

Equal Focal Inverted Pedal Areas Can Vanish in Elliptic Billiards

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Abstract

For an effective even-periodic billiard in an ellipse with a nondegenerate confocal elliptic caustic, invert the orbit vertices about either focus and form the pedal polygon of the inverse polygon with respect to that same focus. Classical half-turn symmetry makes the two ordered pedal polygons congruent and their signed areas equal. We show that their common signed area need not be nonzero. An explicit primitive ten-periodic family of winding number three is governed by an elementary quartic with a unique root. Exact rational interval arithmetic proves opposite area signs at two ellipse aspect ratios; continuity then gives a genuine orbit with both focal inverted-pedal areas zero. Thus the focal area ratio listed as invariant $k_{908,a}$ is one wherever defined, but its literal quotient is not everywhere defined without an additional extension convention. For contrast, we also derive a strictly positive formula for every counterclockwise four-periodic orbit in the stated elliptic-caustic scope.

1 The construction and the domain issue

Let an ellipse have semiaxes $a > b > 0$, center O and foci F_1, F_2 . An elliptic-caustic billiard is an ordered periodic Euclidean billiard orbit whose sidelines are tangent to a fixed nondegenerate confocal ellipse strictly inside the boundary ellipse. The effective period is the period of the orbit, not the length of a list obtained by repeating a shorter cycle. Self-intersecting orbits are allowed.

For an ordered polygon $W = (W_0, \dots, W_{N-1})$, with cyclic indices, write

$$\mathcal{A}(W) = \frac{1}{2} \sum_{i=0}^{N-1} \det(W_i, W_{i+1}). \quad (1)$$

All areas in this article are these ordered signed areas. In particular, cancellation at a self-intersecting polygon is relevant. This agrees with the area convention of Reznik, Garcia and Koiller [1, p. 3]. Their Section 3.3 defines the pedal construction, and Table 10 lists the ratio of the two focal inverted-pedal areas as $k_{908,a}$, with value one for even periods.

Unit inversion about a point F is

$$\mathcal{I}_F(X) = F + \frac{X - F}{\|X - F\|^2}, \quad X \neq F. \quad (2)$$

If $P = (P_i)$ is the billiard orbit, set $Q_i^{(j)} = \mathcal{I}_{F_j}(P_i)$. Let $R_i^{(j)}$ be the perpendicular foot from F_j to the complete sideline $Q_i^{(j)}Q_{i+1}^{(j)}$. Define

$$A_j = \mathcal{A}(R_0^{(j)}, \dots, R_{N-1}^{(j)}). \quad (3)$$

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Thus inversion precedes the pedal operation, and its focus is also the pedal point. These operations are not interchanged.

The distinction between $A_1 = A_2$ and a globally defined quotient $A_1/A_2 = 1$ is the subject of this note.

Theorem 1. *For the construction (2)–(3):*

1. *Every effective even-periodic elliptic-caustic billiard has congruent ordered focal inverted-pedal polygons, differing by a half-turn and a cyclic relabeling. In particular, $A_1 = A_2$.*
2. *There is an ellipse with minor semiaxis 1 and major semiaxis $a_* \in (5/2, 3)$, and a genuine primitive ten-periodic orbit of winding number three with a nondegenerate confocal elliptic caustic, for which $A_1 = A_2 = 0$.*
3. *Consequently the ratio is one wherever the common signed area is nonzero, but the literal quotient is not defined at every orbit in the stated scope.*

Part (1) is a short equivariance consequence of established billiard symmetry, which is attributed below. Part (2) is an existence statement proved using certified endpoint signs, not an assertion that a rounded numerical area is exactly zero. We do not claim a closed-form coordinate for a_* , uniqueness of that parameter, a zero at every phase of its Poncelet family, or a minimum possible period for a common zero.

2 Half-turn equivariance

Lemma 2. *Suppose $N = 2m$ and an ordered polygon satisfies $P_{i+m} = -P_i$. Let its paired inversion and pedal points be F and $-F$. Assume that no vertex is either point and adjacent vertices are distinct. Then*

$$Q_{i+m}^{(-)} = -Q_i^{(+)}, \quad R_{i+m}^{(-)} = -R_i^{(+)}. \quad (4)$$

Its two ordered inverted-pedal polygons therefore have equal signed areas, including when their common area is zero.

