f_j and $O_p^{(j)}$, such as D_j , $\tilde{D}_j$ or their inversions in the unit circle at f_j ; the readings D_j , $\tilde{D}_j$ and $\text{Inv}(D_j; f_j)$ were checked numerically. A reading that used, say, $O_p^{(1)}$ for both foci is not covered; we see no reason for it. We take D_j as the primary reading because its vertices lie on a circle centred at $O_p^{(j)}$ (Theorem 7(b)), as the source says of the dual family.

6 Where the ratios are undefined

This section records where the denominators of Table 1 vanish or the polygons are undefined. Apart from Proposition 6.1 and the exact computation at one member in Section 6.1, the findings are numerical: they were obtained by the verification runs (Section 8) and are reproduced, in 50-digit arithmetic, by the program of this note. We do not prove that any denominator is nonzero elsewhere. A member is given by the parameter t of its first vertex $P_1 = (a \cos t, b \sin t)$ (counter-clockwise); $t + \pi$ gives the same polygon relabelled.

6.1 Row k_{607} : families of zero area

For each $N \in \{6, 8, 10, 12, 16\}$ there is an aspect ratio at which the focal antipedal areas of P vanish at every member computed (Table 2); on these families the ratio $\bar{A}_1^*/\bar{A}_2^*$ is 0/0, although the antipedal polygons are non-degenerate (the smallest $|\sin|$ of the angle between consecutive lines is at least 0.09) and the focal pedal areas $\bar{A}_j$ are at least 1.38. Three of these families, $N = 8, 12, 16$, lie in the class $N \equiv 0 \pmod{4}$ of the row. For $N = 6$, $a = 1$, $b = \frac{1}{2}$ the caustic is $a''^2 = \frac{8}{9}$, $b''^2 = \frac{5}{36}$, the member through $(1, 0)$ is the hexagon with vertices $(\pm 1, 0)$, $(\pm \frac{2}{3}, \pm \frac{\sqrt{5}}{6})$, and in exact arithmetic in $\mathbb{Q}(\sqrt{3}, \sqrt{5})$ its antipedal polygon with respect to $f_1 = (\frac{\sqrt{3}}{2}, 0)$ has six distinct vertices and signed area 0, while its pedal area is $50\sqrt{5}/81$.

N	b/a	remarks
4	none	$\bar{A}_1 \bar{A}_1^* = 16a^4 b^4 / (a^2 + b^2)^2 > 0$ [Fer26a]
6	$\frac{1}{2}$	exact zero at the member through $(a, 0)$
8	0.53941 87523 62378 26881 ...	$(b/a)^2$ is a root of $16x^4 - 4x^3 - 9x^2 + 6x - 1$
10	0.55434 86017 30070 45619 ...	b/a is a root of $8y^5 - 2y^3 + 3y^2 - 1$
12	0.56180 74057 19182 63508 ...	
16	0.56883 13760 37339 91857 ...	

Table 2: Aspect ratios at which the common area $\bar{A}_1^* = \bar{A}_2^*$ of row k_{607} vanishes along the whole convex family ($\tau = 1$, $a = 1$): at the 12 members $t = 0.05 + k\pi/12$ it is below 10^{-48} in 50-digit arithmetic. The polynomials are numerical identifications; for $N = 8$ also a''^2 ($a = 1$) is, numerically, a root of $16y^4 - 56y^3 + 67y^2 - 44y + 16$. For $N = 4$ the area $\bar{A}_1^*$ is positive at $t = 0.37$ for $b/a = 0.05, 0.10, \dots, 0.95$.

The values in Table 2 were located by root-finding in b at the member $t = 0.37$ and agree to 40 digits with those of the verification runs; the polynomials, found by integer-relation search, agree with them to all 50 digits but are numerical identifications. The mechanism is the source's pair of rows $k_{403,b}$ and k_{404} : $\bar{A}_j \bar{A}_j^*$ is constant along the family for $N \equiv 0 \pmod{4}$, and $\bar{A}_j^*/\bar{A}_j$ for $N \equiv 2 \pmod{4}$; both are observed to 50 digits in our computations, and the first is proved in the unrefereed note [Fer26a], whose abstract allows zero antipedal area and gives the value $16a^4 b^4 / (a^2 + b^2)^2$ for $N = 4$. The focal pedal area $\bar{A}_j$ of a convex N -periodic is nonzero, since the focus lies inside the polygon and the triangles spanned by it and consecutive feet are all oriented alike. So if $k_{403,b}$ holds, a zero at one member forces $\bar{A}_j^* = 0$ on the whole family, and the families of Table 2 with $N \equiv 0 \pmod{4}$ are those where its constant vanishes; for $N = 6, 10$ the same rests on k_{404} , which is not

