

Products of the Two Focal Inversion Areas in Elliptic Billiards

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Abstract

For a fixed odd-periodic billiard family in an ellipse, invert the vertices in the unit circles centered at its two foci. We prove that the product of the two resulting signed polygon areas is a positive constant throughout the family. The result includes every admissible coprime star winding and odd repeated traversal. In the complex Jacobi parameter, each area has at most simple poles. For odd periods the two focal pole sets are disjoint, and cyclic reversal forces the other area to vanish at every possible pole. Their product is therefore holomorphic on a compact torus and constant. A real-caustic parity lemma excludes missing odd hyperbolic or degenerate cases. The triangular product was established previously; the present proof addresses the general odd-period conjecture. Exact triangular controls, a circle formula, and an even-period counterexample are included. No absolute-priority claim is made.

1 The focal product conjecture

Let

$$E : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a > b > 0, \quad c = \sqrt{a^2 - b^2}, \quad F_e = (ec, 0) \quad (e = \pm 1).$$

For an ordered closed polygon $W = (W_0, \dots, W_{N-1})$, write

$$S(W) = \frac{1}{2} \sum_{j=0}^{N-1} \det(W_j, W_{j+1}), \quad W_N = W_0.$$

For stars, this is signed shoelace area in trajectory order, not the area of the union of the bounded faces. Unit inversion about F is

$$I_F(X) = F + \frac{X - F}{|X - F|^2}.$$

For a billiard polygon P , set $B_e = S(I_{F_e}(P))$.

Reznik, Garcia and Koiller list the phase independence of B_+B_- for odd N as $k_{903,a}$ in the original invariant table [1, Table 10, p. 12]. Their equation (1) fixes the signed-area convention and Section 3.9 uses the original polygon and boundary ellipse foci. Garcia and Reznik later list the same quantity as k_{807} [3, Table 1]. The triangular formula is already Proposition 7 of Reznik, Garcia and Helman, whose following Conjecture 1 states the odd-period generalization [2]. We do not claim the triangular case as new.

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Theorem 1. *Fix an ellipse E and a confocal caustic supporting a real periodic billiard family with an odd number N of edges, no edge of zero length. Then B_+B_- is positive and independent of the initial phase. The assertion includes self-intersected trajectories and repeated traversals, with the signed-area convention above.*

The caustic and winding are fixed while the phase varies. We first prove the result for a primitive family of odd least period $n \geq 3$ with a nondegenerate elliptic caustic. We then prove that every real odd-period orbit is of this type, up to repetition. Circular boundaries, where the two foci coincide, are treated separately.

The complexification-and-pole-removal method is classical. Roitman, Garcia and Reznik use it to prove related perimeter invariants [4, Theorem 2 and Corollary 1]; their Table 1, entry 18*, still lists the present area product among experimental invariants. The specific mechanism here is cancellation of an area's poles by zeros of the opposite focal area.

2 The billiard parameter

Write the elliptic caustic as

$$C : \frac{x^2}{a_0^2} + \frac{y^2}{b_0^2} = 1, \quad a_0^2 = a^2 - \lambda, \quad b_0^2 = b^2 - \lambda, \quad 0 < \lambda < b^2.$$

Put $k = c/a_0$, $k' = b_0/a_0$, $K = K(k)$ and $K' = K(k')$. All Jacobi functions have modulus $k \in (0, 1)$. Choose $\delta \in (0, K)$ by $\text{cn } \delta = b_0/b$. Then

$$a = \frac{a_0 \text{dn } \delta}{\text{cn } \delta}, \quad b = \frac{b_0}{\text{cn } \delta}, \quad \text{sn}^2 \delta = \frac{\lambda}{b^2}. \quad (1)$$

The classical parametrization [5, Theorem 2] is

$$P(u) = (-a \text{sn } u, b \text{cn } u), \quad P_j(u) = P(u + jh), \quad h = 2\delta. \quad (2)$$

Indeed the tangent to $T(v) = (-a_0 \text{sn } v, b_0 \text{cn } v)$ on C is

$$-\text{sn } v \frac{x}{a_0} + \text{cn } v \frac{y}{b_0} = 1.$$

The addition formulas [6, Section 22.8] show that the tangents at $v = u \pm \delta$ meet at $P(u)$. Consecutive vertices in (2) therefore share the caustic tangent at $u + \delta$. The consistent forward branch is the specular billiard map. This is not uniform spacing in eccentric anomaly.

