

Isogonal Conjugacy and Equal Focal Inverted Pedal Areas

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Abstract

Invert the vertices of a triangle about a point and form the pedal triangle of its inverse with respect to that same point. We derive its ordered signed area in terms of the original circumcircle power and the three vertex distances. The formula proves that every finite isogonal-conjugate pair off the sidelines produces equal, nonzero inverse-pedal areas. For a genuine three-periodic elliptic billiard, the optical property of the ellipse and the billiard reflection law make its two foci such a pair. This gives an elementary proof of invariant $k_{908,b}$: the literal focal area quotient is one for every nondegenerate elliptic-caustic triangular orbit. Two exact orbits in the same ellipse and caustic show that the individual area can nevertheless vary. Classical triangle identities are explicitly separated from the application; no absolute novelty or priority is claimed.

1 The source assertion and the construction

Reznik, Garcia and Koiller list $k_{908,b}$ in Table 10 of [1, p. 12]: for a three-periodic elliptic billiard, the two focal inverted-pedal areas have quotient one. The signed-area convention is [1, p. 3]. The same equality is recorded as Observation 1 in [2, Section 3]; Figure 4 there makes the construction explicit. It is the pedal of the inverse triangle, not the inverse of the original pedal triangle.

For an ordered triangle $T = (A, B, C)$ write

$$D = \mathcal{A}(T) = \frac{1}{2} \{ \det(A, B) + \det(B, C) + \det(C, A) \}.$$

Let its circumcircle have center O and radius $R > 0$, and put $a = |B - C|$, $b = |C - A|$, $c = |A - B|$. These letters denote side lengths; the ellipse semiaxes below will instead be α, β . For a point F distinct from the vertices define unit inversion by

$$\mathcal{I}_F(X) = F + \frac{X - F}{\|X - F\|^2}.$$

Drop perpendiculars from F to the complete sidelines $\mathcal{I}_F(A)\mathcal{I}_F(B)$, $\mathcal{I}_F(B)\mathcal{I}_F(C)$ and $\mathcal{I}_F(C)\mathcal{I}_F(A)$, in that order. Denote their ordered signed area by $I(F)$, and write

$$p_F = \|F - O\|^2 - R^2, \quad d_A = |A - F|, \quad d_B = |B - F|, \quad d_C = |C - F|.$$

Theorem 1. *Let T be noncollinear. If F, G are a finite isogonal-conjugate pair off its sidelines, then $I(F) = I(G) \neq 0$. Both areas have the sign of D .*

Theorem 2. *Let T be a genuine three-periodic Euclidean billiard orbit in an ellipse, with a non-degenerate confocal elliptic caustic and foci F_1, F_2 . Then $I(F_1) = I(F_2) \neq 0$, and consequently $I(F_1)/I(F_2) = 1$.*

No assertion about all odd periods or about the constancy of either individual area is part of Theorem 2.

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2 A signed area identity

Lemma 3. *For an ordered triangle U of circumradius r and a point Q of circumcircle power p , its successive sideline feet have signed area $-pA(U)/(4r^2)$.*

Proof. Normalize the circumcircle to the unit circle. Write its complex vertices as z_1, z_2, z_3 , with $|z_i| = 1$, and write q for Q . Reflection in the line $z_i z_{i+1}$ sends q to $z_i + z_{i+1} - z_i z_{i+1} \bar{q}$. Hence its perpendicular foot is

$$h_i = \frac{1}{2}(z_i + z_{i+1} + q - z_i z_{i+1} \bar{q}).$$

Expanding the cyclic shoelace expression gives

$$\sum_i (h_i \bar{h}_{i+1} - \bar{h}_i h_{i+1}) = \frac{1 - |q|^2}{4} \sum_i (z_i \bar{z}_{i+1} - \bar{z}_i z_{i+1}).$$

Terms linear in q or $\bar{q}$ cancel cyclically. Scaling back, $1 - |q|^2 = -p/r^2$, proving the statement with the indicated ordering. This is the classical signed pedal-area formula. $\square$