proved. For $b/a \in [0.05, 0.99]$ the verification runs found exactly one such aspect ratio for each of $N = 6, 8, 10, 12, 16$ and none for $N = 4$. The proposers appear to have known the case $N = 6$: the video table of [GR21a] lists a video titled “N=6 zero-area antipedal @ a/b=2”; the title does not name the point, but the antipedal polygon of P with respect to O has nonzero area at the members computed there. Whether this is why the source states k_{607} only for $N \equiv 0 \pmod{4}$ we cannot tell; the restriction does not avoid the vanishing. For the outer polygon, row k_{608} , the note [Fer26b] exhibits a 6-periodic at $a/b = 1 + \sqrt{3}$ with both focal antipedal areas of P' equal to zero.

6.2 Row k_{610} : degenerate members and zeros

Proposition 6.1. *If $a''^2 \leq ac$, the directrices $x = \pm a''^2/c$ of $\mathcal{C}$ meet $\mathcal{E}$ in at most four points, which are the vertices of at most two polygons of the family; the antipedal polygons of P'' with respect to f_1 and f_2 are undefined exactly at the members whose polygon is one of these, that is, at the at most $2N$ values of $t \in [0, 2\pi)$ that are eccentric anomalies of their vertices. If $a''^2 > ac$, they are defined at every member.*

Proof. This is Theorem 4: a member is degenerate if and only if one of its vertices lies on a directrix. The polygon of the family through a given point of $\mathcal{E}$ is unique, since Γ and Γ^- contain one point over each point of $\mathcal{E}$ and the clockwise members are the counter-clockwise ones reversed (Steps 1 and 5 in the proof of Lemma A); with each vertex it contains the antipedal one (Lemma A), and the four points form two antipedal pairs. $\square$

For $a = 1$, $b = \frac{1}{2}$, $N = 4$ ($a''^2 = \frac{4}{5} < ac = \frac{\sqrt{3}}{2}$) the degenerate members in $[0, \pi)$ are $t = 0.393010192890$ ($= \arccos \frac{8}{5\sqrt{3}}$), 0.542639102250 , 2.598953551340 and 2.748582460700 ; at $t \pm 10^{-10}$ the two areas are equal and of size $8.5 \cdot 10^8$. For $a = 1.7$, $b = 0.3$, $N = 6$ there are six, among them $t = 1.369917121400$ and 1.771675532189 ; for $a = 1$, $b = 0.8$, $N = 4$ ($a''^2 > ac$) there are none. Besides its poles, the common area $\bar{A}_1^* = \bar{A}_2^*$ has isolated zeros, for instance at $a = 1.7$, $b = 0.3$, $N = 6$, $t = 1.370305463819$ and 1.771287189771 , within $4 \cdot 10^{-4}$ of a degenerate member but at a non-degenerate polygon (Table 3).

6.3 Isolated zeros of the other rows

The common areas of k_{905} and, in star families, k_{906} , and the area $S(R^*)$ of $k_{406,a}$, change sign at isolated members (Table 3): at each listed t the area changes sign between $t \mp 10^{-8}$, both areas are below 10^{-38} at the computed root, and all vertices are finite; at the zero of $S(R^*)$ the area centroid of $k_{406,a}$ is $0/0$. The list is not complete: the verification runs found further zeros of k_{905} for $N = 4$ at many aspect ratios up to $b/a = 0.7$ and for $N = 6, 8$ at small b/a . For a convex member, whose inner polygon P'' is convex, such a zero requires a focus outside P'' : inversion in a circle centred at f_j preserves the orientation of every triangle $f_j P_i'' P_{i+1}''$, so the area of $\text{Inv}(P''; f_j)$ is nonzero when f_j lies inside P'' . At the six zeros of k_{905} in Table 3 both foci lie outside P'' . The runs found no zero of k_{906} in a convex family, and none of k_{609} or of the dual areas of $k_{907,b}$ (both readings) in any family scanned, although these can be small, for instance $|\bar{A}_j''| \approx 1.3 \cdot 10^{-6}$ for $a = 1$, $b = 0.65$, $(N, \tau) = (12, 5)$, and $|A_{j,\text{dual}}| \approx 1.5 \cdot 10^{-4}$ for $b/a = 0.15$, $N = 4$. For none of the seven rows is nonvanishing proved.