For a primitive orbit, reversing direction if necessary gives

$$h = \frac{4\tau K}{n}, \quad \text{gcd}(\tau, n) = 1, \quad 0 < \tau < n/2. \quad (3)$$

The real parametrization is injective modulo $4K$, so the n vertices are distinct. This covers the admitted coprime star windings as well as the convex case.

3 Meromorphic inverse areas

Use the bilinear dot product after complexification, not the Hermitian one. A direct calculation gives

$$(P(u) - F_e) \cdot (P(u) - F_e) = (a + e c \text{sn } u)^2. \quad (4)$$

Define

$$g_e(u) = a + e \operatorname{cn} u, \quad Q_e(u) = \frac{P(u) - F_e}{g_e(u)^2}. \quad (5)$$

The omitted translation by F_e does not change polygon area. For real u , $g_e(u) \geq a - c > 0$, so all inverse vertices are finite. Their areas extend to meromorphic functions

$$B_e(u) = \frac{1}{2} \sum_{j=0}^{n-1} \det(Q_e(u + jh), Q_e(u + (j+1)h)). \quad (6)$$

The vectors P, Q_e have common periods $4K, 4iK'$. Under $u \mapsto u + 2iK'$, $\operatorname{sn} u$ is unchanged and $\operatorname{cn} u$ changes sign [6, Section 22.4]. This reflects each vector in the major axis, and consequently

$$B_e(u + 2iK') = -B_e(u). \quad (7)$$

In particular $2iK'$ is an antiperiod, not an area period.

Lemma 2. *On the torus with periods $4K, 4iK'$, the only possible poles of Q_e are at*

$$Z_e - \delta, \quad Z_e + \delta, \quad Z_e + 2iK' - \delta, \quad Z_e + 2iK' + \delta, \quad Z_- = K + iK', \quad Z_+ = 3K + iK'. \quad (8)$$

Each has order at most two.

Proof. At a Jacobi pole $iK' + 2\ell K + 2riK'$, P has order one, whereas g_e^2 has order two with nonzero leading coefficient. Thus Q_e is holomorphic there and vanishes.

The standard quarter-period formula is

$$P(K + iK' + w) = \left(-\frac{a \operatorname{dn} w}{k \operatorname{cn} w}, -\frac{ibk'}{k \operatorname{cn} w} \right). \quad (9)$$

In particular $\operatorname{sn}(K + iK' + w) = \operatorname{dn} w / (k \operatorname{cn} w)$. Combining this with (1) gives precisely the four displayed zeros of g_e . At each zero,

$$\operatorname{cn}^2 u = -\frac{b^2}{c^2}, \quad \operatorname{dn}^2 u = -\frac{\lambda}{a_0^2},$$

so $g'_e(u) = e c \operatorname{cn} u \operatorname{dn} u \neq 0$. These four distinct roots exhaust the zero divisor, since sn has four simple poles on the chosen torus. No other pole of Q_e is possible. $\square$

Lemma 3. *Every possible pole of B_e is at most simple. At each such phase u_* , both focal area functions satisfy*

$$B_f(u_* - \varepsilon) = -B_f(u_* + \varepsilon), \quad f = \pm 1, \quad (10)$$

as meromorphic identities near $\varepsilon = 0$.

Proof. By cyclic relabeling and (7), it is enough to take $u_* = Z_e - \delta$. Exactly the consecutive vertices $j = 0, 1$ are singular for focus e . The only two roots in their horizontal period row differ by $2\delta = h$, and the primitive grid has n distinct residues modulo $4K$.

