

Polar, Dual and Inversive Polygon Areas of N -Periodics in the Elliptic Billiard

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Abstract

Reznik, Garcia and Koiller listed many quantities found experimentally to be constant along the families of N -periodic billiard trajectories in an ellipse. We treat five rows of their table of inversive objects (records AMR-050-0056, -0057, -0058, -0059 and -0061 of the UnsolvedMath collection), built from a trajectory P and one focus f : the inversion $P^\dagger$ of P in the unit circle at f , the polar polygon (the antipedal polygon of $P^\dagger$), its dual polygon, and the focal pedal and antipedal polygons. For least period N , any winding number, a confocal elliptic caustic and signed areas we prove that the ratios of the polar area to the inversive area (row k_{813}) and to the dual area (row k_{814} , credited by the source to A. Akopyan) are constant for every N , the dual being the polar reciprocal of the polar polygon about the centre of the circle touched by its sides, as in the source's figure, or its point-wise inversion; for $N = 4$ the constants are $2a^4/b^4$ and $2a^6/b^{10}$. Rows k_{815} (odd N) and k_{816} (even N) use a symbol that the source does not define. Read as the area of the pedal polygon of $P^\dagger$, as in the source's row $k_{908,b}$, row k_{815} is false (exact certificate for $N = 3$) and k_{816} is false for $N = 4$ (exact certificate) but true for every even $N \geq 6$; read as the area of the inversion of the focal pedal polygon, k_{815} is false and k_{816} true; read as the focal pedal area, both hold. Row k_{818} ($N \equiv 2 \pmod{4}$) follows from two earlier deposits of the author, doi:10.5281/zenodo.23094118 and doi:10.5281/zenodo.23079799; we reprove it, and its quotient is undefined on the families where the focal antipedal area vanishes identically, which are recorded in earlier deposits (for $N = 6$: exactly $b/a = \frac{1}{2}$). The method is Stachel's Jacobi parametrisation, the phases at which a side of P is an isotropic line through f , and Liouville's theorem. This is an unrefereed note.

1 Introduction

1.1 The five rows

Let $\mathcal{E}$ be the ellipse $x^2/a^2 + y^2/b^2 = 1$ with $a > b > 0$, centre O and foci $(\pm c, 0)$, $c = \sqrt{a^2 - b^2}$, and fix λ with $0 < \lambda < b^2$. The ellipse $\mathcal{C} : x^2/(a^2 - \lambda) + y^2/(b^2 - \lambda) = 1$ is confocal with $\mathcal{E}$ and lies inside it. A *billiard trajectory* in $\mathcal{E}$ is a sequence $(P_l)_{l \in \mathbb{Z}}$ of points of $\mathcal{E}$ with $P_{l+1} \neq P_l$ such that at every P_l the chords $P_{l-1}P_l$ and P_lP_{l+1} are mirror images of each other in the normal of $\mathcal{E}$ at P_l ; it has the *caustic* $\mathcal{C}$ if all its chords are tangent to $\mathcal{C}$. It is an *N -periodic* if $P_{l+N} = P_l$ for all l and $N \geq 3$ is the least such number. If one trajectory with caustic $\mathcal{C}$ is N -periodic, then all are, with the same winding number τ around O (Poncelet's porism; Lemma 2.1); they form a one-parameter family, the *family*. A quantity is *invariant* if it has the same value at every member of the family.

For a closed polygon $W = (W_1, \dots, W_N)$, indices modulo N , the *signed area* is $S(W) = \frac{1}{2} \sum_i W_i \times W_{i+1}$, where $x \times y = x_1y_2 - x_2y_1$; this is the convention of the source [RGK21], and $A := S(P)$. Let

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f be one of the foci. The source (we use its arXiv version [RGK21, v11], whose row codes the corpus uses) defines, for P and the focus f (its f_j): the *inversive polygon* $P^\dagger$, whose vertices are the inversions of the P_i in the unit circle centred at f , with area $A^\dagger$ (its $A_j^\dagger$); the *polar polygon*, the “antipedal of the inversive” with respect to f , whose i -th vertex is the intersection of the lines through $P_i^\dagger$ and $P_{i+1}^\dagger$ perpendicular to $fP_i^\dagger$ and $fP_{i+1}^\dagger$, with area A_{pol} (according to the source its vertices lie on a circle and its sides touch a circle, the “polar caustic”, with centre O_p); the *dual polygon*, the “inversion of the polar with respect to O_p ”, said in the caption of the source’s figure to be inscribed in a circle centred at O_p , with area A_{dual} (the reading is fixed below); the *focal pedal polygon*, whose vertices are the feet of the perpendiculars from f to the side lines P_iP_{i+1} , with area $\bar{A}$ (its $\bar{A}_j$, called $A_{j,\text{ped}}$ in its table of symbols); and the *focal antipedal polygon*, whose i -th vertex is the intersection of the lines through P_i and P_{i+1} perpendicular to fP_i and fP_{i+1} , with area A_{ant} (its $A_{j,\text{ant}}$).

record	row	quantity in the source	N
AMR-050-0056	k_{813}	$A_{j,\text{pol}}/A_{j,\text{inv}}$	all
AMR-050-0057	k_{814}	$A_{j,\text{pol}}/A_{j,\text{dual}}$	all
AMR-050-0058	k_{815}	$A_{j,\text{ped}}^\dagger \cdot A_{j,\text{dual}}$	odd
AMR-050-0059	k_{816}	$A_{j,\text{ped}}^\dagger/A_{j,\text{dual}}$	even
AMR-050-0061	k_{818}	$A_j^\dagger/A_{j,\text{ant}}$	$\equiv 2 \pmod{4}$

Table 1: The five rows of the table of invariants of inversive objects in [RGK21, v11], all marked as not proven there, with the corpus records [ula26]. The records ask to prove that the quantity is constant as P varies through the family.

The dual polygon. The source recalls that “the dual is the inverse of the pedal” [AZ07]. Accordingly we take as dual polygon the polar reciprocal of the polar polygon with respect to the unit circle centred at O_p : its vertices are the poles of the side lines of the polar polygon, equivalently the inversions of the feet of O_p on these lines. These vertices lie on a circle centred at O_p (Lemma 3.1), as the caption says and the figure shows. The inversions of the polar *vertices* in that circle, the *point-wise* reading, are not at a constant distance from O_p (the first verification run found relative spreads 0.05 to 0.37); we treat this reading as well and write $A_{\text{dual}}^{\text{pt}}$ for its area. The source does not give the radius of the circle at O_p ; a radius r multiplies the dual area by r^4 , so no statement below depends on it, and we take radius 1. The deposit [Fer26h] notes, for row k_{811} , that the source’s descriptions of the dual construction are incompatible, and treats both.

Two undefined symbols. The symbol $A_{j,\text{inv}}$ of k_{813} is not in the source’s table of symbols; the corpus record writes $A_j^\dagger$, and the polygon $P^\dagger$ is the one the source calls inversive, so we read it as $A^\dagger$. The symbol $A_{j,\text{ped}}^\dagger$ of k_{815} and k_{816} is not defined in the source; it occurs also in its rows $k_{908,a}$ and $k_{908,b}$. We consider three readings: (i) the area $A_{\text{ped}}^\dagger$ of the pedal polygon of $P^\dagger$ with respect to f ; (ii) the area of the inversion, in the unit circle at f , of the focal pedal polygon (the dagger as in the source’s $A_j^\dagger$, the inversion of the outer polygon), which is the polar polygon by Lemma 3.1, so that the area is A_{pol} ; (iii) the focal pedal area $\bar{A}$, the dagger being a slip. Reading (i) is the only one of the three under which the source’s row $k_{908,b}$, $A_{1,\text{ped}}^\dagger/A_{2,\text{ped}}^\dagger = 1$ for $N = 3$, holds: for reading (i) this is the observation of Reznik, Garcia and Helman on the pedal triangles of the two focus-inversive triangles [RGH22, Section 3, first observation], proved in [Fer26e], while under (ii) and (iii) the quotient varies, with values between 0.43 and 2.3 at the members computed by the verification runs. Reading (i) is also the one of the deposits [Fer26b, Fer26e] on row k_{908} . Under reading (ii) row k_{816} is row k_{814} for even N .

1.2 Results

Every statement holds for both foci (Section 1.4); we write $A^\dagger$, A_{pol} , $\dots$ for the areas with respect to either focus.

Theorem 1 (row k_{813}). *Let $a > b > 0$, let the caustic be a confocal ellipse, let $N \geq 3$ be the least period of the trajectories of the family and τ their winding number (any τ coprime to N), and use signed areas. Then there is a constant $\kappa_1 > 0$, depending only on a , b and λ , such that $A_{\text{pol}} = \kappa_1 A^\dagger$ at every member of the family. In particular $A_{j,\text{pol}}/A_j^\dagger$ is constant along the family, for every $N \geq 3$, every winding number and both foci. For $N = 4$, $\kappa_1 = 2a^4/b^4$.*

Theorem 2 (row k_{814}). *Let $a > b > 0$, let the caustic be a confocal ellipse, let $N \geq 3$ be the least period of the trajectories of the family and τ their winding number (any τ coprime to N), and use signed areas.*

- (a) *There is a constant $\kappa_2 > 0$, depending only on a, b, λ , such that $A_{\text{dual}} = \kappa_2 A^\dagger$ at every member. Hence $A_{j,\text{pol}}/A_{j,\text{dual}} = \kappa_1/\kappa_2$ is constant along the family, for every $N \geq 3$. For $N = 4$ it equals $2a^6/b^{10}$.*
- (b) *For the point-wise reading there is a constant $\kappa_3 > 0$ with $A_{\text{dual}}^{\text{pt}} = \kappa_3 A^\dagger$, so $A_{\text{pol}}/A_{\text{dual}}^{\text{pt}}$ is constant as well. For $N = 3$, $A_{\text{dual}}^{\text{pt}} = A_{\text{dual}}/4$, and for $N = 4$, $A_{\text{dual}}^{\text{pt}} = A_{\text{dual}}/2$.*

Theorem 3 (row k_{815}). *Let $a > b > 0$, let the caustic be a confocal ellipse, let $N \geq 3$ be the least period of the trajectories of the family and τ their winding number (any τ coprime to N), and use signed areas.*

- (A) *Under readings (i) and (ii) the row is false. For $a^2 = 8$, $b^2 = 3$ the 3-periodic family has the caustic $\lambda = 72/25$, and its members*

$$\begin{aligned} M_1 : & (2\sqrt{2}, 0), (-\frac{8}{5}\sqrt{2}, \frac{3}{5}\sqrt{3}), (-\frac{8}{5}\sqrt{2}, -\frac{3}{5}\sqrt{3}), \\ M_2 : & (0, \sqrt{3}), (-\frac{8}{5}\sqrt{3}, -\frac{1}{5}\sqrt{3}), (\frac{8}{5}\sqrt{3}, -\frac{1}{5}\sqrt{3}) \end{aligned}$$

give $A_{\text{ped}}^\dagger A_{\text{dual}} = \frac{243\sqrt{10}}{256000} + \frac{243}{32000}$ and $\frac{6561}{1024000}$ (reading (i)), and $A_{\text{pol}} A_{\text{dual}} = \frac{27\sqrt{10}}{5} + \frac{999}{40}$ and $\frac{729}{40}$ (reading (ii)), with respect to the focus $(-\sqrt{5}, 0)$. Moreover, for every $N \geq 3$ the product $A_{\text{pol}} A_{\text{dual}}$ is not constant along any family.

- (B) *Under reading (iii) the row holds: for every odd N there is a constant $\kappa_4 > 0$, depending only on a, b, λ , such that $\bar{A} A_{\text{dual}} = \kappa_4$ at every member.*

All statements remain true with $A_{\text{dual}}^{\text{pt}}$ in place of A_{dual} .

Theorem 4 (row k_{816}). *Let $a > b > 0$, let the caustic be a confocal ellipse, let $N \geq 3$ be the least period of the trajectories of the family and τ their winding number (any τ coprime to N), and use signed areas. Let N be even.*

- (A) *Under reading (i) the row is false for $N = 4$. For $a = 5$, $b = 4$ the 4-periodic family has the caustic $\lambda = 400/41$, and its members $R_1 = ((5, 0), (0, 4), (-5, 0), (0, -4))$ and $R_2 = 41^{-1/2}((25, 16), (-25, 16), (-25, -16), (25, -16))$ give $A_{\text{ped}}^\dagger/A_{\text{dual}} = \frac{2}{1681}$ and $\frac{1}{800}$ with respect to the focus $(-3, 0)$.*
- (B) *Under reading (i) the row holds for every even $N \geq 6$: there is a real constant κ_5 , depending only on a, b, λ , with $A_{\text{ped}}^\dagger = \kappa_5 A_{\text{dual}}$ at every member.*

(C) Under reading (iii) the row holds for every even $N \geq 4$: there is a constant $\kappa_6 > 0$ with $\bar{A} = \kappa_6 A_{\text{dual}}$ at every member; for $N = 4$, $\kappa_6 = 2a^6/(b^2(a^2 + b^2)^2)$.

Under reading (ii) the quotient is $A_{\text{pol}}/A_{\text{dual}}$, which is constant for every N by Theorem 2. All statements remain true with $A_{\text{dual}}^{\text{pt}}$ in place of A_{dual} . So row k_{816} holds for every even $N \geq 6$ under each of the three readings, and for $N = 4$ it depends on the reading.

For $N = 4$ the inversive pedal area has poles of order three, for even $N \geq 6$ simple poles (Proposition 6.4); this is why (A) and (B) differ. The constant κ_5 can vanish (Remark 6.5).