7 The hypotheses

The period. Lemma B needs only $\sigma P = P^{[k]}$ for some shift k . By Lemma A this holds for every trajectory of even least period m , listed with any multiple $N = gm$ of m vertices, with $k = m/2$. So all seven theorems hold, for instance, for the 4-periodic listed twice or three times ($N = 8, 12$),

row	(a, b)	(N, τ)	t
k_{905}	(1, 0.5)	(4, 1)	0.291168207670, 0.695254222277
k_{905}	(1, 0.65)	(4, 1)	0.215729124391
k_{905}	(1.7, 0.3)	(4, 1)	0.153808897536
k_{905}	(1.7, 0.3)	(6, 1)	0.096212547997, 1.224359996661
k_{610}	(1.7, 0.3)	(6, 1)	1.370305463819, 1.771287189771
k_{906}	(1, 0.5)	(8, 3)	0.032769881931, 0.488000972025
k_{906}	(1, 0.65)	(12, 5)	0.695020665228
$k_{406,a} (S(R^*))$	(1, 0.1)	(12, 1)	0.045714304829

Table 3: Some members at which the common area of a row vanishes (first vertex $P_1 = (a \cos t, b \sin t)$, t rounded to 12 decimals).

although for g even the shift $N/2$ gives $P_{i+N/2} = P_i$ rather than $-P_i$. The hypothesis “even least period” cannot be replaced by “closes after an even number N of reflections”: a trajectory of odd least period has no two antipodal vertices (Remark 3.3(a)), and for the 3-periodic of $a = 1$, $b = \frac{1}{2}$, listed twice or four times ($N = 6$ and $N = 12 \equiv 0 \pmod{4}$), each of the seven identities fails at each of the three members computed: the relative differences of the two areas lie between 0.026 and 1.3, and the vertex centroid of R^* lies at distance 0.95 to 1.45 from O (part (ii) of the program in Section 8). They fail likewise for odd $N = 3, 5, 7$, for which the source does not state them. With the source’s convention, an N -periodic has least period N , so the rows are covered.

The caustic. The records say only “a fixed confocal caustic”, but the setting of [RGK21] is a pair of confocal ellipses. For a hyperbolic caustic the symmetry differs [Sta21, Corollary 4.3]: for $N \equiv 0 \pmod{4}$ the trajectory is symmetric in the minor axis, which also exchanges f_1 and f_2 but reverses orientation, so the argument of Lemma B gives opposite signed areas; the first verification run found the ratio -1 for k_{609} , k_{610} , k_{905} , k_{906} , $k_{907,b}$ at $N = 4$, $(a, b) = (1, 0.5)$, for all six ratio rows at $N = 8$, $(a, b) = (1, 0.8)$, and a vertex centroid of R^* different from O . For $N \equiv 2 \pmod{4}$ the trajectory is centrally symmetric if τ is odd and symmetric in the major axis if τ is even. We do not treat hyperbolic caustics.

Areas and members. All areas are signed, as in the source; the equalities of signed areas imply those of the absolute values. Nothing else is assumed: $a > b > 0$ is arbitrary, and the theorems hold at every member, counter-clockwise or clockwise, and for convex and star N -periodics alike. There are no 2-periodics with an elliptic caustic (a 2-periodic runs along a chord through O), so the smallest case is $N = 4$.

8 Computations and reproducibility

All lemmas, theorems and Proposition 6.1 are proved without computation. The computations test the statements and locate the exceptional sets of Section 6. The reproducibility package has five parts.

The exact program of this note, `check_note.py` (Python with `mpmath`, 50 significant digits, about 16 seconds), tests the final statements. Trajectories are computed by taking at each vertex the tangent to $\mathcal{C}$ with $\mathcal{C}$ on the left; the caustic of a family (a, b, N, τ) is found by bisection on the total turning of the vertices (for $N = 4$ it agrees with $a^2/\sqrt{a^2 + b^2}$ to 50 digits). (i) Theorems 1–7 on 25 families: $(a, b) \in \{(1, 0.5), (1, 0.8), (1.7, 0.3), (1, 0.65)\}$ with $N = 4, \dots, 12$ even and $\tau = 1$; the star families $(N, \tau) = (8, 3), (10, 3), (12, 5)$ for $b = 0.5, 0.8, 0.65$ ($a = 1$); and the 4-periodic of $(1, 0.5)$ listed twice