Formula (9) is even in w . Translation by $2K$ negates P , so the same evenness holds at Z_+ . Therefore the two singular original vertices have a common finite value P_* . Expand $V = P - F_e$ at either root as $V_0 + V_1\varepsilon + \dots$. Equation (4) gives

$$V_0 \cdot V_0 = 0, \quad V_0 \cdot V_1 = 0.$$

Moreover $V_0 \neq 0$, since its x coordinate is eb^2/c . In complex dimension two the orthogonal complement of a nonzero isotropic vector is its own span. Hence V_1 is parallel to V_0 . Since g_e has a simple zero, both negative Laurent coefficients of V/g_e^2 are parallel to V_0 , for each of the two singular vertices.

It follows that the determinant involving both singular vertices has neither a fourth-order nor a third-order pole. All other edge determinants have at most one singular vertex, so B_e has order at most two.

Evenness of P about Z_e also makes Q_f even about Z_e for either focus f . Because $h = 2\delta$,

$$Q_f(u_* - \varepsilon + jh) = Q_f(u_* + \varepsilon + (1 - j)h).$$

The permutation $j \mapsto 1 - j$ reverses cyclic order modulo n , and signed area changes sign. This proves (10). Its Laurent expansion has only odd powers of ε ; the possible double-pole coefficient vanishes. Every possible pole is thus simple or removable. Cyclic translates and the second imaginary row obey the same argument. $\square$

4 Cancellation between the two foci

Cyclic relabeling in (6) gives period h for each B_e . Together with period $4K$ and (3), Bezout's identity yields the smaller real period

$$L = \frac{4K}{n}.$$

Work on the compact torus

$$\mathcal{T} = \mathbb{C}/(L\mathbb{Z} + 4iK'\mathbb{Z}).$$

Since $h = \tau L$, the two real-separated roots of each focus in (8) collapse to one point on $\mathcal{T}$. Thus the only possible poles of B_- are

$$r = K + iK' - \delta, \quad r + 2iK',$$

and the only possible poles of B_+ are these two points translated by $2K$. For odd n ,

$$2K = \frac{n}{2}L \equiv \frac{L}{2} \pmod{L\mathbb{Z}}.$$

The two focal pole sets are therefore disjoint.

At any possible pole of B_e , the function B_{-e} is holomorphic. Lemma 3 applies to both labels, so B_{-e} is odd about that phase and hence vanishes there. Its zero cancels the at-most-simple pole of B_e . Exchanging the labels removes every possible pole of B_+B_- . The product is holomorphic on the compact torus $\mathcal{T}$, and is therefore constant.

For the forward real elliptic-caustic orbit, every directed edge has C on its left, including star trajectories. Both foci lie strictly inside C , since $a_0^2 - c^2 = b_0^2 > 0$. Consequently

$$\det(P_j - F_e, P_{j+1} - F_e) > 0.$$

Each inverse determinant is this positive quantity divided by two positive squared focal distances. Thus each B_e is positive in this orientation, and the constant product is positive. Reversal negates both areas and preserves the product.

5 Real caustics and repeated traversals

We verify that the elliptic-caustic proof omits no real odd case.

Lemma 4. *Every nondegenerate real periodic orbit of odd length in a noncircular ellipse has a nondegenerate elliptic caustic.*

Proof. Write a chord line as $\alpha x + \beta y = w$, normalized by $\alpha^2 + \beta^2 = 1$. Its confocal caustic parameter satisfies the dual tangency equation

$$w^2 = (a^2 - \lambda)\alpha^2 + (b^2 - \lambda)\beta^2, \quad \lambda = a^2\alpha^2 + b^2\beta^2 - w^2. \quad (11)$$

A nonzero chord is not a supporting tangent of the strictly convex boundary, so $\lambda > 0$, and (11) gives $\lambda \leq a^2$.