Theorem 5 (row k_{818}). *Let $a > b > 0$, let the caustic be a confocal ellipse, let $N \geq 3$ be the least period of the trajectories of the family and τ their winding number (any τ coprime to N), and use signed areas. Let $N \equiv 2 \pmod{4}$, put $\alpha = a^2$, $\beta = b^2$ and*

$$R := \frac{\alpha\beta(3\beta - \alpha) + \lambda(\alpha - \beta)^2}{2\alpha\beta(\beta - \lambda)}.$$

- (a) $A_{\text{ant}} = RA$ and $A^\dagger = \mu A$ at every member, with a constant $\mu > 0$ depending only on a, b, λ .
- (b) If $R \neq 0$, then $A_{\text{ant}} \neq 0$ at every member, and $A_j^\dagger/A_{j,\text{ant}} = \mu/R$ along the family, for both foci.
- (c) $R = 0$ if and only if $\lambda(\alpha - \beta)^2 = \alpha\beta(\alpha - 3\beta)$; then $A_{\text{ant}} = 0$ at every member, and the quotient of the row is undefined at every member. This requires $b/a < 1/\sqrt{3}$. For $N = 6$ the caustic is $\lambda = a^2b^2/(a+b)^2$ and $R = (2b-a)(a+b)^2/(b^2(2a+b))$, so for $N = 6$ the case $R = 0$ occurs exactly when $b/a = \frac{1}{2}$.

Part (a) is the quotient of two published identities of the author: $A_{\text{ant}} = RA$ for every even N is [Fer26d, Lemma 4], and that $A/A^\dagger$ is a positive constant for $N \equiv 2 \pmod{4}$ is [Fer26c, Theorem 1]. So Theorem 5(a), (b) is a scoped corollary of these two deposits; we give an independent proof of the two proportionalities with the method of this note in Section 7. Numerically, for winding number 1 and $a = 1$, R vanishes at $b = 0.5543486017\dots$ ($N = 10$), $b = 0.5661108682\dots$ ($N = 14$) and $b = 0.5706649669\dots$ ($N = 18$) (Section 8); on these families, too, the quotient of row k_{818} is undefined at every member.

Table 2 summarises the answers.

row	reading	answer	where
k_{813}	—	constant for every N ; $2a^4/b^4$ for $N = 4$	Theorem 1
k_{814}	either dual	constant for every N ; $2a^6/b^{10}$ for $N = 4$ (reciprocal dual)	Theorem 2
k_{815}	(i)	false: exact certificate for $N = 3$; varies for every odd N tested	Theorem 3(A)
k_{815}	(ii)	false for every N	Theorem 3(A)
k_{815}	(iii)	constant for every odd N	Theorem 3(B)
k_{816}	(i)	false for $N = 4$ (exact); constant for every even $N \geq 6$	Theorem 4(A), (B)
k_{816}	(ii)	constant for every even N	Theorem 2
k_{816}	(iii)	constant for every even N	Theorem 4(C)
k_{818}	—	constant if $R \neq 0$; undefined at every member if $R = 0$	Theorem 5

Table 2: The answers. The readings (i)–(iii) of $A_{j,\text{ped}}^\dagger$ are those of Section 1.1; the verdicts on k_{815} and k_{816} do not depend on the reading of “dual”.

1.3 What is new and what is not

The five rows, and the experimental evidence for them, are due to Reznik, Garcia and Koiller [RGK21]. The caption of their table credits row k_{814} to a private communication of A. Akopyan (the mark itself is printed at row k_{811} , and the title of the communication, “Sum of squared sidelengths of focus-polar polygon is invariant”, concerns a sum of squared side lengths, the quantity of row k_{811}). The parametrisation by Jacobi elliptic functions (Section 2) is Stachel’s canonical form [Sta22, Theorem 2]; the central symmetry of trajectories of even period, which we use, is [Sta21, Corollary 4.2(i)]; the caustic property of confocal conics is classical [Tab05]. The caustics of the families with $N = 4$ and $N = 6$, with their members through $(a, 0)$, and the values $AA^\dagger = 4$ for $N = 4$ and $A/A^\dagger = 4a^3b^4/((2a-b)(a+b)^2)$ for $N = 6$ are due to Garcia and Reznik [GR22, Section 3, Appendix B]. The polar polygons of a family form a Poncelet family of bicentric polygons whose pedal polygons with respect to the limiting point f are the inversive polygons $P^\dagger$: this is the bicentric family of Roitman, Garcia and Reznik [RGR21], who proved that its sum of cosines and the perimeter of $P^\dagger$ are invariant by complexifying the phase and applying Liouville’s theorem, which is also the method used here. Row $k_{908,b}$ is one of the source’s; Reznik, Garcia and Helman [RGH22], who study the focus-inversive triangles, state the same observation (Section 3 there).

Row k_{818} follows from the deposits [Fer26c, Fer26d] of the author, as explained after Theorem 5; the product row k_{817} was obtained from [Fer26d] and a further deposit in the same way in [Fer26f]. The families on which the focal antipedal area vanishes identically appear in the deposits [Fer26a] (Table 2 there: $N = 6$ at $b/a = \frac{1}{2}$ and $N = 10$ at $b/a = 0.5543486017\dots$, with $N = 8, 12, 16$; obtained numerically) and [Fer26i], which credits an earlier observation of a zero-area antipedal polygon for $N = 6$ at aspect ratio 2. The proposers had observed this case: the list of animations of [RGK21, v11] has one titled “ $N = 6$, $a/b = 2$ antipedal polygon has zero signed area”, and that of [GR22] one titled “ $N=6$ zero-area antipedal @ $a/b = 2$ ”; the titles do not name the point, which [Fer26i] identifies from the animation as a focus. For k_{818} we add only the independent proof and the statements for $N = 6$, which combine the caustic and the value of $A/A^\dagger$ given in [GR22] with [Fer26d, Lemma 4]. Related deposits of the author are [Fer26b, Fer26e] on row k_{908} , which use reading (i) of $A_{j,ped}^\dagger$, [Fer26h] on row k_{811} , which also shows that the point-wise dual is the midpoint polygon of the reciprocal dual (we use this in the proof of Theorem 2(b)), and [Fer26g], which applies the same Jacobi method to pedal areas.

To our knowledge the following is new: the proofs of rows k_{813} and k_{814} for every N (Theorems 1 and 2, including the point-wise reading), the analysis of rows k_{815} and k_{816} under the three readings with the exact certificates (Theorems 3 and 4), and the independent proof of the proportionalities behind row k_{818} . The searches on which this is based are described in the paragraph “Scope and priority” in Section 8; we make no priority claim.

1.4 Conventions

Least period. “ N -periodic” means least period N : in the source each N has its own family, and its column “which N ” is meaningless for a trajectory of smaller period listed several times (a control in Section 8 shows that a 3-periodic listed twice does not satisfy the even- N rows). Trajectories with winding number $\tau > 1$ (star polygons) are included.

Orientation and foci. Reversing the order of the vertices changes the sign of every signed area and leaves ratios and products of two areas unchanged; we use the counterclockwise order. The reflection $(x, y) \mapsto (-x, y)$ maps $\mathcal{E}$, $\mathcal{C}$ and the family to themselves, exchanges the foci, commutes with all constructions and changes the sign of every signed area; so each row holds for one focus if and only if it holds for the other. We work with $f = (-c, 0)$.

Signed areas, ratios. Antipedal polygons may be self-intersecting, as the source notes, and signed

areas may vanish; a ratio is considered at the members where its denominator is nonzero. The areas A , $A^\dagger$, A_{pol} , A_{dual} , $A_{\text{dual}}^{\text{pt}}$ and $\bar{A}$ are positive at every member (Lemma 3.2).

Caustic. The caustic is a confocal ellipse, as in the source; hyperbolic caustics are not treated.

Versions of the source. The published version of [RGK21], titled “Fifty new invariants”, lists 56 invariants; its table of inversive objects has only the rows k_{801} – k_{804} (sums of inverse focal distances, perimeter and sum of cosines of the inversive polygon). The five rows, and the polar and dual polygons, occur only in the arXiv version, whose row codes the corpus uses.

2 The parametrisation

Throughout, $\alpha = a^2$, $\beta = b^2$, $0 < \lambda < \beta$, $c = \sqrt{\alpha - \beta}$, $e = c/a$, and

$$k^2 = \frac{\alpha - \beta}{\alpha - \lambda} \in (0, 1), \quad k'^2 = 1 - k^2, \quad K = K(k), \quad K' = K(k').$$

The functions sn , cn , dn have modulus k . For $x, y \in \mathbb{C}^2$ we write $x \cdot y = x_1 y_1 + x_2 y_2$ (bilinear, also for complex vectors), $x \times y = x_1 y_2 - x_2 y_1$ and $x^\perp = (-x_2, x_1)$, so that $x^\perp \cdot y^\perp = x \cdot y$ and $x^\perp \cdot y = x \times y$. A vector $\nu \neq 0$ with $\nu \cdot \nu = 0$ is *isotropic*; then ν is a multiple of $(1, \pm i)$, and $x \cdot \nu = 0$ implies $x \in \mathbb{C}\nu$. We use the following facts [DLM26, §§22.2, 22.4, 22.6, 22.8, 22.16].

(J1) sn , cn , dn are meromorphic, with simple poles exactly at $\Pi := iK' + 2K\mathbb{Z} + 2iK'\mathbb{Z}$; dn has simple zeros exactly at $\Pi' := K + iK' + 2K\mathbb{Z} + 2iK'\mathbb{Z}$; $4K$ and $4iK'$ are periods of all three.

(J2) $\text{sn}(w + 2K) = -\text{sn } w$, $\text{cn}(w + 2K) = -\text{cn } w$, $\text{sn}(w + 2iK') = \text{sn } w$, $\text{cn}(w + 2iK') = -\text{cn } w$, $\text{dn}(w + 2iK') = -\text{dn } w$, and

$$\begin{aligned} \text{sn}(w + iK') &= \frac{1}{k \text{sn } w}, & \text{cn}(w + iK') &= \frac{-i \text{dn } w}{k \text{sn } w}, & \text{dn}(w + iK') &= \frac{-i \text{cn } w}{\text{sn } w}, \\ \text{sn}(w + K + iK') &= \frac{\text{dn } w}{k \text{cn } w}, & \text{cn}(w + K + iK') &= \frac{-ik'}{k \text{cn } w}. \end{aligned}$$

(J3) $\text{sn}^2 + \text{cn}^2 = 1$, $\text{dn}^2 + k^2 \text{sn}^2 = 1$, and with $D = 1 - k^2 \text{sn}^2 t \text{sn}^2 \Delta$,

$$\text{sn}(t \pm \Delta) = \frac{\text{sn } t \text{cn } \Delta \text{dn } \Delta \pm \text{sn } \Delta \text{cn } t \text{dn } t}{D}, \quad \text{cn}(t \pm \Delta) = \frac{\text{cn } t \text{cn } \Delta \mp \text{sn } t \text{sn } \Delta \text{dn } t \text{dn } \Delta}{D}.$$

(J4) For real u , $\text{sn } u = \sin \text{am } u$ and $\text{cn } u = \cos \text{am } u$, where am is real analytic, $\text{am}' = \text{dn} > 0$ on $\mathbb{R}$, and $\text{am}(u + 2K) = \text{am } u + \pi$.

Let $\Delta \in (0, K)$ be the number with $\text{sn}^2 \Delta = \lambda/\beta$, put $s_0 = \text{sn } \Delta$, $c_0 = \text{cn } \Delta$, $d_0 = \text{dn } \Delta$ (all positive), $h := 2\Delta$ and $D(t) := 1 - k^2 s_0^2 \text{sn}^2 t$. Then

$$c_0^2 = \frac{\beta - \lambda}{\beta}, \quad d_0^2 = \frac{\alpha(\beta - \lambda)}{\beta(\alpha - \lambda)}, \quad \frac{c d_0}{a c_0} = k, \quad \beta d_0^2 = \alpha(1 - k^2). \quad (1)$$

For $u, t, v \in \mathbb{C}$ define

$$P(u) := (-a \text{sn } u, b \text{cn } u), \quad m(t) := \left(-\frac{d_0 \text{sn } t}{a c_0}, \frac{\text{cn } t}{b c_0} \right), \quad n(v) := \left(-\frac{\text{sn } v}{a}, \frac{\text{cn } v}{b} \right).$$

For real v the point $P(v)$ lies on $\mathcal{E}$ and $\{X : n(v) \cdot X = 1\}$ is the tangent of $\mathcal{E}$ at $P(v)$. A line $px + qy = 1$ touches $\mathcal{C}$ if and only if $(\alpha - \lambda)p^2 + (\beta - \lambda)q^2 = 1$.

Lemma 2.1 (chords and trajectories). (a) $m(t) \cdot P(t - \Delta) = m(t) \cdot P(t + \Delta) = 1$ for all $t \in \mathbb{C}$. For real t the line $L(t) := \{X : m(t) \cdot X = 1\}$ passes through the two distinct points $P(t \pm \Delta)$ of $\mathcal{E}$ and is tangent to $\mathcal{C}$.

(b) For real u the points $P_l := P(u + lh)$, $l \in \mathbb{Z}$, form a billiard trajectory with caustic $\mathcal{C}$ which runs counterclockwise around O ; the chord $P_l P_{l+1}$ lies on $L(u + lh + \Delta)$. It is N -periodic if and only if $Nh = 4\tau K$ with an integer τ coprime to N ; then $0 < \tau < N/2$ is its winding number. Every billiard trajectory with caustic $\mathcal{C}$ is of this form after a reversal of its order.