Proof. Substitution gives

$$\mathcal{I}_{-F}(-X) = -F + \frac{-X + F}{\|-X + F\|^2} = -\mathcal{I}_F(X).$$

This proves the first identity in (4). Inversion is an involution off its pole, so adjacent inverse vertices remain distinct. For distinct U, V , the perpendicular foot from F is

$$\pi_{UV}(F) = U + \frac{(F - U) \cdot (V - U)}{\|V - U\|^2}(V - U). \quad (5)$$

Negating F, U, V preserves its scalar coefficient and negates its result. This proves the second identity on every cyclic edge, including the last-to-first edge. Cyclic relabeling preserves (1), as does a half-turn because $\det(-U, -V) = \det(U, V)$. $\square$

To apply the lemma, translate O to zero so that the actual foci are $F, -F$. The canonical elliptic-caustic parametrization in Stachel's Theorem 2 [2] assigns vertex phases $u_0 + 2i\Delta$, where $\Delta = 2\tau K/N$ and the effective period satisfies $\gcd(N, \tau) = 1$. Here K is the corresponding complete elliptic quarter-period and τ is the winding number. If N is even, τ is odd. Moving by $N/2$ vertices

adds $2\tau K$ to the phase, and both the sn and cn coordinates change sign. Hence $P_{i+N/2} = -P_i$. This is prior mathematical input, not a new symmetry theorem.

Both foci lie strictly inside the boundary ellipse, so no boundary vertex is an inversion pole. Effective adjacent orbit vertices are distinct. Lemma 2 proves Theorem 1(1). It does not prove nonvanishing.

3 An explicit ten-periodic elliptic-caustic family

The remaining argument is elementary. We normalize the minor semiaxis to 1 and will ultimately use $5/2 \leq a \leq 3$.

3.1 A unique continuous parameter

For $a \geq 1$, define

$$E(a, q) = a^4 - (a^2 - 1)^2 q^4 + 2a(a + 1)q\{a^2 - (a^2 + 1)q^2\}. \quad (6)$$

Lemma 3. *For every $a \geq 1$, there is exactly one root $q(a) \in (0, 1)$ of $E(a, q) = 0$. Its q derivative is negative, and $q(a)$ is a continuous smooth function of a on the interval used below.*

Proof. The endpoint values are $E(a, 0) = a^4 > 0$ and $E(a, 1) = -2a - 1 < 0$, while

$$E_{qq}(a, q) = -12(a^2 - 1)^2 q^2 - 12a(a + 1)(a^2 + 1)q < 0 \quad (q > 0). \quad (7)$$

Strict concavity gives a unique crossing from the positive to the negative sign. At that crossing the derivative is less than the secant slope from $q = 0$ to the root and is therefore negative. The implicit function theorem yields the asserted continuous smooth root. $\square$

For this root put

$$t = \sqrt{1 - q^2}, \quad D = a^2 - (a^2 - 1)q^2 = a^2 t^2 + q^2, \quad X = \frac{a^2 - (a^2 + 1)q^2}{D}, \quad Y = \frac{2atq}{D}. \quad (8)$$

Then $D > 0$ and $X^2 + Y^2 = 1$. At $q_0 = a/\sqrt{a^2 + 1}$, the braces in (6) vanish and

$$E(a, q_0) = \frac{4a^6}{(a^2 + 1)^2} > 0.$$

The unique-root signs give $q > q_0$, hence

$$X < 0, \quad D < 2q^2 < 2aq, \quad Y > t > 0. \quad (9)$$

In particular, X and Y are strictly between -1 and 1 in their respective negative and positive ranges.

Define

$$\begin{aligned} P_0 &= (a, 0), & P_1 &= (aX, Y), & P_2 &= (-aq, -t), \\ P_3 &= (aq, -t), & P_4 &= (-aX, Y), & P_{i+5} &= -P_i \quad (0 \leq i \leq 4). \end{aligned} \quad (10)$$

All ten points lie on $x^2/a^2 + y^2 = 1$. Their five y -levels $0, \pm t, \pm Y$ are distinct by (9); at each level the two x -values are distinct. Thus all ten vertices are distinct.