and three times; five members each, in both orientations (250 labelled polygons). It checks closure and Lemma A (largest defect $1.1 \cdot 10^{-45}$), Theorem 1 (below $1.5 \cdot 10^{-47}$, relative), the equalities of Theorems 2–7 (largest relative difference $1.1 \cdot 10^{-43}$, in the star family (12, 5), whose areas reach $7.7 \cdot 10^3$), with the readings D_j , $\tilde{D}_j$, $\text{Inv}(D_j; f_j)$ of $k_{907,b}$, the sibling rows $k_{604,b}$, $k_{605,b}$, $k_{903,b}$, $k_{904,b}$, the ellipse $\mathcal{E}'$ and its foci, the scaling by r^4 , and the circles and the identity $\Pi_j = \text{Ant}(\text{Inv}(P; f_j); f_j)$ of Theorem 7. (ii) Negative controls: odd $N = 3, 5, 7$; the 3-periodic listed two and four times; and the points $(-c, b/10)$, $f'_2 + (0, b/10)$, $(0, b/10)$ in place of f_2 , f'_2 , O for $N = 6, 8, 10$, at three members each. Each row fails at each member of each control, with relative deviations between $4 \cdot 10^{-5}$ ($N = 7$, $b/a = 0.8$, a nearly circular orbit) and order 1. (iii) The exceptional sets: Table 2 with the exact $N = 6$ member in $\mathbb{Q}(\sqrt{3}, \sqrt{5})$ and the identifications for $N = 8, 10$; the constancy of $\bar{A}_1 \bar{A}_1^*$ and $\bar{A}_1^* / \bar{A}_1$ on four families; Table 3; and the degenerate members of Proposition 6.1, obtained from the trajectories through the points where the directrices meet $\mathcal{E}$. All 64 checks pass.

The program with which the results were first obtained (original/): `sym7_check.py`, 60 digits, 22 even families with nine members each, odd and repeated controls, with its recorded output. *The programs of the first two verification runs* described below, with their recorded outputs (`verification_run_A/`, `verification_run_B/`). *The programs of the third verification run*, with their recorded outputs (`independent_run_2/`).

Verification. The results were first obtained, with complete proofs of Lemma A, Lemma B and the seven applications, and with the program in `original/`. Three independent verification runs, all AI-assisted, followed; each wrote its own programs. The first two examined the results before this note was written and re-ran the original program, whose output was reproduced; the third examined the final text. The first run compared the rows, the definitions and Stachel’s theorem numbers with the tex sources of [RGK21, v11] and [Sta21]; re-derived both lemmas and every definedness argument and found the criterion of Theorem 4; computed caustics by Cayley’s criterion and trajectories by the reflection law (340 polygons, both orientations, turning numbers up to 5); checked the $N = 6$ member of Section 6.1 with a computer algebra system; and found, by scans in the aspect ratio, the families of Table 2 for $N = 8, 12$, the zeros of k_{905} and of $S(R^*)$, the degenerate members of k_{610} , and the behaviour of hyperbolic caustics and repetitions (Section 7). The second run read the definitions independently, wrote its programs before reading those of the first write-up, computed the identities in 46 family runs ($N \leq 16$) with two independent billiard maps, scanned 360 members per family, found the families of Table 2 for $N = 6, 8, 10, 12, 16$ with the polynomials for $N = 8, 10$ and the zeros of k_{905} , k_{610} and, in star families, k_{906} , showed that its programs detect wrong inputs, checked the bibliography and repeated the literature search. Neither of these two runs found a false statement among the lemmas and theorems. Both found the numerical statements of the first write-up wrong: it had stated that, apart from the k_{607} family $N = 6$, $b/a = \frac{1}{2}$, no common value came near zero, and had suggested that this family explains the restriction $N \equiv 0 \pmod{4}$. All corrections they required were made: the exceptional sets of Section 6; in the proof of Lemma A, the argument that the reflected chord lies on the other tangent (Corollary 3.2), which had been used before it was proved, and the clockwise case; the hypothesis on the period (least period N implies an odd turning number, not conversely); the position of the foci of $\mathcal{E}'$; Stachel’s theorem number; the wording on the source’s note after its table of pairs and on its journal version; the reading condition for $k_{907,b}$; and three citations. This note was then written; the derivation of the caustic property from Lemma 3.1, the proof of the invariance of Ω , Proposition 6.1 and the exact computation in $\mathbb{Q}(\sqrt{3}, \sqrt{5})$ were added at this stage, and the program `check_note.py` tests the final statements.