Proof. (a) With $s = \text{sn } t$, $c_t = \text{cn } t$, $d_t = \text{dn } t$, (J3) gives $c_0 m(t) \cdot P(t \pm \Delta) = [d_0 s(sc_0 d_0 \pm s_0 c_t d_t) + c_t(c_t c_0 \mp s s_0 d_t d_0)]/D = c_0(s^2 d_0^2 + c_t^2)/D = c_0$, because $s^2 d_0^2 + c_t^2 = 1 - k^2 s_0^2 s^2$. For real t , $P(t + \Delta) = P(t - \Delta)$ would force $2\Delta \in 4K\mathbb{Z}$ by (J4). Tangency: $(\alpha - \lambda)d_0^2 s^2/(\alpha c_0^2) + (\beta - \lambda)c_t^2/(\beta c_0^2) = 1$, since $(\alpha - \lambda)d_0^2/\alpha = (\beta - \lambda)/\beta = c_0^2$ by (1). (b) By (a), the two chords at P_l lie on $L(u + lh \pm \Delta)$; these are distinct tangents from P_l to $\mathcal{C}$ (otherwise one line would contain three distinct points of $\mathcal{E}$), so they are the two tangents from P_l to $\mathcal{C}$, and these are mirror images of each other in the normal of $\mathcal{E}$ at P_l , by the classical reflection property of confocal conics [Tab05] (a direct verification in this parametrisation is [Fer26g, Lemma 2.2]). By (J4) the eccentric angle of P_l increases with l , by less than π per step since $0 < h < 2K$. $P_{l+N} = P_l$ for all l if and only if $Nh \in 4K\mathbb{Z}$; if $Nh = 4\tau K$ and $g = \gcd(\tau, N) > 1$, then N/g is a period, and if $\gcd(\tau, N) = 1$, every period is a multiple of N . The eccentric angle increases by $2\pi\tau$ along a period, and $\tau/N = \Delta/(2K) < \frac{1}{2}$. Finally, if (P_l) is any trajectory with caustic $\mathcal{C}$ and $P_0 = P(u)$, u real, then $P_1 = P(u \pm h)$, as the chord $P_0 P_1$ is one of the two tangents from P_0 ; after a reversal $P_1 = P(u + h)$, and the reflection law and induction give $P_l = P(u + lh)$ for all l . $\square$

Part (b) is Stachel's canonical parametrisation [Sta22, Theorem 2]: the modulus is the numerical eccentricity of $\mathcal{C}$, the vertex parameters advance by 2Δ , and $\Delta = 2\tau K/N$ for N -periodics. From now on N and τ are fixed with $Nh = 4\tau K$, $\gcd(\tau, N) = 1$, and for $u \in \mathbb{C}$ we put

$$u_l := u + lh, \quad t_l := u_l + \Delta, \quad P_l := P(u_l), \quad m_l := m(t_l), \quad n_l := n(u_l) \quad (l \in \mathbb{Z}/N);$$

these depend only on l modulo N , as P , m , n have the period $4K$. By Lemma 2.1, the family is $\{(P_l)_l : u \in \mathbb{R}\}$, and a quantity is invariant if and only if it is constant as a function of the real phase u .

Lemma 2.2 (the focus). Let $f = (-c, 0)$ and $r(u) := a - c \text{sn } u$.

(a) $m(t) \cdot f = k \text{sn } t$ and $n(v) \cdot f = e \text{sn } v$.

(b) $(P(u) - f) \cdot (P(u) - f) = r(u)^2$; for real u , $r(u) = |P(u) - f| \geq a - c > 0$.

(c) r has simple zeros exactly at the points of $\Xi_1 := \{K - \Delta + iK', K + \Delta + iK'\} + 4K\mathbb{Z} + 2iK'\mathbb{Z}$, and $P(u) - f$ is a nonzero isotropic vector there; $r = 2a$ on $\Xi_2 := \Xi_1 + 2K$.

(d) $1 - k \text{sn } t$ has double zeros exactly at the points of $\Pi'_1 := K + iK' + 4K\mathbb{Z} + 2iK'\mathbb{Z}$, and equals 2 on $\Pi'_2 := \Pi'_1 + 2K$; $\Pi' = \Pi'_1 \cup \Pi'_2$.

(e) $\Xi_1 = \Pi'_1 \pm \Delta$ and $\Xi_2 = \Pi'_2 \pm \Delta$.

Proof. (a) $m \cdot f = c d_0 \text{sn } t/(ac_0) = k \text{sn } t$ by (1), and $n \cdot f = c \text{sn } v/a$. (b) For $(x, y) \in \mathbb{C}^2$ with $y^2 = \beta(1 - x^2/\alpha)$ one has $(x+c)^2 + y^2 = (a+ex)^2$; put $x = -a \text{sn } u$. (c) $r = 0$ means $\text{sn } u = a/c$. By (J2), $\text{sn}(K + iK' \pm \Delta) = d_0/(kc_0) = a/c$, by (1). The function sn is elliptic of order 2 for the periods $4K$, $2iK'$, and $K + iK' \pm \Delta$ are not congruent modulo these periods ($2\Delta \notin 4K\mathbb{Z}$), so these are all the solutions, and they are simple. There $P - f$ is isotropic by (b) and nonzero, as its first coordinate is $c - a^2/c \neq 0$.

On Ξ_2 , $\text{sn} = -a/c$ by (J2). (d) By (J2), $\text{sn}(K + iK' + w) = \text{dn } w / (k \text{cn } w) = (1 + \frac{1}{2}k'^2 w^2 + O(w^4))/k$; so $1 - k \text{sn}$ has a double zero at $K + iK'$ and, sn being of order 2, no other zero modulo $4K\mathbb{Z} + 2iK'\mathbb{Z}$. On Π'_2 , $\text{sn} = -1/k$. (e) is the computation in (c). $\square$

At $t \in \Pi'_1$ the line $L(t)$ passes through f (by (a)) and has an isotropic normal ($m \cdot m = 0$ by Lemma 2.3(iv)): it is one of the two isotropic lines through f , which touch every conic with focus f . Its two “endpoints” $P(t \pm \Delta)$ coincide (Lemma 2.4(b)) at the point of contact with $\mathcal{E}$.

Lemma 2.3 (closed forms). *For all $t \in \mathbb{C}$, with f as in Lemma 2.2,*

- (i) $P(t - \Delta) \times P(t + \Delta) = 2ab s_0 c_0 \text{dn } t / D(t)$;
- (ii) $(P(t - \Delta) - f) \times (P(t + \Delta) - f) = 2ab s_0 c_0 \text{dn } t (1 - k \text{sn } t) / D(t)$;
- (iii) $(P(t + \Delta) - P(t - \Delta)) \cdot (P(t + \Delta) - P(t - \Delta)) = 4\alpha s_0^2 \text{dn}^4 t / D(t)^2$;
- (iv) $m(t) \cdot m(t) = \text{dn}^2 t / (\beta c_0^2)$.

For real $v < w$ with $w - v < 2K$: $P(v) \times P(w)$, $m(v) \times m(w)$ and $n(v) \times n(w)$ are positive.

Proof. Write $\text{sn}_\pm = \text{sn}(t \pm \Delta)$ etc. By (J3), $\text{sn}_+ \text{cn}_- - \text{sn}_- \text{cn}_+ = 2s_0 c_0 \text{dn } t / D$, which is (i) since $P(t - \Delta) \times P(t + \Delta) = ab(\text{sn}_+ \text{cn}_- - \text{sn}_- \text{cn}_+)$. For (ii) add $f \times (P(t - \Delta) - P(t + \Delta)) = -bc(\text{cn}_- - \text{cn}_+) = -2bc s_0 d_0 \text{sn } t \text{dn } t / D$ and use $cd_0/(ac_0) = k$. For (iii), $P(t + \Delta) - P(t - \Delta) = -(2s_0 \text{dn } t / D)(a \text{cn } t, b d_0 \text{sn } t)$ and $\alpha \text{cn}^2 t + \beta d_0^2 \text{sn}^2 t = \alpha \text{dn}^2 t$ by (1). (iv): $c_0^2 m \cdot m = d_0^2 \text{sn}^2 t / \alpha + \text{cn}^2 t / \beta = \text{dn}^2 t / \beta$. Finally, each of the three cross products is a positive constant times $\text{sn } w \text{cn } v - \text{sn } v \text{cn } w = \sin(\text{am } w - \text{am } v)$, and $0 < \text{am } w - \text{am } v < \pi$ by (J4). $\square$

Lemma 2.4 (parities and shifts). *Let ϕ be one of P , m , n , and $R := \text{diag}(1, -1)$.*

- (a) For $z \in \Pi$: $\phi(z + w) = -\phi(z - w)$.
- (b) For $Z \in \Pi'$: $\phi(Z + w) = \phi(Z - w)$ and $\text{sn}(Z + w) = \text{sn}(Z - w)$.
- (c) $\phi(w + 2K) = -\phi(w)$, $\phi(w + 2iK') = R\phi(w)$, and sn is unchanged under $w \mapsto w + 2iK'$.

Moreover ϕ is holomorphic outside Π , with simple poles at Π .

Proof. Each ϕ is $(c_1 \text{sn}, c_2 \text{cn})$ with constants $c_1 c_2 \neq 0$. By (J2), $\text{sn}(iK' + w)$ and $\text{cn}(iK' + w)$ are odd in w , and $\text{sn}(K + iK' + w)$, $\text{cn}(K + iK' + w)$ are even in w ; a translation by $2pK + 2qiK'$ multiplies sn by $(-1)^p$ and cn by $(-1)^{p+q}$, which gives (a) and (b) on all of Π and Π' . (c) is (J2), and the last statement is (J1). $\square$

3 The derived polygons as functions of the phase

Signed areas do not change under translations, so we compute all polygons in coordinates centred at $f = (-c, 0)$. Put $p_l := P_l - f$, $r_l := r(u_l)$, $z_p := (-c/\beta, 0)$, $R_* := a/\beta$, and define, for $u \in \mathbb{C}$ and $l \in \mathbb{Z}/N$,

$$\begin{aligned} X_l &:= \frac{p_l}{r_l^2}, & Y_l &:= \frac{m_l}{1 - k \text{sn } t_l}, & T_l &:= \frac{n_l}{1 - e \text{sn } u_l}, & D_l &:= z_p + \frac{T_l - z_p}{R_*^2}, \\ V_l &:= \frac{1}{2}(D_l + D_{l+1}), & v_l &:= \frac{(1 - k \text{sn } t_l) m_l}{m_l \cdot m_l}, & w_l &:= \frac{r_{l+1}^2 p_l^\perp - r_l^2 p_{l+1}^\perp}{p_l \times p_{l+1}}, & q_l &:= \frac{w_l}{w_l \cdot w_l}. \end{aligned}$$

With $S(W) = \frac{1}{2} \sum_l W_l \times W_{l+1}$ we set $A := S(p)$, $A^\dagger := S(X)$, $A_{\text{pol}} := S(Y)$, $A_T := S(T)$, $A_{\text{dual}} := S(D)$, $A_{\text{dual}}^{\text{pt}} := S(V)$, $\bar{A} := S(v)$, $A_{\text{ant}} := S(w)$ and $A_{\text{ped}}^\dagger := S(q)$. All are meromorphic functions of $u \in \mathbb{C}$, and $A_{\text{dual}} = A_T / R_*^4$, since D is the image of T under a homothety of ratio R_*^{-2} .

Lemma 3.1 (identification). *Let u be real, and let $\nu_l := p_l/r_l$ be the unit vector from f to P_l .*

- (a) X_l is the inversion of P_l in the unit circle at f .
- (b) Y_l is the pole, with respect to the unit circle at f , of the side line $P_l P_{l+1}$; it is also the intersection of the lines through X_l and X_{l+1} perpendicular to fX_l and fX_{l+1} , and the inversion of ν_l , which is the foot of f on the line $P_l P_{l+1}$. So (Y_l) is the polar polygon of the source, and the inversion of the focal pedal polygon is the polar polygon.
- (c) The side $Y_{l-1}Y_l$ lies on the polar line $\ell_l = \{Z : p_l \cdot Z = 1\}$ of P_l , and $\ell_l = \{Z : (Z - z_p) \cdot \nu_l = R_*\}$. So all sides of the polar polygon touch the circle with centre z_p and radius R_* , at the points $T_l = z_p + R_* \nu_l$, which are the poles of the tangents of $\mathcal{E}$ at the P_l . In the original coordinates: $O_p = (1 + b^{-2})f$ and the polar caustic has radius a/b^2 .
- (d) $D_l = z_p + \nu_l/R_*$ is the pole of ℓ_l with respect to the unit circle at O_p ; so (D_l) is the dual polygon of Section 1.1, inscribed in the circle with centre O_p and radius b^2/a . The inversion of Y_l in the unit circle at O_p is $V_l = \frac{1}{2}(D_l + D_{l+1})$; so (V_l) is the point-wise dual polygon, the midpoint polygon of the dual polygon.
- (e) w_l is the vertex of the focal antipedal polygon between P_l and P_{l+1} ; it is the pole of the line $X_l X_{l+1}$ with respect to the unit circle at f , and q_l is the foot of f on that line. So (q_l) is the pedal polygon of $P^\dagger$ with respect to f , and $A_{\text{ped}}^\dagger$ is the area of reading (i).