3.2 Reflection at every vertex

The first three forward unit directions are

$$v_0 = (-q, t), \quad v_1 = (X, -Y), \quad v_2 = (1, 0). \quad (11)$$

Indeed

$$P_1 - P_0 = \frac{2aq}{D}v_0.$$

For the next edge, direct substitution gives

$$a(q + X)Y + (t + Y)X = \frac{tE(a, q)}{D^2} = 0. \quad (12)$$

This is $\det(P_2 - P_1, v_1) = 0$. The multiplier is positive because both the edge and v_1 have negative y -components; explicitly it is $(t + Y)/Y$. Also $P_3 - P_2 = 2aqv_2$.

At P_1 an outward normal is $n_1 = (X/a, Y)$, and

$$\det(v_0 - v_1, n_1) = -\frac{aqY + (a + 1)XY + tX}{a} = -\frac{tE(a, q)}{aD^2} = 0. \quad (13)$$

The parallel coefficient of $v_0 - v_1$ along n_1 is positive, since their second components are $t + Y$ and Y . Two distinct unit vectors whose difference is parallel to a normal satisfy the reflection law

$$v_{\text{out}} = v_{\text{in}} - 2\frac{v_{\text{in}} \cdot n}{\|n\|^2}n. \quad (14)$$

The correct incoming orientation also follows from $v_0 \cdot n_1 = -qX/a + tY > 0$.

At P_2 use $n_2 = (-q/a, -t)$. The exact identity

$$v_1 - v_2 = (X - 1, -Y) = \frac{2aq}{D}n_2 \quad (15)$$

proves reflection by the same unit-vector argument. At P_0 , the incoming velocity (q, t) and outgoing velocity $(-q, t)$ are reflections across the vertical tangent. Reflection in the coordinate axes, with reversal of path order, and the half-turn in (10) supply the remaining vertices. Thus (10) is an actual closed Euclidean billiard, not just a symmetric inscribed polygon.

3.3 One strictly nested confocal elliptic caustic

Let $\lambda = q^2$ and let the caustic have squared semiaxes

$$a_c^2 = a^2 - q^2, \quad b_c^2 = 1 - q^2 = t^2. \quad (16)$$

They are positive, strictly smaller than the boundary squared semiaxes, and satisfy $a_c^2 - b_c^2 = a^2 - 1$. The caustic is therefore nondegenerate, elliptic and confocal with the boundary.

A line $n \cdot Z = d$ is tangent to this centered ellipse precisely when $d^2 = a_c^2 n_x^2 + b_c^2 n_y^2$. The line P_0P_1 has normal (t, q) and offset at , so its tangency identity is

$$(at)^2 = (a^2 - q^2)t^2 + t^2q^2.$$

The line P_2P_3 is the horizontal tangent $y = -t$. For P_1P_2 , use normal (Y, X) and evaluate its offset at P_2 . Its squared-offset residual is

$$\begin{aligned} & (aqY + tX)^2 - (a^2 - q^2)Y^2 - t^2X^2 \\ &= Y\{(q^2 - a^2t^2)Y + 2aqtX\} = 0, \end{aligned} \quad (17)$$

by (8). The coordinate symmetries establish all remaining tangencies. Each contact is on the actual chord segment: the caustic lies strictly inside the boundary ellipse, so its contact with the sideline lies between that line's two boundary intersections. Thus one fixed confocal elliptic caustic serves the entire orbit.

3.4 Effective period and winding

Let α be the angle of (q, t) and β the angle of $(-X, Y)$, both in $(0, \pi/2)$. The continuously unwrapped arguments in normalized boundary coordinates $(x/a, y)$ for the first six vertices are

$$0, \quad \pi - \beta, \quad \pi + \alpha, \quad 2\pi - \alpha, \quad 2\pi + \beta, \quad 3\pi. \quad (18)$$

Every increment is strictly between 0 and π . The second half repeats these increments, so the total argument increase is 6π : the winding number is three. All ten vertices are distinct, so the effective period is ten. In particular, $\gcd(10, 3) = 1$. This is a permitted self-intersecting elliptic-caustic orbit, not a hyperbolic caustic or a padded odd cycle.

4 Certified signs and the exact common zero

For (10), let

$$F(a) = (\sqrt{a^2 - 1}, 0)$$

be the actual right focus. Let $A(a)$ be the signed area obtained by inverting the vertices about $F(a)$, projecting $F(a)$ onto their successive inverse sidelines, and applying (1).

