

A Pentagonal Counterexample to a Poncelet Squared-Side Criterion

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Abstract

We construct a fixed circle and a strictly nested noncircular ellipse admitting a Poncelet family of convex primitive pentagons with constant sum of squared side lengths. The circle center is neither the ellipse center nor either focus. This disproves the necessity direction of a 2026 center-or-focus criterion, without contradicting its triangle results. The parameters are specified by an isolated cubic root, and all contacts occur on the actual edge segments. A projectivity agrees with unit inversion on a bicentric circumcircle and preserves the complete nested configuration. Constancy follows from an explicit geometric application of the published cyclic master identity of Khare, Lakshminarayan and Sukhatme. Exact standard-library computations certify the displayed algebraic member; no new general Jacobi identity or absolute-priority claim is made.

1 The criterion and an explicit counterexample

For an ordered polygon $W = (W_0, \dots, W_{N-1})$ define

$$\mathcal{S}(W) = \sum_{i=0}^{N-1} |W_{i+1} - W_i|^2, \quad W_N = W_0.$$

A Poncelet family here has vertices on one fixed circle and edges tangent to one fixed strictly nested conic, with a fixed tangent branch. “Primitive” refers to the least period, not the length of a repeated vertex list.

Murad proposes in [1, Section 6, Conjecture 1] that, for a fixed family of cyclic n -gons tangent to a central conic, constancy of $\mathcal{S}$ is equivalent to coincidence of the circumcenter with the conic center or a conic focus. The displayed statement is about a fixed n and its family; it does not require simultaneous invariance for every n . We give a counterexample to its necessity already for strictly convex primitive $n = 5$. This concerns the proposed generalization, not the proved triangle results in [1, 4].

Theorem 1. *Let t be the unique root in $(1/10, 1/5)$ of*

$$g(t) = 25t^3 - 50t^2 - 40t + 8 = 0, \tag{1}$$

and put $r = 4/5 - 2t^2$. The circle and ellipse

$$\Gamma : \left(u + \frac{5}{24}\right)^2 + v^2 = \left(\frac{25}{24}\right)^2, \tag{2}$$

$$\mathcal{E} : \left(\frac{24}{25}\right)^2 (u^2 + v^2) = r^2 \left(1 - \frac{2}{5}u\right)^2 \tag{3}$$

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are strictly nested and admit a Poncelet family of convex primitive pentagons. For every member W of this family,

$$\mathcal{S}(W) = \frac{175}{18} - \frac{625}{48}t. \quad (4)$$

The center of Γ is neither the center nor either focus of $\mathcal{E}$.

The proof separates three issues: an invariant for vertex inverses of bicentric polygons, a nonsingular projective transfer to a circle/ellipse pair, and an exact five-periodic configuration. Neither a small numerical residual nor one closed polygon alone establishes the invariant.

2 Vertex inversion of a bicentric family

Let Q_i be a primitive N -periodic polygon between strictly nested circles. Write its consecutive sidelines as

$$\ell_i = \{X : (X - C) \cdot n_i = r\}, \quad |n_i| = 1, \quad Q_i = \ell_i \cap \ell_{i+1},$$

where the inner disk of center C and radius r lies to the left of each directed edge. Normals are coherent outward normals, not independently chosen unsigned directions. Define

$$D_i = C + \frac{n_i}{r}, \quad I_i = C + \frac{Q_i - C}{|Q_i - C|^2}.$$

The first polygon inverts the feet of the C -pedal of Q ; the second inverts its vertices. Solving the two tangent equations gives

$$Q_i - C = \frac{r(n_i + n_{i+1})}{1 + n_i \cdot n_{i+1}}, \quad I_i = C + \frac{n_i + n_{i+1}}{2r} = \frac{D_i + D_{i+1}}{2}. \quad (5)$$

The denominator cannot vanish at a finite genuine intersection of two distinct support lines. Also $|Q_i - C| \geq r > 0$, so every inversion is defined. In particular,

$$\mathcal{S}(I) = \frac{N - T_2}{2r^2}, \quad T_2 = \sum_{i=0}^{N-1} n_{i-1} \cdot n_{i+1}. \quad (6)$$

Lemma 2. *The trace T_2 , and hence $\mathcal{S}(I)$, is constant on a primitive finite bicentric Poncelet family with $N \geq 3$. Fixed repeated traversals multiply the sums. Reversal preserves them.*

Proof. We spell out the geometric mapping to a published cyclic identity, including its pole cancellation. The analytic principle is the MI-II master identity of Khare, Lakshminarayan and Sukhatme [2, Section 2.1.2, equation (26) in the arXiv version]; it is not a new general Jacobi identity.