Proof. We use that the polar of the point with position vector x (relative to f) with respect to the unit circle at f is the line $\{Z : x \cdot Z = 1\}$, so that the pole of a line $\{Z : y \cdot Z = 1\}$ is y , and that the foot of f on this line is $y/(y \cdot y)$, the inversion of the pole. (a) holds as $p_l \cdot p_l = r_l^2$. (b) By Lemma 2.1(a) and Lemma 2.2(a), the side line $L(t_l) = \{X : m_l \cdot X = 1\}$ is $\{Z : m_l \cdot Z = 1 - k \operatorname{sn} t_l\} = \{Z : Y_l \cdot Z = 1\}$, with pole Y_l and foot $Y_l/(Y_l \cdot Y_l) = \nu_l$. The line through X_l perpendicular to fX_l is $\{Z : X_l \cdot Z = X_l \cdot X_l\} = \{Z : p_l \cdot Z = 1\}$, the polar ℓ_l of P_l ; the polars of P_l and P_{l+1} meet at the pole of the line $P_l P_{l+1}$ (La Hire), which is Y_l . (c) With $P_l = (x, y)$ we have $p_l \cdot z_p = -c(x + c)/\beta$, so $1 - p_l \cdot z_p = (\alpha + cx)/\beta = a r_l/\beta$, as $r_l = a + ex$ (Lemma 2.2(b)). For $Z \in \ell_l$ this gives $(Z - z_p) \cdot \nu_l = (1 - z_p \cdot p_l)/r_l = a/\beta = R_*$. The point of contact is the foot $z_p + R_* \nu_l$ of z_p on ℓ_l . The tangent of $\mathcal{E}$ at P_l is $\{Z : n_l \cdot Z = 1 - e \operatorname{sn} u_l\} = \{Z : T_l \cdot Z = 1\}$, with pole T_l ; T_l lies on ℓ_l , since $T_l \cdot p_l = (n_l \cdot P_l - n_l \cdot f)/(1 - e \operatorname{sn} u_l) = 1$, and a direct computation with $p_l = (c - a \operatorname{sn} u_l, b \operatorname{cn} u_l)$ gives $(1 - e \operatorname{sn} u_l)(T_l - z_p) \times p_l = \operatorname{sn} u_l \operatorname{cn} u_l ((a - ce)/b - b/a) = 0$, because $a - ce = (a^2 - c^2)/a = b^2/a$. Hence $T_l = z_p + R_* \nu_l$. (d) The pole of $\{(Z - z_p) \cdot \nu_l = R_*\}$ with respect to the unit circle at z_p is $z_p + \nu_l/R_* = z_p + (T_l - z_p)/R_*^2 = D_l$. The angle θ_l from ν_l to ν_{l+1} lies in $(0, \pi)$, since $p_l \times p_{l+1} > 0$ by Lemma 2.3(ii) and (J4) (dn , D , $1 - k \operatorname{sn}$ are positive on $\mathbb{R}$). Solving the equations of ℓ_l and ℓ_{l+1} gives $Y_l - z_p = R_*(\nu_l + \nu_{l+1})/(1 + \nu_l \cdot \nu_{l+1})$ with $1 + \nu_l \cdot \nu_{l+1} = 1 + \cos \theta_l > 0$, hence $|Y_l - z_p|^2 = 2R_*^2/(1 + \nu_l \cdot \nu_{l+1})$ and $z_p + (Y_l - z_p)/|Y_l - z_p|^2 = z_p + (\nu_l + \nu_{l+1})/(2R_*) = V_l$. (e) The line through P_l perpendicular to fP_l is $\{Z : p_l \cdot Z = r_l^2\} = \{Z : X_l \cdot Z = 1\}$, the polar of X_l . Using $p^\perp \cdot p = 0$ and $p_{l+1}^\perp \cdot p_l = -p_l \times p_{l+1}$, one checks $w_l \cdot p_l = r_l^2$ and $w_l \cdot p_{l+1} = r_{l+1}^2$. So w_l is the antipedal vertex and the pole of $X_l X_{l+1}$ (La Hire), and $q_l = w_l/(w_l \cdot w_l)$ is the foot of f on $X_l X_{l+1} = \{Z : w_l \cdot Z = 1\}$. $\square$

That the point-wise dual is the midpoint polygon of the reciprocal dual is observed, for bicentric polygons in general, in [Fer26h]. The polar polygon is a bicentric polygon: its vertices lie on the polar reciprocal of $\mathcal{C}$ and its sides touch that of $\mathcal{E}$, both circles because f is a focus; this is the bicentric family of [RGR21], and $P^\dagger$ is its pedal polygon with respect to the limiting point f .

Lemma 3.2 (positivity). *For every real phase, A , $A^\dagger$, A_{pol} , A_T , A_{dual} , $A_{\text{dual}}^{\text{pt}}$ and $\bar{A}$ are positive. More precisely, with $\theta_l \in (0, \pi)$ the angle $P_l f P_{l+1}$,*

$$A_{\text{dual}} = \frac{b^4}{2a^2} \sum_l \sin \theta_l, \quad A_{\text{dual}}^{\text{pt}} = \frac{b^4}{2a^2} \sum_l \cos \frac{\theta_l}{2} \cos \frac{\theta_{l+1}}{2} \sin \frac{\theta_l + \theta_{l+1}}{2}.$$

Proof. Since $0 < h < 2K$, the last statement of Lemma 2.3 gives $P_l \times P_{l+1} > 0$, $m_l \times m_{l+1} > 0$, and $p_l \times p_{l+1} > 0$ by Lemma 2.3(ii). Hence $A > 0$, $A^\dagger = \frac{1}{2} \sum_l (p_l \times p_{l+1}) / (r_l^2 r_{l+1}^2) > 0$, $A_{\text{pol}} = \frac{1}{2} \sum_l (m_l \times m_{l+1}) / ((1 - k \operatorname{sn} t_l)(1 - k \operatorname{sn} t_{l+1})) > 0$ and $\bar{A} > 0$, as $1 - k \operatorname{sn}$ and $m \cdot m$ are positive on $\mathbb{R}$. By Lemma 3.1(d), $D_l - z_p = \nu_l / R_*$ and $V_l - z_p = (\nu_l + \nu_{l+1}) / (2R_*)$, with $R_*^{-2} = b^4 / a^2$; so $A_{\text{dual}} = \frac{1}{2R_*^2} \sum \nu_l \times \nu_{l+1}$ and, by $\sin x + \sin y + \sin(x+y) = 4 \cos \frac{x}{2} \cos \frac{y}{2} \sin \frac{x+y}{2}$, $(\nu_l + \nu_{l+1}) \times (\nu_{l+1} + \nu_{l+2}) = 4 \cos \frac{\theta_l}{2} \cos \frac{\theta_{l+1}}{2} \sin \frac{\theta_l + \theta_{l+1}}{2}$. All terms are positive, and $A_T = R_*^4 A_{\text{dual}}$. $\square$

Let $\mathcal{F}$ be the set of meromorphic functions F on $\mathbb{C}$ with

$$F(u+h) = F(u), \quad F(u+4K) = F(u), \quad F(u+2iK') = -F(u).$$

Lemma 3.3 (periods and signs). *All nine areas defined above lie in $\mathcal{F}$. If N is even, they are also invariant under $u \mapsto u + 2K$, and $P_{l+N/2} = -P_l$.*

Proof. $u \mapsto u+h$ shifts the indices. P , m , n , sn have the period $4K$. By Lemma 2.4(c), $u \mapsto u+2iK'$ replaces P_l , m_l , n_l by RP_l , Rm_l , Rn_l and leaves $\operatorname{sn} u_l$, $\operatorname{sn} t_l$, r_l unchanged; R fixes f and z_p , preserves dot products, satisfies $(Rx)^\perp = -R(x^\perp)$ and $Rx \times Ry = -x \times y$, so every derived vertex is replaced by its image under R and every area changes sign. If N is even, τ is odd, so $\frac{N}{2}h = 2\tau K \in 2K + 4K\mathbb{Z}$ and $u \mapsto u+2K$ shifts the indices by $N/2$; with Lemma 2.4(c) this gives $P_{l+N/2} = P(u_l + 2K) = -P_l$, the central symmetry of [Sta21, Corollary 4.2(i)]. $\square$

Lemma 3.4 (folding). *Suppose that $t_l(u_0) \in \Pi'$ for some $u_0 \in \mathbb{C}$ and some l . Then $F(u_0 - w) = -F(u_0 + w)$ for each of the nine areas F , as meromorphic functions of w . Consequently the Laurent series of F at u_0 has only odd powers of w : a pole of order at most 2 is simple, and if F is holomorphic at u_0 , then $F(u_0) = 0$.*

Proof. Let $Z := t_l(u_0)$. Then $u_{l+j}(u_0 \pm w) = Z + (j - \frac{1}{2})h \pm w$ and $t_{l+j}(u_0 \pm w) = Z + jh \pm w$, so Lemma 2.4(b) gives $P_{l+1-j}(u_0 - w) = P_{l+j}(u_0 + w)$, and likewise for n , $\operatorname{sn} u$, r , while $m_{l-j}(u_0 - w) = m_{l+j}(u_0 + w)$ and $\operatorname{sn} t_{l-j}(u_0 - w) = \operatorname{sn} t_{l+j}(u_0 + w)$. The vertices X_i , T_i , D_i are functions of the data at the vertex i , and Y_i , v_i , V_i , w_i , q_i are symmetric functions of the data at the two vertices i , $i+1$ of the side i (for w_i this is visible from its definition). So each derived polygon at $u_0 - w$ is the polygon at $u_0 + w$ listed in the reverse cyclic order ($i \mapsto 2l+1-i$ for vertices, $i \mapsto 2l-i$ for sides), and reversing the order of a closed polygon changes the sign of its signed area. The identity holds wherever both sides are defined, hence as meromorphic functions. $\square$

Remark 3.5 (phases with a vertex at infinity). If instead $u_j(u_0) \in \Pi$, Lemma 2.4(a) gives $P_{j-i}(u_0 - w) = -P_{j+i}(u_0 + w)$: the polygon is reversed and reflected in O , which exchanges the two foci. For even N the central symmetry $P_{l+N/2} = -P_l$ undoes the reflection, and the nine areas are odd at such phases as well; for odd N no such parity holds in general (the first two verification runs found nonzero even Laurent coefficients for $N = 5$ and 7). The proofs below use neither fact.

4 The singular phases

Let $G := h\mathbb{Z} + 4K\mathbb{Z} + 2iK'\mathbb{Z}$. Since $\gcd(\tau, N) = 1$, $h\mathbb{Z} + 4K\mathbb{Z} = \frac{4K}{N}\mathbb{Z}$, so $G = \frac{4K}{N}\mathbb{Z} + 2iK'\mathbb{Z}$. Put $\xi := K - \Delta + iK' \in \Xi_1$ and

$$\mathcal{I}_1 := \xi + G, \quad \mathcal{I}_2 := \xi + 2K + G, \quad \mathcal{J} := iK' + h\mathbb{Z} + 2K\mathbb{Z} + 2iK'\mathbb{Z}.$$

Lemma 4.1 (cosets). (a) $\mathcal{I}_1$ is the set of phases u at which $u_l \in \Xi_1$ for some l , equivalently $t_l \in \Pi'_1$ for some l ; likewise $\mathcal{I}_2$ with Ξ_2, Π'_2 ; and $\mathcal{J}$ is the set of u with $u_l \in \Pi$ for some l . No real phase lies in $\mathcal{I}_1 \cup \mathcal{I}_2 \cup \mathcal{J}$.

(b) $\mathcal{I}_1 = \mathcal{I}_2$ if N is even, and $\mathcal{I}_1 \cap \mathcal{I}_2 = \emptyset$ if N is odd.

(c) $\mathcal{J} \cap (\mathcal{I}_1 \cup \mathcal{I}_2) = \emptyset$ if N is odd or $N \equiv 0 \pmod{4}$, and $\mathcal{J} = \mathcal{I}_1 = \mathcal{I}_2$ if $N \equiv 2 \pmod{4}$.

(d) At every point of $\mathcal{I}_1 \cup \mathcal{I}_2$ the hypothesis of Lemma 3.4 holds.

(e) Let $u_0 \in \mathcal{I}_1$ and let l be such that $u_l(u_0) \in \xi + 4K\mathbb{Z} + 2iK'\mathbb{Z}$. Then $u_i \in \Xi_1$ exactly for $i \in \{l, l+1\}$, and $t_i \in \Pi'_1$ exactly for $i = l$. If N is even, $u_i \in \Xi_2$ exactly for $i \in \{l + \frac{N}{2}, l + \frac{N}{2} + 1\}$ and $t_i \in \Pi'_2$ exactly for $i = l + \frac{N}{2}$; if N is odd, no u_i lies in Ξ_2 and no t_i in Π'_2 . If $N \equiv 2 \pmod{4}$, $u_i \in \Pi$ exactly for $i \in \{j_1, j_2\}$, $j_1 = l - \frac{N-2}{4}$, $j_2 = l + \frac{N+2}{4}$; otherwise no u_i lies in Π . For odd N the same holds at $u_0 \in \mathcal{I}_2$ with Ξ_1, Π'_1 and Ξ_2, Π'_2 exchanged.