Lemma 4. *If $p_F \neq 0$, then*

$$I(F) = \frac{Dp_F^2}{4R^2 d_A^2 d_B^2 d_C^2}. \quad (1)$$

Proof. Translate F to the origin and write u, v, w for the original vertices. Their circumcircle equation is $|X|^2 - 2O \cdot X + p_F = 0$. Unit inversion changes it to

$$p_F |Y|^2 - 2O \cdot Y + 1 = 0.$$

The inverse circle has center O/p_F , radius $R/|p_F|$, and power $1/p_F$ at the origin. Let D_{inv} be the signed area of its ordered inverse triangle. Multiplication of its shoelace expression by $|u|^2 |v|^2 |w|^2$ gives

$$2D_{\text{inv}} |u|^2 |v|^2 |w|^2 = \det(u, v) |w|^2 + \det(v, w) |u|^2 + \det(w, u) |v|^2.$$

Substitute $|u|^2 = 2O \cdot u - p_F$ and cyclically. The dot-product terms cancel by $\det(u, v)w + \det(v, w)u + \det(w, u)v = 0$, so

$$D_{\text{inv}} = -\frac{Dp_F}{d_A^2 d_B^2 d_C^2}. \quad (2)$$

Apply Lemma 3 to this inverse circle and triangle. Its factor is $-(1/p_F)/(4R^2/p_F^2) = -p_F/(4R^2)$. Combining this with (2) gives (1), including its sign. $\square$

3 Isogonal conjugacy cancels the focal factors

Normalize the barycentric coordinates of F as (x, y, z) , $x + y + z = 1$. Off-sideline membership means $xyz \neq 0$. The standard isogonal-coordinate rule is

$$G = \left(\frac{a^2}{xS}, \frac{b^2}{yS}, \frac{c^2}{zS} \right), \quad S = \frac{a^2}{x} + \frac{b^2}{y} + \frac{c^2}{z}. \quad (3)$$

Finiteness of G means $S \neq 0$. To recover the rule directly, reflection in a vertex-angle bisector interchanges the two cevian sine ratios. At A this gives $y_G/z_G = b^2 z/(c^2 y)$, and cyclically; normalization then gives (3).

The elementary circle-power identity in normalized barycentrics gives

$$p_F = -(a^2yz + b^2zx + c^2xy) = -xyzS, \quad p_G = -\frac{a^2b^2c^2}{xyzS^2}. \quad (4)$$

For example, the first identity follows by expanding

$$|x(A - O) + y(B - O) + z(C - O)|^2 = R^2 - (a^2yz + b^2zx + c^2xy).$$

The second follows by substituting (3); here the sum $x + y + z = 1$ is essential. Both powers are nonzero.

There is also the vertex-distance identity

$$|G - A|^2 = \frac{b^2c^2}{y^2z^2S^2}|F - A|^2, \quad (5)$$

with cyclic counterparts. Indeed set $u = B - A$, $v = C - A$. Then $F - A = yu + zv$ and $G - A = (b^2u/y + c^2v/z)/S$. Squaring, using $|u|^2 = c^2$ and $|v|^2 = b^2$, proves (5) including its mixed dot-product term. Multiplying the three cyclic identities yields

$$\frac{\prod_{V=A,B,C} |G - V|^2}{\prod_{V=A,B,C} |F - V|^2} = \frac{a^4b^4c^4}{x^4y^4z^4S^6} = \left(\frac{p_G}{p_F}\right)^2. \quad (6)$$

Substitution in (1) proves Theorem 1. Its denominator is positive, $D \neq 0$ and both powers are nonzero, so the common area is nonzero and has the sign of D .

4 Why the two billiard foci are isogonal

Let the boundary ellipse have semiaxes $\alpha > \beta > 0$ and a nondegenerate confocal elliptic caustic of semiaxes $\alpha_c, \beta_c > 0$. The focal squared distance is $\alpha^2 - \beta^2 = \alpha_c^2 - \beta_c^2$; thus both foci lie strictly inside the caustic. They cannot lie on any of its supporting lines, which are precisely the triangle sidelines. They are finite and distinct from the boundary vertices.