Proposition 4. *The following strict rational bounds hold:*

$$\frac{6439}{10^7} < A(5/2) < \frac{6440}{10^7}, \quad -\frac{2813}{10^6} < A(3) < -\frac{2812}{10^6}. \quad (19)$$

Proof by exact rational interval arithmetic. The dependency-free certificate supplied with the verification artifacts is

`certify_n10_axis_endpoint_areas.py.`

It performs the following finite procedure at each rational endpoint a .

Start with $q \in [0, 1]$. Repeat 180 times: bisect at the rational midpoint q_m , evaluate the polynomial (6) exactly, and retain the subinterval indicated by its sign. Lemma 3 makes this a valid isolating interval for the unique root. If a midpoint is a root, retain the singleton. Enclose the positive square roots in (8) and $\sqrt{a^2 - 1}$ by squared rational comparisons and 180 bisections. For an interval radicand, use the lower square-root bound at its lower endpoint and the upper bound at its upper endpoint.

Propagate the ten vertices, inversions (2), projections (5), and shoelace sum (1) by rational interval arithmetic. After each operation, round the lower endpoint down and the upper endpoint up to the dyadic grid $2^{-224}\mathbb{Z}$. This is enclosing arithmetic, not rounding to a floating-point value. The certificate explicitly verifies that every interval used as a division denominator excludes zero. It also checks distinctness of all ten vertices by strictly positive pair-distance lower bounds.

The resulting rational area intervals are strictly contained in the two intervals of (19). These contain approximately 0.0006439307696928117 and -0.0028127735995823080 , respectively, but those decimal displays play no role in the sign proof. The exact isolating intervals, intermediate denominator bounds and final rational endpoints are retained in the verification receipts. The complete procedure uses Python's standard-library `Fraction` arithmetic and no floating-point operations. $\square$

The root $q(a)$, the vertices and their actual focus vary continuously with a on $[5/2, 3]$. No inversion pole occurs: the foci are strictly inside the boundary ellipse and every input vertex is on its boundary. All adjacent vertices remain distinct by (9); inversion is injective off its pole, so every inverse sideline has distinct endpoints. All projection denominators are therefore nonzero throughout the interval. Consequently $A(a)$ is continuous there.

Proposition 4 and the intermediate value theorem give $a_* \in (5/2, 3)$ with $A(a_*) = 0$. Sections 3.2–3.4 prove that its orbit has effective period ten, winding three, and the nondegenerate confocal elliptic caustic (16). Its antipodal matching exchanges the two focal constructions, so Lemma 2 gives $A_1 = A_2 = 0$ at this orbit. This proves Theorem 1(2).

5 Interpretation and scope

The intermediate zero is established by varying the ellipse aspect ratio along an exact source-valid family. We do not infer a sign change as phase varies within one fixed ellipse’s Poncelet family. Nor is an approximate parameter from numerical root finding substituted for the existence argument.

At the orbit just proved to exist, the literal ratio is 0/0. Thus global nonvanishing, and an everywhere-defined quotient without an additional convention, are false in the signed-area scope. Equality of the two areas remains true. On the nonzero-area domain division gives $A_1/A_2 = 1$, establishing Theorem 1(3). Defining a removable extension equal to one at a common zero is possible, but is an extra convention, not a division identity or a consequence of nonvanishing.

The historical table entry motivates the question; its proof-status marker is not evidence of present novelty. We make no absolute priority claim for the symmetry corollary or the domain correction. This article was developed with AI assistance in mathematical exploration, symbolic calculations and certificate preparation. It is self-audited and unrefereed; independent human review and formal proof-assistant verification are not asserted.

A A positive formula for every four-periodic orbit

The higher-period zero contrasts with an all-orbit four-periodic formula. Here coordinates remain positively oriented; reversing orbit order changes the sign of every area.

Proposition 5. *For a counterclockwise effective four-periodic elliptic-caustic billiard, choose orthonormal coordinates along two successive outward boundary normals. Write the chosen focus as (f, g) and put*

$$h^2 = b^2 + f^2, \quad k^2 = b^2 + g^2, \quad h, k > 0.$$

Its focal inverted-pedal signed area is

$$\frac{4b^2(hk)^3}{a^6(a^2 + b^2)^2} > 0. \quad (20)$$

Proof. Half-turn symmetry gives $P, Q, -P, -Q$. If v_1, v_2 are the appropriate two unit side directions, the successive outward normals are proportional to $v_2 - v_1$ and $v_1 + v_2$. Their dot product is zero. The ellipse’s shape matrix in this normal frame is $M = b^2 I + FF^\top$, with $F = (f, g)$ and $a^2 = b^2 + f^2 + g^2$. Its support contacts are exactly

$$P = (h, fg/h), \quad Q = (fg/k, k). \quad (21)$$

This covers every nondegenerate orbit under consideration.