Proof. (a) $u_l \in \Xi_1$ means $u \in \{\xi, \xi + h\} - lh + 4K\mathbb{Z} + 2iK'\mathbb{Z} \subset \xi + G$, and conversely; $t_l \in \Pi'_1$ means $u \in \xi - lh + 4K\mathbb{Z} + 2iK'\mathbb{Z}$. The same with $+2K$ for $\mathcal{I}_2$. All three sets lie on the lines $\text{Im } u \in K' + 2K'\mathbb{Z}$. (e) $(i-l)h$ is real, and the real elements of $\Xi_1 - \xi$ are $\{0, h\} + 4K\mathbb{Z}$; $(i-l)h \in 4K\mathbb{Z}$ if and only if $N \mid i-l$, and $(i-l)h \in h + 4K\mathbb{Z}$ if and only if $N \mid i-l-1$. So $u_i \in \Xi_1$ exactly for $i \in \{l, l+1\}$, and $t_i = u_i + \Delta$ lies in $\Pi'_1 = \xi + \Delta + 4K\mathbb{Z} + 2iK'\mathbb{Z}$ exactly for $i = l$. If N is even, τ is odd and $u_{l+N/2} = u_l + 2\tau K \in u_l + 2K + 4K\mathbb{Z}$, and the same counting applies to Ξ_2, Π'_2 . If N is odd, $u_i \in \Xi_2$ would need $2\tau(i-l)$ or $2\tau(i-l-1)$ to equal N times an odd number, which is impossible by parity. Finally, $u_j \in \Pi$ means $(j-l)h + K - \Delta \in 2K\mathbb{Z}$, that is $2\tau(2(j-l)-1) = N(2s+1)$ for some integer s . This is impossible for odd N (parity) and for $N \equiv 0 \pmod{4}$ (then τ is odd, and the left side is $\equiv 2$, the right side $\equiv 0 \pmod{4}$). For $N = 2N'$ with N' odd it means $N' \mid 2(j-l)-1$, that is $j-l \equiv \frac{N'+1}{2} \pmod{N'}$, which gives j_1 and j_2 . The case $u_0 \in \mathcal{I}_2$, N odd, is the same computation shifted by $2K$. (b) $2K \in G$ if and only if $2K \in \frac{4K}{N}\mathbb{Z}$, that is, N is even. (c) $\mathcal{J}$ and $\mathcal{I}_1 \cup \mathcal{I}_2 = \xi + h\mathbb{Z} + 2K\mathbb{Z} + 2iK'\mathbb{Z}$ are cosets of $h\mathbb{Z} + 2K\mathbb{Z} + 2iK'\mathbb{Z}$, so they are equal or disjoint, and by (e) they meet if and only if $N \equiv 2 \pmod{4}$; then N is even and $\mathcal{I}_1 = \mathcal{I}_2$ by (b). (d) For $u_0 \in \mathcal{I}_1$, $t_l(u_0) \in \Pi'_1 \subset \Pi'$; likewise for $\mathcal{I}_2$. $\square$

Lemma 4.1(b), (c) is the source's column "which N ": ratio rows where the singular sets coincide, product rows where they are disjoint.

Lemma 4.2 (singular vertices). As functions of u :

- (a) X_i has poles exactly where $u_i \in \Xi_1$, of order 2, and is holomorphic (with value 0) where $u_i \in \Pi$.
- (b) Y_i has poles exactly where $t_i \in \Pi'_1$, of order 2, and is holomorphic where $t_i \in \Pi$.
- (c) T_i and D_i have poles exactly where $u_i \in \Xi_1$, simple, and are holomorphic where $u_i \in \Pi$.
- (d) v_i has poles exactly where $t_i \in \Pi'_2$, of order 2, and is holomorphic where $t_i \in \Pi'_1$ or $t_i \in \Pi$.
- (e) w_i is holomorphic except where $u_i \in \Pi$ or $u_{i+1} \in \Pi$, where its pole has order at most 1. If $t_i(u_0) \in \Pi'_1$, then $w_i(u_0 + w)$ is even in w , and $w_i(u_0) = \mu\nu$ with $\mu \neq 0$ and $\nu := p_i(u_0)$, a nonzero isotropic vector.

(f) $w_i \cdot w_i = r_i^2 r_{i+1}^2 \operatorname{dn}^2 t_i / (\beta c_0^2 (1 - k \operatorname{sn} t_i)^2)$. If none of $r_i, r_{i+1}, \operatorname{dn} t_i$ vanishes at a phase, then q_i is holomorphic there. If w_i is holomorphic at a phase and $w_i \cdot w_i$ has a zero of order m there, then q_i has a pole of order at most m . If $t_i(u_0) \in \Pi'_1$, then $w_i \cdot w_i$ has a zero of order exactly 2 at u_0 .

Proof. Near a point $z \in \Pi$ we have $P(z+w) = \rho/w + O(1)$ with $\rho \cdot \rho \neq 0$ (from (J2): $\rho = \pm(a, \pm ib)/k$, so $\rho \cdot \rho = c^2/k^2$), m and n have simple poles, and $1 - k \operatorname{sn}, 1 - e \operatorname{sn}, \operatorname{dn}$ have simple poles, while $D = 1 - k^2 s_0^2 \operatorname{sn}^2$ has a double pole. (a) r^2 has double zeros exactly on Ξ_1 , where $p \neq 0$ (Lemma 2.2(c)); at Π , $p \cdot p = r^2$ has a double pole and $X = p/r^2 = O(w)$. (b) Lemma 2.2(d), $m(K + iK') \neq 0$, and at Π both m and $1 - k \operatorname{sn}$ have simple poles. (c) $1 - e \operatorname{sn} = r/a$ has simple zeros exactly on Ξ_1 , and $n \neq 0$ there; at Π both n and $1 - e \operatorname{sn}$ have simple poles. (d) $m \cdot m = \operatorname{dn}^2 / (\beta c_0^2)$ has double zeros exactly on Π' ; on Π'_1 the factor $1 - k \operatorname{sn}$ has a double zero as well, and on Π'_2 it equals 2; at Π the quotient is bounded. (e) By Lemma 2.3(ii), $p_i \times p_{i+1} = 2ab s_0 c_0 \operatorname{dn} t_i (1 - k \operatorname{sn} t_i) / D(t_i)$ vanishes exactly where $t_i \in \Pi'$ (simply on Π'_2 , triply on Π'_1 , since $D = c_0^2$ on Π'), is finite and nonzero where $t_i \in \Pi$, and has simple poles exactly where $D(t_i) = 0$; by (J2) this means $t_i \in \Pi \pm \Delta$, that is $u_i \in \Pi$ or $u_{i+1} \in \Pi$. There the numerator of w_i is $O(w^{-2})$, so $w_i = O(w^{-1})$. Where $t_i(u_0) \in \Pi'_2$, the two endpoints coincide (Lemma 2.4(b)) and $r_i = r_{i+1} = 2a$ at u_0 , so the numerator vanishes at u_0 and w_i is holomorphic. Where $t_i(u_0) \in \Pi'_1$, put $\varphi(w) = p_i(u_0 + w)$ and $\delta(w) = r_i(u_0 + w)^2$; by Lemma 2.4(b), $p_{i+1}(u_0 + w) = \varphi(-w)$ and $r_{i+1}(u_0 + w)^2 = \delta(-w)$. So the numerator $\delta(-w)\varphi(w)^\perp - \delta(w)\varphi(-w)^\perp$ and the denominator $\varphi(w) \times \varphi(-w)$ are odd in w , and w_i is even. Writing $\delta = \delta_e + \delta_o$ and $\varphi^\perp = \varphi_e + \varphi_o$ for the even and odd parts, the numerator is $2(\delta_e \varphi_o - \delta_o \varphi_e)$; since δ has a double zero, $\delta_e = O(w^2)$ and $\delta_o = O(w^3)$, while $\varphi_o = O(w)$; so the numerator is $O(w^3)$, and the denominator has a zero of order exactly 3. Hence w_i is holomorphic at u_0 . By (f), $w_i \cdot w_i$ has a zero of order exactly 2 at u_0 ; an even function vanishing at u_0 would make it $O(w^4)$, so $w_i(u_0) \neq 0$. As $w_i \cdot p_i = r_i^2$, we get $w_i(u_0) \cdot \nu = 0$, and ν is isotropic and nonzero (Lemma 2.2(c)), so $w_i(u_0) \in \mathbb{C}\nu$. (f) Using $x^\perp \cdot y^\perp = x \cdot y$, $w_i \cdot w_i = r_i^2 r_{i+1}^2 (p_{i+1} - p_i) \cdot (p_{i+1} - p_i) / (p_i \times p_{i+1})^2$; insert Lemma 2.3(ii), (iii). At $t_i(u_0) \in \Pi'_1$ the order is $2 + 2 + 2 - 4 = 2$. If $r_i, r_{i+1}, \operatorname{dn} t_i$ do not vanish at a phase where w_i is finite, then $w_i \cdot w_i$ is finite and nonzero there (at $t_i \in \Pi$ the quotient $\operatorname{dn}^2 / (1 - k \operatorname{sn})^2$ is finite and nonzero); where $u_i \in \Pi$ (say), $w_i \cdot w_i$ has a double pole, as r_i^2 has one and the other factors are finite and nonzero ($t_i \in \Pi + \Delta$ is not in $\Pi \cup \Pi'$), so $q_i = O(w)$. The bound on the pole order is clear. $\square$

Lemma 4.3 (comparison). (a) Every entire function in $\mathcal{F}$ is 0.

(b) Let $\mathcal{C} = \xi_0 + G$ for some $\xi_0 \in \mathbb{C}$, and let $F, \Phi \in \mathcal{F}$ be holomorphic outside $\mathcal{C}$, with at most simple poles on $\mathcal{C}$, and $\Phi \neq 0$. Then $\operatorname{Res}_{\xi_0} \Phi \neq 0$, and $F = \kappa \Phi$ with $\kappa = \operatorname{Res}_{\xi_0} F / \operatorname{Res}_{\xi_0} \Phi$.

(c) If $F, \Phi \in \mathcal{F}$ and $F\Phi$ is entire, then $F\Phi$ is constant.

Proof. (a) F has the periods $4K$ and $4iK'$, so it is bounded, hence constant by Liouville's theorem, and $F(u + 2iK') = -F(u)$ gives $F = 0$. (b) For $g = jh + 4Kp + 2iK'q \in G$ we have $F(u + g) = (-1)^q F(u)$, so $\operatorname{Res}_{\xi_0 + g} F = (-1)^q \operatorname{Res}_{\xi_0} F$, and likewise for Φ . If $\operatorname{Res}_{\xi_0} \Phi = 0$, then Φ is entire, so $\Phi = 0$ by (a). The function $F - \kappa \Phi \in \mathcal{F}$ has at most simple poles, with zero residues, on $\mathcal{C}$, so it is entire and vanishes by (a). (c) $F\Phi$ has the periods $4K$ and $2iK'$, so it is bounded and constant. $\square$

Since $\mathcal{I}_1$, and for even N also $\mathcal{I}_2 = \mathcal{I}_1$ and (for $N \equiv 2 \pmod{4}$) $\mathcal{J} = \mathcal{I}_1$, are cosets of G , Lemma 4.3(b) applies to functions whose poles lie on one of them. For odd N , $\mathcal{J}$ is the union of the two cosets $iK' + G$ and $iK' + 2K + G$.

5 Rows k_{813} and k_{814}

In this section and the next, u_0 denotes a point of $\mathcal{I}_1$ and l the index of Lemma 4.1(e), so that $u_l, u_{l+1} \in \Xi_1$ and $t_l \in \Pi'_1$ at u_0 .

Proof of Theorem 1. By Lemma 3.3, $A^\dagger$ and A_{pol} lie in $\mathcal{F}$. By Lemma 4.2(a) and Lemma 4.1(a), $A^\dagger$ is holomorphic outside $\mathcal{I}_1$, and at u_0 only X_l and X_{l+1} are singular, with poles of order 2. In $2A^\dagger = \sum_i X_i \times X_{i+1}$ the terms $X_{l-1} \times X_l$ and $X_{l+1} \times X_{l+2}$ have poles of order at most 2, and $X_l \times X_{l+1} = (p_l \times p_{l+1})/(r_l^2 r_{l+1}^2)$ has order at most 1: its numerator has a zero of order 3 at u_0 (Lemma 2.3(ii), Lemma 2.2(d); $\text{dn } t_l$ has a simple and $1 - k \text{sn } t_l$ a double zero), and its denominator a zero of order 4. (For $N = 3$ the three terms are $X_{l+2} \times X_l$, $X_l \times X_{l+1}$ and $X_{l+1} \times X_{l+2}$, with the same bounds.) So $A^\dagger$ has a pole of order at most 2 at u_0 , hence at most 1 by Lemma 3.4 and Lemma 4.1(d). By Lemma 4.2(b), A_{pol} is holomorphic outside $\mathcal{I}_1$, and at u_0 only Y_l is singular (Lemma 4.1(e)), with a pole of order 2; so $2A_{\text{pol}} = Y_l \times (Y_{l+1} - Y_{l-1}) + (\text{holomorphic})$ has a pole of order at most 2, hence at most 1. As $A^\dagger > 0$ on real phases (Lemma 3.2), Lemma 4.3(b) with $\mathcal{C} = \mathcal{I}_1$ and $\Phi = A^\dagger$ gives $A_{\text{pol}} = \kappa_1 A^\dagger$, and $\kappa_1 = A_{\text{pol}}/A^\dagger > 0$ on real phases. The parametrisation, hence κ_1 , depends only on a, b, λ (which determine $\Delta/K = 2\tau/N$, hence N and τ). The value for $N = 4$ is Proposition 5.1. $\square$

Proof of Theorem 2(a). By Lemma 4.2(c), $A_T \in \mathcal{F}$ is holomorphic outside $\mathcal{I}_1$, and at u_0 exactly T_l and T_{l+1} are singular, with simple poles. So the terms $T_{l-1} \times T_l$ and $T_{l+1} \times T_{l+2}$ have poles of order at most 1 and $T_l \times T_{l+1}$ of order at most 2; hence A_T has a pole of order at most 2 at u_0 , and at most 1 by Lemma 3.4. Lemma 4.3(b) gives $A_T = \kappa A^\dagger$, and $A_{\text{dual}} = A_T/R_*^4 = \kappa_2 A^\dagger$ with $\kappa_2 = \kappa/R_*^4 > 0$ by Lemma 3.2. So $A_{\text{pol}}/A_{\text{dual}} = \kappa_1/\kappa_2$. $\square$

Proof of Theorem 2(b). By Lemma 3.1(d), $V_l = \frac{1}{2}(D_l + D_{l+1})$, so

$$2A_{\text{dual}}^{\text{pt}} = \frac{1}{4} \sum_l (D_l + D_{l+1}) \times (D_{l+1} + D_{l+2}) = A_{\text{dual}} + \frac{1}{4}B, \quad B := \sum_l D_l \times D_{l+2}.$$

For $N = 3$, $B = \sum_l D_l \times D_{l-1} = -2A_{\text{dual}}$, so $A_{\text{dual}}^{\text{pt}} = A_{\text{dual}}/4$; for $N = 4$ the terms of B cancel in pairs, so $A_{\text{dual}}^{\text{pt}} = A_{\text{dual}}/2$. Let $N \geq 4$. Like the areas, B is invariant under translations, lies in $\mathcal{F}$ (the proof of Lemma 3.3 applies), and is holomorphic outside $\mathcal{I}_1$. At u_0 only D_l and D_{l+1} are singular, with simple poles, and no term $D_i \times D_{i+2}$ contains both, since $\{i, i+2\} = \{l, l+1\}$ would need $N \mid 1$ or $N \mid 3$. So B has at most simple poles, and Lemma 4.3(b) gives $B = \kappa' A^\dagger$. Hence $A_{\text{dual}}^{\text{pt}} = \kappa_3 A^\dagger$ with $\kappa_3 = \frac{1}{2}\kappa_2 + \frac{1}{8}\kappa'$, and $\kappa_3 = A_{\text{dual}}^{\text{pt}}/A^\dagger > 0$ on real phases by Lemma 3.2. $\square$

The positivity of $A_{\text{dual}}^{\text{pt}}$ (Lemma 3.2), and therefore $\kappa_3 \neq 0$, is what makes row k_{814} hold under the point-wise reading as well; the first verification run had proved the proportionality, and all three runs found $\kappa_3 > 0$ in every billiard family they tested.