The caustic is inside the orbit triangle, not an exterior ellipse tangent only to extended sides. Each contact lies strictly between its two boundary vertices: the caustic is strictly inside the outer ellipse, whose only intersections with the chord line are those vertices. For any triangle side, the contacts on the other two sides lie strictly in its inward half-plane. Since a convex ellipse lies on one side of a supporting line, that ellipse must lie in this inward half-plane. Doing this on all three sides places it inside the triangle. In particular the focus barycentrics are positive.

The boundary ellipse satisfies $|P - F_1| + |P - F_2| = 2\alpha$. Its gradient is the sum of the two unit focal directions, so the normal bisects those directions. The billiard reflection law makes this same normal bisect the two triangle side rays at P . Reflection in the bisector therefore exchanges the two focal rays at every vertex. Consequently F_1, F_2 are isogonal conjugates of the triangle. Theorem 1 now proves Theorem 2, including the quotient's nonvanishing denominator.

Reversing the orbit ordering reverses both signed areas, not the quotient. Replacing unit inversion by a common radius $\rho > 0$ scales both areas by ρ^4 . Neither observation changes the result.

5 Exact controls and a varying individual area

Take the boundary ellipse $x^2/8 + y^2/5 = 1$, its foci $(\pm\sqrt{3}, 0)$ and caustic parameter $\lambda = 40/9$. The caustic squared semiaxes are $32/9, 5/9$. Two ordered orbits are

$$\begin{aligned} T_h &= ((2\sqrt{2}, 0), (-4\sqrt{2}/3, 5/3), (-4\sqrt{2}/3, -5/3)), \\ T_v &= ((0, \sqrt{5}), (-8/3, -\sqrt{5}/3), (8/3, -\sqrt{5}/3)). \end{aligned}$$

Their side lengths are respectively $(5, 10/3, 5)$ and $(4, 16/3, 4)$, giving the same perimeter $40/3$. A direct exact check verifies the ellipse equations, unit-velocity reflection, and tangency to the same caustic, including contact parameters strictly in $(0, 1)$. All three vertices in each orbit are distinct, so their effective period is three.

Direct inversion, orthogonal projection and signed shoelace evaluation give

$$I_h(F_1) = I_h(F_2) = \frac{\sqrt{2}}{72}, \quad I_v(F_1) = I_v(F_2) = \frac{5\sqrt{5}}{576}.$$

Their squares differ by $1/110592$. Thus an individual inverse-pedal area can vary within a single Poncelet family even though its paired focal quotient is always one.

The accompanying source package contains dependency-free exact checkers. One uses rational arithmetic and $\mathbb{Q}(\sqrt{55})$ to test the triangle identities, 400 finite isogonal pairs, orientation and translation controls, a nonisogonal control and a source-valid billiard triangle. Another computes the two orbits above in $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$, with rational sign enclosures. The analytic proofs, not the number of tests or floating-point residuals, establish the theorems.

The original equal-area observation belongs to the source authors; the signed pedal formula, isogonal coordinate map and circle-power identities are classical ingredients. This note gives the explicit derivation and its three-periodic application without an absolute priority claim. The manuscript was prepared with AI assistance and mathematical checks by the author; it is self-audited and unrefereed, with no independent human review or formal proof-assistant certification asserted.

References

- [1] D. Reznik, R. Garcia and J. Koiller, *Eighty New Invariants in the Elliptic Billiard*, arXiv:2004.12497v11 (2020), Table 10. [arXiv:2004.12497v11](https://arxiv.org/abs/2004.12497v11).
- [2] D. Reznik, R. Garcia and M. Helman, *The Talented Mr. Inversive Triangle in the Elliptic Billiard*, arXiv:2012.03020v2 (2021), Section 3, Observation 1 and Figure 4; European Journal of Mathematics **8** (2022), 1550–1565. [doi:10.1007/s40879-021-00489-2](https://doi.org/10.1007/s40879-021-00489-2).