First normalize $b = 1$. All real f, g are covered by $r, s > 0$ through

$$f = \frac{r^2 - 1}{2r}, \quad h = \frac{r^2 + 1}{2r}, \quad g = \frac{s^2 - 1}{2s}, \quad k = \frac{s^2 + 1}{2s}. \quad (22)$$

Conversely, $r = h + f > 0$, $s = k + g > 0$, since $(h - f)(h + f) = (k - g)(k + g) = 1$. Set $H = a^2$. The four inverse vertices relative to the focus are

$$\begin{aligned} U_0 &= H^{-1}(h^2(h + f), -hg(h + f)), \\ U_1 &= H^{-1}(-fk(k + g), k^2(k + g)), \\ U_2 &= H^{-1}(-h^2(h - f), -hg(h - f)), \\ U_3 &= H^{-1}(-fk(k - g), -k^2(k - g)). \end{aligned} \tag{23}$$

For consecutive U_i, U_{i+1} , the pedal foot from zero is

$$W_i = \frac{\det(U_i, U_{i+1})}{\|U_{i+1} - U_i\|^2} (U_{i+1,y} - U_{i,y}, -(U_{i+1,x} - U_{i,x})). \tag{24}$$

For clarity, an explicit common-denominator calculation is recorded next. Put $R = r^2$, $S = s^2$, and

$$K_0 = R^2S + RS^2 + 4RS + R + S, \quad B_0 = \frac{16r^5s^5(R + 1)^2(S + 1)^2}{(R + S)^3(RS + 1)^3K_0^2}.$$

Substitution of (23)–(24) gives the four successive determinants $\det(W_i, W_{i+1}) = B_0L_i$, where

$$\begin{aligned} L_0 &= (R + 1)(R^2S + RS^2 + 2RS - R + S), \\ L_1 &= -R(S + 1)(R^2S - RS^2 - 2RS - R - S), \\ L_2 &= S(R + 1)(R^2S - RS^2 + 2RS + R + S), \\ L_3 &= (S + 1)(R^2S + RS^2 + 2RS + R - S). \end{aligned} \tag{25}$$

Expanding these four polynomials shows $\sum_i L_i = 8RS(R + 1)(S + 1)$. Hence their half-sum is

$$\mathcal{A}(W) = \frac{64r^7s^7(r^2 + 1)^3(s^2 + 1)^3}{(r^2 + s^2)^3(r^2s^2 + 1)^3(r^4s^2 + r^2s^4 + 4r^2s^2 + r^2 + s^2)^2} > 0. \tag{26}$$

The displayed numerator and denominator factors are all strictly positive. This is a rational identity from the actual feet, not a positivity inference from finite samples.

Finally,

$$(r^2 + s^2)(r^2s^2 + 1) = 4r^2s^2a^2, \quad K_0 = 4r^2s^2(a^2 + 1), \quad (r^2 + 1)(s^2 + 1) = 4rs hk.$$

Equation (26) becomes $4(hk)^3/[a^6(a^2 + 1)^2]$ in the normalized coordinates. Restoring b scales a unit focal inversion and its pedal coordinates by $1/b$, and their area by $1/b^2$, giving (20). $\square$

This four-periodic nonvanishing theorem does not imply positivity or nonvanishing for higher effective even periods. The ten-periodic construction in the main text proves precisely that distinction.

References

- [1] D. Reznik, R. Garcia and J. Koiller, *Eighty New Invariants in the Elliptic Billiard*, arXiv:2004.12497v11 (2020). <https://arxiv.org/abs/2004.12497v11>. See the signed-area convention on p. 3, Section 3.3 and Table 10.
- [2] H. Stachel, *On the Motion of Billiards in Ellipses*, arXiv:2105.03624v2 (2021). <https://arxiv.org/abs/2105.03624v2>. See Theorem 2 for the canonical elliptic-caustic parametrization.