Proposition 5.1 ($N = 4$). *Let $N = 4$. Then $\lambda = a^2 b^2 / (a^2 + b^2)$, and*

$$\frac{A_{\text{pol}}}{A^\dagger} = \frac{2a^4}{b^4}, \quad \frac{A_{\text{pol}}}{A_{\text{dual}}} = \frac{2a^6}{b^{10}}, \quad \frac{\bar{A}}{A_{\text{dual}}} = \frac{2a^6}{b^2(a^2 + b^2)^2}$$

at every member of the family.

Proof. For $N = 4$ the only winding number is $\tau = 1$, so $h = K$, and the member with $u = 0$ is the rhombus $P_0 = (0, b)$, $P_1 = (-a, 0)$, $P_2 = (0, -b)$, $P_3 = (a, 0)$. Its side $x/a + y/b = 1$ touches $\mathcal{C}$ if and only if $(\alpha - \lambda)/\alpha + (\beta - \lambda)/\beta = 1$, which gives λ (this caustic and the rhombus are in [GR22, Appendix B.2]). In coordinates centred at f : $p_i = (c, b)$, $(c - a, 0)$, $(c, -b)$, $(a + c, 0)$ with $r_i = a, a - c, a, a + c$; the inversive vertices are $(c, \pm b)/a^2$ and $(\mp 1/(a \mp c), 0)$; the polar vertices, the

poles of the side lines $\pm x/a \pm y/b = 1$, are $Y_0 = \frac{1}{a-c}(-1, \frac{a}{b})$, $Y_1 = \frac{1}{a-c}(-1, -\frac{a}{b})$, $Y_2 = \frac{1}{a+c}(1, -\frac{a}{b})$, $Y_3 = \frac{1}{a+c}(1, \frac{a}{b})$; the contact points are $T_i = (0, \frac{1}{b})$, $(-\frac{1}{a-c}, 0)$, $(0, -\frac{1}{b})$, $(\frac{1}{a+c}, 0)$; and the focal feet are $v_i = Y_i/(Y_i \cdot Y_i)$. The shoelace formula gives

$$\begin{aligned} A &= 2ab, & A^\dagger &= \frac{2}{ab}, & A_{\text{pol}} &= \frac{4a^3}{b^5}, & A_T &= \frac{2a}{b^3}, \\ A_{\text{dual}} &= \frac{A_T}{R_*^4} = \frac{2b^5}{a^3}, & \bar{A} &= \frac{4a^3b^3}{(a^2+b^2)^2}, & A_{\text{ant}} &= 4ab, \end{aligned}$$

using $a^2 - c^2 = b^2$. The three ratios are constant along the family by Theorems 1 and 2 and Theorem 4(C), so their values at the rhombus are their values at every member. $\square$

At the rhombus, $AA^\dagger = 4$, the value of the source's row $k_{804,b}$, given for $N = 4$ in [GR22, Section 3.2], and $A_{\text{ant}}/A = 2$, which is R of Theorem 5 at $\lambda = a^2b^2/(a^2+b^2)$; by [Fer26d, Lemma 4], $A_{\text{ant}} = 2A$ along the whole family. For $N = 3$, $a^2 = 8$, $b^2 = 3$, the exact program of this note gives $A_{\text{pol}}/A^\dagger = 512/3$ and $A_{\text{pol}}/A_{\text{dual}} = 2560/81$ at the members M_1, M_2 of Theorem 3, and $A_{\text{dual}}^{\text{pt}} = A_{\text{dual}}/4$.

6 Rows k_{815} and k_{816}

Lemma 6.1 (focal pedal area). $\bar{A} \in \mathcal{F}$ is holomorphic outside $\mathcal{I}_2$ and has at most simple poles on $\mathcal{I}_2$.

Proof. By Lemma 4.2(d), v_i is singular only where $t_i \in \Pi'_2$, that is on $\mathcal{I}_2$. At $u_1 \in \mathcal{I}_2$ exactly one side parameter t_{i_0} lies in Π'_2 (Lemma 4.1(e); for even N , $i_0 = l + \frac{N}{2}$), v_{i_0} has a pole of order 2, and $v_{i_0 \pm 1}$ are holomorphic. So $2\bar{A} = v_{i_0} \times (v_{i_0+1} - v_{i_0-1}) + (\text{holomorphic})$ has a pole of order at most 2, hence at most 1 by Lemma 3.4. $\square$

Lemma 6.2 (inversive pedal area). $A_{\text{ped}}^\dagger \in \mathcal{F}$ is holomorphic outside $\mathcal{I}_1 \cup \mathcal{I}_2$. If N is odd or $N \geq 6$ is even, its poles are at most simple. If $N = 4$, its poles have order at most 3.

Proof. By Lemma 4.2(f), q_i is holomorphic wherever none of $r_i, r_{i+1}, \text{dn } t_i$ vanishes, that is, unless $u_i \in \Xi_1$, $u_{i+1} \in \Xi_1$ or $t_i \in \Pi'$; such phases lie in $\mathcal{I}_1 \cup \mathcal{I}_2$.

Odd N , $u_1 \in \mathcal{I}_2$. By Lemma 4.1(e) only the side i_0 with $t_{i_0} \in \Pi'_2$ has a vanishing factor ($\text{dn } t_{i_0}$, simply; its vertices lie in Ξ_2 , where $r = 2a$), no vertex is at infinity, and q_{i_0} has a pole of order at most 2. As in Lemma 6.1, $A_{\text{ped}}^\dagger$ has a pole of order at most 2, hence at most 1.

The point $u_0 \in \mathcal{I}_1$. Let $\nu := p_l(u_0) = p_{l+1}(u_0)$, a nonzero isotropic vector, and $w = u - u_0$. Call the side $i \in \{l-1, l+1\}$ *ordinary* if its two vertices are finite at u_0 and $t_i(u_0) \notin \Pi \cup \Pi'$. We claim: *if the side $i \in \{l-1, l+1\}$ is ordinary, then w_i is holomorphic at u_0 , $w_i \cdot w_i$ has a zero of order exactly 2, and $w_i = (\mu_i + \sigma_i w)\nu + O(w^2)$ with $\mu_i \neq 0$.* Indeed, exactly one of r_i, r_{i+1} vanishes, simply (r_l for $i = l-1$, r_{l+1} for $i = l+1$), and $\text{dn } t_i, 1 - k \text{sn } t_i$ are finite and nonzero, which gives the order; let i' be the other vertex of the side ($l-1$, resp. $l+2$); then $w_i \cdot p_{i'} = r_{i'}^2 \neq 0$, so $w_i(u_0) \neq 0$, while $w_i \cdot p_l = r_l^2$ (resp. $w_i \cdot p_{l+1} = r_{l+1}^2$) vanishes at u_0 , so $w_i(u_0) \cdot \nu = 0$ and $w_i(u_0) = \mu_i \nu$; writing $w_i = \mu_i \nu + w \varrho + O(w^2)$ we get $w_i \cdot w_i = 2\mu_i w \nu \cdot \varrho + O(w^2)$, so $\nu \cdot \varrho = 0$ and $\varrho \in \mathbb{C}\nu$. Together with Lemma 4.2(e) ($w_l = \mu\nu + O(w^2)$, and $w_l \cdot w_l$ has a zero of order exactly 2) this gives: *for $i, j \in \{l-1, l, l+1\}$, with the sides $l \pm 1$ among them ordinary, $w_i \times w_j = O(w^2)$ and hence $q_i \times q_j = (w_i \times w_j)/((w_i \cdot w_i)(w_j \cdot w_j)) = O(w^{-2})$, since $\nu \times \nu = 0$.*

Odd N , $u_0 \in \mathcal{I}_1$. No vertex is at infinity (Lemma 4.1(c)), no t_i lies in Π'_2 , and $t_{l \pm 1} = t_l \pm h \notin \Pi \cup \Pi'$ (this would need $h \in K\mathbb{Z}$, i.e. $N \mid 4\tau$). So the singular feet are q_{l-1}, q_l, q_{l+1} (for $N = 3$ these are all the feet), each with a pole of order at most 2, the sides $l \pm 1$ are ordinary, and every term of

$2A_{\text{ped}}^\dagger = \sum q_i \times q_{i+1}$ containing two singular feet is $O(w^{-2})$ by the claim. So $A_{\text{ped}}^\dagger$ has a pole of order at most 2 at u_0 , hence at most 1.

Even $N \geq 6$, $u_0 \in \mathcal{I}_1 = \mathcal{I}_2$. The vanishing factors occur for the sides $l-1, l, l+1$ (r_l or r_{l+1}) and $l + \frac{N}{2}$ (dn, simply; its vertices lie in Ξ_2 and are finite, as $j_1, j_2 \notin \{l + \frac{N}{2}, l + \frac{N}{2} + 1\}$). So $q_{l+N/2}$ has a pole of order at most 2, and since $N \geq 6$ its neighbours $q_{l+N/2 \pm 1}$ are not among q_{l-1}, q_l, q_{l+1} . If $N \equiv 2 \pmod{4}$, the feet of the sides containing j_1 or j_2 are holomorphic by Lemma 4.2(f), except when the other vertex of the side is l or $l+1$, which happens only for $N = 6$ ($j_1 = l-1, j_2 = l+2$). If $N \geq 8$, the vertices $l-1, l+2$ are finite and $t_{l \pm 1} \notin \Pi \cup \Pi'$ ($N \neq 4$), so the sides $l \pm 1$ are ordinary, and all terms of $2A_{\text{ped}}^\dagger$ are $O(w^{-2})$: those with two singular feet by the claim, the others because they contain one foot with a pole of order at most 2. If $N = 6$, then $w_{l-1} = \varrho/w + \sigma + O(w)$ by Lemma 4.2(e), and $w_{l-1} \cdot w_{l-1}$ is finite and nonzero at u_0 (r_{l-1}^2 has a double pole, r_l^2 a double zero, and $t_{l-1} \in \Pi + \Delta$), so q_{l-1} has a pole of order at most 1. From $w_{l-1} \cdot p_l = r_l^2 = O(w^2)$ and $p_l = \nu + O(w)$ we get $\varrho \cdot \nu = 0$, so $\varrho \in \mathbb{C}\nu$ and $w_{l-1} \times w_l = (\varrho/w) \times \mu\nu + O(1) = O(1)$; hence $q_{l-1} \times q_l = O(1)/(O(1)O(w^2)) = O(w^{-2})$. The side $l+1$ is treated in the same way, and the remaining terms contain at most one singular foot, of order at most 2. In all cases $A_{\text{ped}}^\dagger$ has a pole of order at most 2 at u_0 , hence at most 1.

$N = 4$. Then $\mathcal{J} \cap \mathcal{I}_1 = \emptyset$, all w_i are holomorphic at u_0 , and each $w_i \cdot w_i$ has a zero of order exactly 2 (for $i = l \pm 1$, $t_i \in \Pi$, where $\text{dn}^2/(1 - k \text{sn})^2$ is finite and nonzero; for $i = l+2$, $\text{dn } t_i$ has a simple zero). So each q_i has a pole of order at most 2, $A_{\text{ped}}^\dagger$ one of order at most 4, hence at most 3 by Lemma 3.4. $\square$

	M_1 ($N = 3$)	M_2 ($N = 3$)	R_1 ($N = 4$)	R_2 ($N = 4$)
A	$\frac{54\sqrt{6}}{25}$	$\frac{144}{25}$	40	$\frac{1600}{41}$
$A^\dagger$	$\frac{\sqrt{15+4\sqrt{6}}}{64}$	$\frac{9}{64}$	$\frac{1}{10}$	$\frac{41}{400}$
A_{pol}	$\frac{8\sqrt{15+32\sqrt{6}}}{3}$	24	$\frac{125}{256}$	$\frac{1025}{2048}$
A_{dual}	$\frac{27\sqrt{15+108\sqrt{6}}}{320}$	$\frac{243}{320}$	$\frac{2048}{125}$	$\frac{10496}{625}$
$\bar{A}$	$\frac{96\sqrt{6-24\sqrt{15}}}{625}$	$\frac{216}{625}$	$\frac{32000}{1681}$	$\frac{800}{41}$
A_{ant}	$36\sqrt{6} - 9\sqrt{15}$	96	80	$\frac{3200}{41}$
$A_{\text{ped}}^\dagger$	$\frac{3\sqrt{6}}{800}$	$\frac{27}{3200}$	$\frac{4096}{210125}$	$\frac{328}{15625}$

Table 3: Exact signed areas at the members of Theorems 3 and 4, with respect to the focus $(-\sqrt{5}, 0)$, resp. $(-3, 0)$; the point-wise dual areas are $A_{\text{dual}}/4$ ($N = 3$) and $A_{\text{dual}}/2$ ($N = 4$). Computed by the exact program of this note (Section 8).