The third run examined the text of the note and the package line by line, as if nothing had been checked, and these four additions first. It read [RGK21, v11], [Sta21], [BT22] and [GR21a] again in the tex sources of the arXiv versions cited and the abstracts of [Fer26a, Fer26b, Kri26], re-derived every

proof, and recomputed the statements with its own programs, whose methods differ from those used before: the caustic is found from the classical elliptic-integral formula for the rotation number, the trajectories are generated by the reflection law alone, every chord being tested for tangency, and $\mathcal{E}'$ is fitted to outer vertices. It checked Lemma A and Theorems 1–7 on 400 labelled polygons in 40 families ($N \leq 18$, turning numbers up to 7, both orientations, repetitions; largest relative difference $3 \cdot 10^{-42}$); Lemma 3.1, Corollary 3.2, the invariance of Ω (through the Jacobian of the billiard map) and the invariant measure of Step 3 (the vertices of a trajectory are equally spaced in its coordinate); Table 2 by root-finding at another member (agreement to the 20 printed decimals), Table 3 (to the 12 printed decimals) and the degenerate members of Proposition 6.1, including the boundary case $a''^2 = ac$; and it repeated the exact computation of Section 6.1 with a computer algebra system. It also repeated the literature search, read the survey [GR21b], and re-ran from the archive the program of this note, the original program and five programs of the first two runs: the outputs were reproduced, except that one recorded output of the second run predates a change of its program in the four families with $N = 4$ (described in the package). It found no wrong lemma, theorem or proposition. The corrections it required were made: the numbers for the repeated 3-periodic in Section 7, which were wrong (a normalised quantity had been reported as a distance, and maxima over the members as a range), and the corresponding output of `check_note.py`; one sentence each in the proofs of Lemma A (the coordinates on $\mathcal{M}$), Theorem 6(b) (every family of 4-periodics has the caustic of the rhombus) and Proposition 6.1; two wordings in Section 6; and the record of the survey and of the searches. No proof depends on a computation.

Scope and priority. The seven rows, their names and the observation that they hold are due to Reznik, Garcia and Koiller [RGK21]. The central symmetry of even periodic trajectories is classical; it is printed in [Sta21, Corollary 4.2(i)] and used as a well known fact in [BT22], whose proof of the rows $k_{202,a}$ and $k_{202,b}$ pairs the vertices i and $i + N/2$ in the same way. The source marks its rows $k_{604,b}$, $k_{605,b}$, $k_{903,b}$, $k_{904,b}$ as proved by this symmetry and states the argument after its table of pairs, for $k_{604,a}$ and $k_{604,b}$. Nothing here is deep: the note makes the one-line argument complete and checkable for the seven rows, with its exact hypotheses, and records where the ratios are undefined; the exceptional sets are computed, not proved, except for Proposition 6.1 and the exact zero at one member. Earlier unrefereed notes on sibling rows, by the author and by A. Kriebel, are related, for instance [Fer26a] on $k_{403,b}$, [Kri26] on k_{606} and [Fer26b] on k_{608} . No printed proof of any of the seven rows was found in the sources searched: the source, its follow-up [GR21a] and the survey [GR21b], [Sta21, BT22], the arXiv listings found by queries on antipedal, inversive and pedal polygons in elliptic billiards, on centrally symmetric billiards and on the works of the proposers and of Stachel, the works citing [RGK21] in OpenAlex (19) and Semantic Scholar (31), zbMATH Open, Crossref, and the about twenty unrefereed 2026 deposits on Zenodo, which treat sibling rows only. A search that finds nothing is not a proof of novelty, and we claim no priority.

What was read: [RGK21] in the tex source of arXiv v11 (preliminaries, antipedal polygons, pairs of pedal polygons, inversive objects, symbol table), not the journal version, whose table numbering may differ; [Sta21] in arXiv v2 (Section 4 and the statement of Theorem 3.6); [BT22] in arXiv v2 (the proof of the theorem on k_{202}); [GR21a] in arXiv v3 (abstract and table of videos; the videos were not watched); the abstracts of the Zenodo deposits, and the section on prior observations of [Fer26b]; the survey [GR21b] in its open-access edition, whose text was searched: it treats 3-periodics and contains none of the seven rows. Bibliographic data were checked with Crossref and, for Zenodo, DataCite. The searches were made on 15 October 2026 by the first write-up and the three verification runs, with the public interfaces of the databases named above; the third run also made one general web search.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has

reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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