If $b^2 < \lambda < a^2$, then $w^2 < c^2\alpha^2$. Thus $\alpha \neq 0$ and the line crosses the x -axis strictly between the foci, at $(w/\alpha, 0)$, inside the ellipse. Its direction vector $(-\beta, \alpha)$ has nonzero y component. Its two boundary endpoints consequently have opposite nonzero y coordinates. Every step alternates this sign, making a closed odd sequence impossible.

At $\lambda = b^2$, the same argument holds with $|w/\alpha| = c < a$ when $\alpha \neq 0$. If $\alpha = 0$, (11) forces $w = 0$: the chord is the major axis, and reflection traps the trajectory in the axial two-cycle. At $\lambda = a^2$, (11) reduces to $w^2 = -c^2\beta^2$, so $\beta = w = 0$ and only the minor-axis two-cycle remains. Finally $\lambda = 0$ is grazing, with no nonzero chord. Hence an odd nondegenerate orbit must have $0 < \lambda < b^2$. $\square$

If a listed odd cycle of length N traverses a primitive n -cycle d times, then $N = dn$ and both n, d are odd. Its shoelace sums repeat each edge exactly d times, so

$$B_e^{(N)} = dB_e^{(n)}, \quad B_+^{(N)}B_-^{(N)} = d^2B_+^{(n)}B_-^{(n)}.$$

Phase independence follows from the primitive theorem. Together with Lemma 4, this proves Theorem 1.

For a circular boundary of radius a , both foci coincide with the center and $Q = P/a^2$. A primitive regular star of winding τ has

$$S(P) = \frac{na^2}{2} \sin \frac{2\pi\tau}{n}, \quad B_+B_- = \frac{n^2}{4a^4} \sin^2 \frac{2\pi\tau}{n}.$$

For odd n the product is positive; repetitions scale it as above. If both inversion circles have radius ρ instead of one, each area acquires a factor ρ^4 and the product a factor ρ^8 .

6 Exact controls and the parity boundary

The archived standard-library checker uses rational arithmetic and quadratic extensions only. On the ellipse $a^2 = 21, b^2 = 16$ with $\lambda = 336/25$, it verifies two primitive triangular phases

$$H = ((\sqrt{21}, 0), (-3\sqrt{21}/5, 16/5), (-3\sqrt{21}/5, -16/5)), \\ V = ((0, 4), (-21/5, -8/5), (21/5, -8/5)).$$

They are not cyclic relabelings or central negations of one another. Exact boundary incidence, specular reflection with unit velocities, and internal tangency to the same caustic are checked. At H ,

$$B_{\pm} = \sqrt{21} \left(\frac{7}{576} \mp \frac{\sqrt{105}}{12096} \right),$$

whereas at V , $B_+ = B_- = 1/18$. Both products are $1/324$, consistent with the published triangular formula [2, Proposition 7]. The checker also verifies cyclic relabeling, reversal, central negation and repeat counts 3, 5, 7. The circle of radius two gives $27/256$ for a triangle.

Oddness is essential. On $a^2 = 4, b^2 = 1$ with $\lambda = 4/5$, two genuine primitive four-periodic phases are

$$D = ((2, 0), (0, 1), (-2, 0), (0, -1))$$

and the rectangle with cyclic vertices $(\pm 2/\sqrt{5}, \pm 2/\sqrt{5})$. Their focal inverse-area products are respectively 1 and $25/16$. The exact checker verifies the common caustic and reflection law. This refutes deletion of the odd-period hypothesis, not the odd-period conjecture.

Separate high-precision diagnostics sampled 42 odd families, including 27 star families, at five phases each. Those samples are numerical evidence only. Neither they nor the exact finite controls replace the all-period meromorphic argument.

This is an unrefereed preprint. AI-assisted tools supported literature searches, algebraic exploration, exact checks, and manuscript preparation; the author is responsible for the claims. The proof has been self-audited, not independently human-reviewed or verified in a proof assistant. The bounded literature search does not certify absolute novelty or priority.

References

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