Proof of Theorem 3. (A) The exact program of this note verifies that the vertices of M_1 and M_2 lie on $\mathcal{E}$ ($a^2 = 8, b^2 = 3$), that every side touches $\mathcal{C}$ with $\lambda = 72/25$, and the reflection law at every vertex; by Lemma 2.1(b), M_1 and M_2 are members of the 3-periodic family (for $N = 3$ the winding number is 1). It computes the areas of Table 3 in exact arithmetic, by the constructions of the source (inversion, feet, intersections of lines), and confirms the identifications of Lemma 3.1. From the table, $A_{\text{ped}}^\dagger A_{\text{dual}} = \frac{243\sqrt{10}}{256000} + \frac{243}{32000}$ at M_1 and $\frac{6561}{1024000}$ at M_2 , whose difference $\frac{243}{204800} + \frac{243\sqrt{10}}{256000}$ is positive; and $A_{\text{pol}} A_{\text{dual}} = \frac{27\sqrt{10}}{5} + \frac{999}{40}$ at M_1 and $\frac{729}{40}$ at M_2 . For every N , Theorems 1 and 2 give $A_{\text{pol}} A_{\text{dual}} = \kappa_1 \kappa_2 (A^\dagger)^2$. If this were constant, $A^\dagger$, which is positive on $\mathbb{R}$, would be constant on $\mathbb{R}$ and hence on $\mathbb{C}$; but $A^\dagger$ has poles, by Lemma 4.3(b) and the proof of Theorem 1. (B) Let N be odd. By Lemma 6.1 and the proof of Theorem 2(a), $\bar{A}$ and A_{dual} lie in $\mathcal{F}$, with at most simple poles, on $\mathcal{I}_2$ and $\mathcal{I}_1$ respectively, and these sets are disjoint (Lemma 4.1(b)). At $u_0 \in \mathcal{I}_1$, $\bar{A}$ is holomorphic and vanishes (Lemma 3.4), so $\bar{A} A_{\text{dual}}$ is holomorphic; at $u_1 \in \mathcal{I}_2$, A_{dual} is holomorphic and vanishes, so

the product is holomorphic. Hence $\bar{A}A_{\text{dual}}$ is entire, and constant by Lemma 4.3(c); it is positive by Lemma 3.2. Finally, $A_{\text{dual}}^{\text{pt}} = (\kappa_3/\kappa_2)A_{\text{dual}}$ is a positive constant multiple of A_{dual} , so all statements hold with $A_{\text{dual}}^{\text{pt}}$. $\square$

At M_1 and M_2 the corrected product is $\bar{A}A_{\text{dual}} = 6561/25000$, and $A_{\text{pol}}/A^\dagger = 512/3$, $A_{\text{pol}}/A_{\text{dual}} = 2560/81$ at both.

Remark 6.3 (why reading (i) fails for odd N). For odd N , Lemma 6.2 and Lemma 3.4 show that $A_{\text{ped}}^\dagger A_{\text{dual}}$ is holomorphic on $\mathcal{I}_2$, where $A_{\text{ped}}^\dagger$ has at most simple poles and A_{dual} vanishes, and outside $\mathcal{I}_1 \cup \mathcal{I}_2$. On $\mathcal{I}_1$ both factors have at most simple poles. So, by Lemma 4.3, the product is constant if and only if the residues of $A_{\text{ped}}^\dagger$ on $\mathcal{I}_1$ vanish, and otherwise it has double poles on $\mathcal{I}_1$. By Theorem 3(A) the residue is not zero for $N = 3$: the obstruction is a double pole on the f -isotropic set $\mathcal{I}_1$, not the $\mathcal{I}_2$ -poles of $A_{\text{ped}}^\dagger$. For odd $N \geq 5$ we have only numerical evidence: the verification runs computed nonzero residues of $A_{\text{ped}}^\dagger$ on $\mathcal{I}_1$ for $N = 5$ and 7 , and the product varies in every odd family tested (Section 8).

Proof of Theorem 4. (A) As in the proof of Theorem 3(A), the exact program verifies that R_1 and R_2 are members of the 4-periodic family of $a = 5$, $b = 4$, with $\lambda = 400/41$, and computes Table 3: $A_{\text{ped}}^\dagger/A_{\text{dual}} = 2/1681$ at R_1 and $1/800$ at R_2 . (The values of $A_{\text{ped}}^\dagger$ agree with the closed formula for 4-periodics in [Fer26b, Appendix].) (B) Let $N \geq 6$ be even. By Lemma 6.2, $A_{\text{ped}}^\dagger \in \mathcal{F}$ has at most simple poles, on $\mathcal{I}_1 \cup \mathcal{I}_2 = \mathcal{I}_1$, and so has A_T , with $A_T \neq 0$. Lemma 4.3(b) gives $A_{\text{ped}}^\dagger = \kappa A_T = \kappa_5 A_{\text{dual}}$ with $\kappa_5 = \kappa R_*^4$. (C) Let N be even. By Lemma 6.1, $\bar{A}$ has at most simple poles, on $\mathcal{I}_2 = \mathcal{I}_1$, and Lemma 4.3(b) gives $\bar{A} = \kappa_6 A_{\text{dual}}$, with $\kappa_6 > 0$ by Lemma 3.2; for $N = 4$ see Proposition 5.1. The statements with $A_{\text{dual}}^{\text{pt}}$ follow as in Theorem 3. $\square$

Proposition 6.4 (the pole of order three). *For $N = 4$, $A_{\text{ped}}^\dagger$ has a pole of order exactly 3 at every point of $\mathcal{I}_1$.*

Proof. The order is at most 3 (Lemma 6.2) and odd (Lemma 3.4), and it is the same at all points of $\mathcal{I}_1 = \xi + G$, as $A_{\text{ped}}^\dagger(u + g) = \pm A_{\text{ped}}^\dagger(u)$ for $g \in G$. If it were at most 1, Lemma 4.3(b) would give $A_{\text{ped}}^\dagger = \kappa A_T$, and $A_{\text{ped}}^\dagger/A_{\text{dual}}$ would be constant, contrary to Theorem 4(A). $\square$

For $N = 4$ the cancellation of the claim in the proof of Lemma 6.2 is not available for the terms containing q_{l+2} , whose antipedal vertex the first verification run found to be a multiple of the other isotropic direction; all three runs observed the order 3 numerically.

Remark 6.5 (κ_5 can vanish). Theorem 4(B) says that $A_{\text{ped}}^\dagger/A_{\text{dual}}$ is constant; the constant can be 0. The deposit [Fer26b, Theorem 1] gives an ellipse with $b = 1$, $5/2 < a < 3$, and a 10-periodic of winding number 3 in it at which $A_{\text{ped}}^\dagger = 0$. Since $A_{\text{dual}} > 0$, Theorem 4(B) gives $\kappa_5 = 0$ for that family, so $A_{\text{ped}}^\dagger$ vanishes at every member. The first verification run located the family at $a = 2.5441592248\dots$ and found $|A_{\text{ped}}^\dagger/A_{\text{dual}}| < 10^{-21}$ at four members.

7 Row k_{818}

Proof of Theorem 5. (a) *As a corollary.* Let $N \equiv 2 \pmod{4}$. The trajectories are centrally symmetric (Lemma 3.3), so [Fer26d, Lemma 4], which is stated for centrally paired elliptic-caustic trajectories, with the antipedal polygon defined by the lines $(P_i - f) \cdot (X - P_i) = 0$ and signed areas in trajectory order, gives $A_{\text{ant}} = RA$ at every member, with R as in Theorem 5 (written there with $D = \alpha\beta$). For

least period $N \geq 6$, $N \equiv 2 \pmod{4}$, and every coprime winding number, [Fer26c, Theorem 1] gives that $A/A^\dagger$ is a positive constant along the family, the same for both foci; so $A^\dagger = \mu A$ with $\mu > 0$.

Independent proof of the proportionalities. By Lemma 4.1(c), $\mathcal{C} := \mathcal{J} = \mathcal{I}_1 = \mathcal{I}_2$ is a coset of G . The function $A \in \mathcal{F}$ is holomorphic outside $\mathcal{C}$, as P_i has simple poles exactly where $u_i \in \Pi$; at $u_0 \in \mathcal{C}$ the vertices at infinity are j_1 and j_2 , which are $N/2 \geq 3$ apart (Lemma 4.1(e)), so no term $P_i \times P_{i+1}$ contains two of them, and A has at most a simple pole; and $A > 0$ on $\mathbb{R}$. By the proof of Theorem 1, which allows vertices at infinity (X is holomorphic at Π), $A^\dagger$ has at most simple poles, on $\mathcal{I}_1 = \mathcal{C}$. By Lemma 4.2(e), w_i is holomorphic except where $u_i \in \Pi$ or $u_{i+1} \in \Pi$, where its pole has order at most 1; so A_{ant} is holomorphic outside $\mathcal{C}$, every term $w_i \times w_{i+1}$ has a pole of order at most 2 at u_0 , and so has A_{ant} ; by Lemma 3.4, the order is at most 1. Lemma 4.3(b) with $\Phi = A$ gives $A_{\text{ant}} = \kappa_a A$ and $A^\dagger = \mu A$, with $\mu = A^\dagger/A > 0$ on $\mathbb{R}$. That $\kappa_a = R$ is [Fer26d, Lemma 4]; our argument gives κ_a only as a quotient of residues.

(b) $A > 0$ at every member (Lemma 3.2), so $A_{\text{ant}} = RA \neq 0$ if $R \neq 0$, and $A^\dagger/A_{\text{ant}} = \mu/R$. Both foci: Section 1.4.

(c) As $\beta - \lambda > 0$, $R = 0$ if and only if $\lambda(\alpha - \beta)^2 = \alpha\beta(\alpha - 3\beta)$, and as the left side is positive this needs $\alpha > 3\beta$. Then $A_{\text{ant}} = 0$ while $A^\dagger > 0$ at every member. Now let $N = 6$, so $\tau = 1$, and consider the member with $P_0 = P(-K) = (a, 0)$. By Lemma 3.3, $P_3 = -P_0$ and $P_2 = -P_5$. The normal of $\mathcal{E}$ at P_0 is the x -axis, so by the reflection law the chords P_0P_1 and P_0P_5 are mirror images in the x -axis, and P_5 is the mirror image of $P_1 = (x_1, y_1)$. Hence $P_2 = (-x_1, y_1)$, and the side P_1P_2 is the line $y = y_1$, which touches $\mathcal{C}$ only if $y_1^2 = \beta - \lambda$; with $y_1^2 = \beta(1 - x_1^2/\alpha)$ this gives $\lambda = \beta x_1^2/\alpha$. The side P_0P_1 , the line $px + qy = 1$ with $p = 1/a$, $q = (a - x_1)/(ay_1)$, touches $\mathcal{C}$ if and only if $(\alpha - \lambda)/\alpha + (\beta - \lambda)(a - x_1)^2/(\alpha y_1^2) = 1$, that is $(a - x_1)^2 = \lambda = \beta x_1^2/\alpha$. So $a - x_1 = \pm \frac{b}{a} x_1$, and as $x_1 \leq a$ only the sign $+$ is possible: $x_1 = a^2/(a + b)$ and $\lambda = a^2 b^2/(a + b)^2$, the caustic and the member given in [GR22, Appendix B.4]. Inserting this into R gives $R = (2b - a)(a + b)^2/(b^2(2a + b))$, which vanishes if and only if $b = a/2$. $\square$

For $a = 1$, $b = \frac{1}{2}$ we get $\lambda = \frac{1}{9}$ and the member H with vertices $(\pm 1, 0)$, $(\pm \frac{2}{3}, \pm \frac{\sqrt{5}}{6})$; the exact program of this note computes its focal antipedal polygon from the definition: six distinct vertices, signed area exactly 0 (a cancellation; the polygon is self-intersecting, as the source allows), while $A = 5\sqrt{5}/9$, $A^\dagger = 15\sqrt{5}/2$, $A_{\text{pol}}/A^\dagger = 64/15$, $A_{\text{ped}}^\dagger/A_{\text{dual}} = 40/3$ and $\bar{A}/A_{\text{dual}} = 320/27$. For $N \geq 10$ we know no closed form of λ ; the exceptional aspect ratios quoted after Theorem 5 were located numerically, for winding number 1 only. For $N = 6$, [GR22, Section 3.4] gives $A/A^\dagger = 4a^3 b^4/((2a - b)(a + b)^2)$ (unit inversion circle; the value is also quoted in [Fer26c]), so for $b \neq a/2$ the constant of row k_{818} is

$$\frac{A^\dagger}{A_{\text{ant}}} = \frac{\mu}{R} = \frac{4a^2 - b^2}{4a^3 b^2 (2b - a)},$$

e.g. 35/16 for $(a, b) = (1, 0.8)$; the third verification run (Section 8) confirmed this formula at the member through $(a, 0)$ with general a, b . For $N = 10$, $(a, b) = (1, \frac{1}{2})$ the constant is $-74.4175\dots$

8 Computations and reproducibility

The certificates of Theorem 3(A) and Theorem 4(A) and the zero area at the hexagon H are exact evaluations of the definitions at explicit points, in exact arithmetic; they were made by the program of this note and, independently, by each of the three verification runs described below. No other proof depends on a computation. The other computations test the statements on trajectories constructed without the parametrisation of Section 2, and check that they fail where their hypotheses fail. The reproducibility package has five parts.

The program of this note (`check_note.py`, `mpmath` and `sympy`, about 15 seconds; 171 checks, no failure). (i) Trajectories are built by the reflection law alone (the first chord leaves with Joachimsthal constant J , found by bisection so that the eccentric angle advances by $2\pi\tau$ in N reflections; $\lambda = (Jab)^2$ is used only for checks), with 50 digits. For 33 families, $(a, b) \in \{(1, 0.6), (1.5, 1), (2.6, 0.9)\}$ and $(N, \tau) \in \{(3, 1), (4, 1), (5, 1), (5, 2), (6, 1), (7, 2), (8, 1), (8, 3), (9, 2), (10, 3), (12, 5)\}$, seven members each, with all polygons built from the source's definitions (the circle touched by the sides of the polar polygon fitted from three sides): closure and tangency hold to $5 \cdot 10^{-42}$; the relative spreads of $A_{\text{pol}}/A^\dagger$, $A_{\text{pol}}/A_{\text{dual}}$ and $A_{\text{pol}}/A_{\text{dual}}^{\text{pt}}$ are at most $7 \cdot 10^{-36}$ (at most 10^{-42} unless $k^2 > 0.9999$); the identifications of Lemma 3.1 hold to $2 \cdot 10^{-43}$; the areas of Lemma 3.2 are positive; $A_{\text{dual}}^{\text{pt}} = A_{\text{dual}}/4$ for $N = 3$; the caustics for $N = 4, 6$ are $a^2b^2/(a^2 + b^2)$ and $a^2b^2/(a + b)^2$; and for $N = 4$ the constants of Proposition 5.1, $A_{\text{dual}}^{\text{pt}} = A_{\text{dual}}/2$ and $AA^\dagger = 4$ hold to $2 \cdot 10^{-47}$. (ii) The certificate of Theorem 3(A): M_1 , M_2 certified exactly (on $\mathcal{E}$, tangency, reflection law), Table 3 in exact arithmetic, the products under readings (i)–(iii) with both duals. (iii) The certificate of Theorem 4(A) likewise, with the formula of [Fer26b, Appendix] for $A_{\text{ped}}^\dagger$; the rhombus with symbolic a, b (Proposition 5.1); and for $N = 6, 8$ ($\tau = 1, 3$; two shapes) the quotient of k_{816} is constant to 10^{-41} under all readings and both duals, while for $N = 4$ it varies under reading (i) (relative spreads 0.22, 0.61). (iv) Row k_{818} : for $N = 6$ ($b/a = 0.8, 0.3/1.7$), $N = 10$ ($\tau = 1, 3$) and $N = 14$, A_{ant}/A equals R and $A^\dagger/A$ is constant, to $5 \cdot 10^{-48}$; the hexagon H is certified exactly and its focal antipedal area is 0; the hexagon with symbolic a, b confirms $\lambda = a^2b^2/(a + b)^2$ and the formula for R for $N = 6$; and for winding number 1, $a = 1$, the roots of $R = 0$ are found at $b = 0.5543486017301$ ($N = 10$), 0.5661108682363 ($N = 14$) and 0.5706649669806 ($N = 18$), where A_{ant}/A is below 10^{-40} at three members. (v) Controls: for $N = 5, 7$ (two shapes) the products and quotients of rows k_{815} (reading (i)), k_{816} (readings (i), (iii)) and k_{818} vary (relative spreads 0.01 to 0.25), as do k_{815} for $N = 6, 8, 12$ and k_{818} for $N = 8, 12$; for pentagons that do not close (Joachimsthal constant perturbed by 10^{-3}) $A_{\text{pol}}/A^\dagger$ varies; and a 3-periodic listed twice does not satisfy row k_{816} .

The programs with which the results were first obtained (`original/`), with their recorded outputs: a screen of 57 families ($N = 3$ to 16, winding numbers up to 5, three aspect ratios, nine members, 50 digits) under all readings, Laurent coefficients at the singular phases, the pole orders of the individual vertices, numerical checks of the closed forms of Section 2, the exact certificates, the exceptional hexagon, and scans of A_{ant}/A . Non-invariants become nearly constant as N grows: in this screen two false candidates had relative spreads $2.9 \cdot 10^{-10}$ and $7.8 \cdot 10^{-12}$ ($b/a = 0.8$, $N = 14$ and 16), while the true invariants were constant to 10^{-39} or better. One program, a symbolic reduction for $N = 4$, did not finish within its time limit, here and in the second verification run; nothing in this note uses it (Proposition 5.1 is proved at the rhombus).

The programs of the first two verification runs (`verification_run_A/`, `verification_run_B/`), described in the next paragraph, with their recorded outputs.

The programs of the third verification run (`independent_run_2/`), with their recorded outputs: exact arithmetic in multiquadratic number fields, without floating point, for the members M_1 , M_2 , R_1 , R_2 , H (certification, all entries of Table 3, the identifications of Lemma 3.1); a symbolic computation for the rhombus and for the member of the 6-periodic family through $(a, 0)$ with general a, b (Proposition 5.1, Theorem 5(c) and the constant of row k_{818} for $N = 6$); a screen of 48 families ($N = 3$ to 14, windings up to 5, 50 digits) with trajectories built from tangent lines of the caustic, λ found by closure and O_p fitted from three sides; Laurent coefficients at the singular phases of 11 families; the roots of $R = 0$ for $N = 10, 14, 18$; and a program that re-runs the whole package from the archive.

Verification. The results were first obtained, with the proofs of Theorem 1, Theorem 2(a), Theorem 3(B), Theorem 4(B), (C) and of the proportionalities in Theorem 5, the certificates, and the programs in `original/`. Two independent verification runs, both AI-assisted, followed; each wrote its

own programs from the source's definitions. The first run compared the rows and definitions with the tex source of [RGK21, v11]; re-derived every lemma and proof, including the case analysis behind Theorem 4(B); recomputed the certificates in exact arithmetic; screened 84 families at 50 digits; computed Laurent coefficients at the singular phases of twelve families (pole orders, residue quotients equal to the invariants to 15 digits, the pole of order three for $N = 4$); proved that the point-wise dual area is a constant multiple of $A^\dagger$; evaluated the constants for $N = 4$ at the symbolic rhombus; and found the deposits [Fer26c, Fer26d] behind row k_{818} , checking $A_{\text{ant}}/A = R$ in 66 even families. The second run built trajectories by the reflection law in 87 families with three negative controls, at 60 digits (110 for ill-conditioned stars); recomputed the certificates, verifying the reflection law exactly; evaluated its constructions at complex phases and scanned the pole locations on grids; located the exceptional aspect ratios for $N = 6, 10, 14, 18$; read the abstracts of the 38 Zenodo deposits on billiards of the two authors who have deposited notes on these tables, which led to [Fer26a, Fer26i]; checked the bibliography; and re-ran the programs of the first write-up, reproducing all recorded outputs except that of the symbolic program mentioned above. Neither run found a gap in the proofs. The corrections they required were all made: row k_{818} is presented as a corollary of [Fer26c, Fer26d], with R and its exceptional set explicit and [Fer26a, Fer26i] cited for the exceptional families; the reading of $A_{j,\text{ped}}^\dagger$ is named in every statement on k_{815} and k_{816} , and reading (ii) is treated; the explanation of the failure of k_{815} under reading (i) was corrected (Remark 6.3); a statement on the journal version was corrected; the constants for $N = 4$ are proved at the rhombus, not by the unfinished program; a statement on the separation of true and false candidates and a typographical error ($AA^\dagger = 4$, $A^\dagger A_{\text{ant}} = 8$ for $N = 4$) were corrected; the point-wise dual and the possible vanishing of κ_5 were added; Remark 3.5 was restricted to odd N ; and the deposits, the data of [RGH22], the credit to Akopyan and the wording on row k_{908} were completed or corrected. When this note was written, the following were added: the positivity of $A_{\text{dual}}^{\text{pt}}$ (Lemma 3.2), which completes Theorem 2(b); the failure of reading (ii) of k_{815} for every N ; the caustic and the formula for R for $N = 6$ in Theorem 5(c); Proposition 6.4; the treatment of odd N in Lemma 6.2 and the conclusion of Remark 6.3 for $N = 3$; and the comparison with the formula of [Fer26b, Appendix].

A third independent verification run, also AI-assisted, then examined the text of this note, these additions first. It re-derived every lemma, proposition and proof, among them the positivity of $A_{\text{dual}}^{\text{pt}}$, the failure of reading (ii), the derivation for $N = 6$, Proposition 6.4, all cases of Lemma 6.2 and the coset argument of Lemma 4.1(c); compared the rows and definitions again with the tex source of [RGK21, v11] and its figure, and examined the published version of [RGK21]; read the deposits [Fer26c, Fer26d] in full and the cited passages of the other sources again, and listed the Zenodo deposits of A. Kriebel again; and wrote its own programs: exact arithmetic in multiquadratic number fields for the certificates and Table 3, a symbolic computation for the rhombus and for the 6-periodic through $(a, 0)$ with general a, b , a screen of 48 families with trajectories built from tangent lines of the caustic, Laurent coefficients at the singular phases of 11 families (pole orders, residue quotients, the pole of order three for $N = 4$, the nonzero residues of $A_{\text{ped}}^\dagger$ on $\mathcal{I}_1$ for $N = 3, 5, 7$), and the roots of $R = 0$. It re-ran all programs of the package from the archive and obtained the recorded outputs; the output of `check_note.py` differs only in its running time, and the symbolic program mentioned above and a 110-digit variant of a program of the second run were not run again. It found no error in the mathematics. The corrections it required were made: the published version of the source is described (Section 1.4); [GR22] is credited for the caustics and members for $N = 4$ and $N = 6$ and for the values of $AA^\dagger$ and $A/A^\dagger$ there, and the constant of row k_{818} for $N = 6$ is stated (Section 7); the attribution of row $k_{908,b}$ and the title of Akopyan's communication are given exactly; the description of the dual in the abstract was made precise for star polygons; and the searches were extended. The program `check_note.py` tests the statements of the note except the constant of row k_{818} for $N = 6$, which the programs of the third run test.

Scope and priority. The note proves rows k_{813} and k_{814} for every ellipse, every confocal elliptic caustic, every least period $N \geq 3$ and every winding number, with either reading of “dual”. It decides rows k_{815} and k_{816} under each of the three readings of the undefined symbol $A_{j,ped}^\dagger$, except that under reading (i) the failure of k_{815} is proved for $N = 3$ only and was observed for every odd N tested. It proves row k_{818} on every family where its quotient is defined, and identifies the families where it is not ($R = 0$): exactly $b/a = \frac{1}{2}$ for $N = 6$, and, numerically, one aspect ratio each for $N = 10, 14, 18$ and winding number 1. Hyperbolic caustics are not treated, and walks that repeat a trajectory of smaller period are excluded. The credits are those of Section 1.3: the rows are due to Reznik, Garcia and Koiller [RGK21], who credit k_{814} to A. Akopyan; the parametrisation and the central symmetry to Stachel [Sta22, Sta21]; the bicentric polar family and its Liouville method to Roitman, Garcia and Reznik [RGR21]; the caustics and members for $N = 4, 6$ and the values of $AA^\dagger$ ($N = 4$) and $A/A^\dagger$ ($N = 6$) to Garcia and Reznik [GR22]; and the observation behind $k_{908,b}$ is stated also by Reznik, Garcia and Helman [RGH22]. Row k_{818} follows from the author’s deposits [Fer26c, Fer26d]; the exceptional families appear in [Fer26a, Fer26i] and in the proposers’ animations; related deposits are [Fer26b, Fer26e, Fer26h, Fer26g]. To our knowledge new are the proofs of k_{813} and k_{814} for every N , the analysis of k_{815} and k_{816} under the three readings with the exact certificates, and the independent proof of the proportionalities behind k_{818} . No printed or deposited proof of any of the five rows was found in the sources searched: the source in its arXiv version [RGK21, v11] and in its published version, which does not contain the five rows; the book [GR21] (full text searched: it treats the focus-inversive triangles and their polar family, but not these areas); the 19 works citing the source in OpenAlex (titles read); the arXiv listings for queries on the elliptic billiard together with inversive, polar, dual, antipedal or bicentric polygons, on areas of Poncelet polygons and on the works of the proposers; queries to zbMATH Open and Crossref; the 2026 Zenodo deposits of A. Ferudun and A. Kriebel (all records listed, the abstracts of those on billiards read; the newest of A. Kriebel, of 15 October 2026, treat rows $k_{605,a}$ and $k_{904,a}$) and the 2026 works that OpenAlex returns for “elliptic billiard”; and five web searches. The searches were made on 15 October 2026 by the first write-up and by the three verification runs; Semantic Scholar answered with a rate limit. A search that finds nothing is not a proof of novelty, and we make no priority claim.

What was read: [RGK21] in the tex source of arXiv v11 (the section on inversive objects with its two tables and the figure of the polar and dual polygons, the definitions of signed area and of pedal and antipedal polygons, the table of symbols, the list of animations), and its published version on the journal’s website (the list of invariants and the section and table on inversive objects; the rest was searched for polar and dual polygons); Theorem 2 of [Sta22] and Corollary 4.2 of [Sta21] in the arXiv versions v2; Section 3 of [RGH22], the abstract and the section on the five polygon families of [RGR21], and the list of animations, Sections 3.2 and 3.4 and Appendix B of [GR22], in the arXiv sources; the text of the book [GR21], searched for these polygons and read where it treats the focus-inversive triangles; the deposits [Fer26c, Fer26d] in full, in their deposited source files (whose checksums agree with the Zenodo records); of [Fer26a, Fer26i, Fer26b, Fer26e, Fer26h, Fer26g] the statements and the passages cited, and the notice [Fer26f]; and Chapter 22 of [DLM26]. The books [Tab05, AZ07] were not read for this note. Bibliographic data were checked with Crossref and, for the deposits, with DataCite.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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