Translate the outer-circle center to the origin and rotate its center line. Jacobi's bicentric parametrization, as used in [3, Section 2], has the form

$$Q_i = R(\cos 2\phi_i, \sin 2\phi_i), \quad \phi_i = \text{am}(u + ih, k), \quad h = \frac{2\tau K}{N}, \quad \gcd(\tau, N) = 1. \quad (7)$$

Here $K = K(k)$ and $K' = K(\sqrt{1 - k^2})$, and the tangent branch fixes an admissible winding $0 < \tau < N$. For a nonconcentric strictly nested pair, $0 < k < 1$; explicitly $k^2 = 4Rd/((R + d)^2 - r^2)$, where d is the distance between centers. We use the real geometric period $2K$ of $Q(u)$, not $4K$. Amplitudes and indices in (7) are lifted: individual sn and cn may change sign after N shifts, while the normal products below do not.

The chord direction is

$$Q_i - Q_{i-1} = 2R \sin(\phi_i - \phi_{i-1}) J(\cos(\phi_{i-1} + \phi_i), \sin(\phi_{i-1} + \phi_i)),$$

where J is counterclockwise rotation through $\pi/2$. The lifted difference is in $(0, \pi)$, so its coefficient is positive. The left-disk convention identifies the outward normal as

$$n_i = (\cos(\phi_{i-1} + \phi_i), \sin(\phi_{i-1} + \phi_i)). \quad (8)$$

This is a local tangent statement and does not require a winding polygon to bound a globally convex region.

Put

$$E(z) = \text{cn}(z, k) + i \text{sn}(z, k), \quad H(z) = \text{cn}(z, k) - i \text{sn}(z, k).$$

The identity $\text{cn}^2 + \text{sn}^2 = 1$ gives $EH = 1$ meromorphically. Writing $E_j = E(u + jh)$, the complex representation of (8) is $E_{i-1}E_i$. The real cosine therefore continues algebraically as

$$T_2(u) = \frac{1}{2} \sum_{j=0}^{N-1} (\mathcal{R}_j + \mathcal{R}_j^{-1}), \quad \mathcal{R}_j = \frac{E_{j-2}E_{j-1}}{E_jE_{j+1}}. \quad (9)$$

No complex conjugation is used away from the real axis. Replacing each reciprocal E^{-1} by H makes each summand a degree-four polynomial in shifted sn and cn.

Under $z \mapsto z + 2K$, both functions E, H negate. Under $z \mapsto z + 2iK'$, they transform to $-H, -E$, respectively. Thus each reciprocal sum in (9) has periods $2K$ and $2iK'$. It belongs to the $P = Q = 0$ class of [2]; its parameter there is k^2 . Coprimality reindexes its sampling as the $2K/N$ grid, and $Nh = 2\tau K$ makes the finite sum genuinely cyclic.

For $N \geq 4$, the four shifts $-2h, -h, 0, h$ are distinct modulo $2K$. The possible poles are simple and have real offsets on that grid. MI-II expresses the cyclic sum as a constant plus terms from principal parts of orders at least two; these orders are absent. The total simple residue is zero over the period parallelogram, so the sum is constant. Here is the same cancellation directly, to make the principal-part claim explicit.

At every common pole z of sn, cn the odd-reflection identities give $E(z - s) = -E(z + s)$ and $H(z - s) = -H(z + s)$ [6, Tables 22.4.1 and 22.4.3]. Exactly one of E, H has a simple pole and the other a simple zero there, since $EH = 1$. Away from these poles neither has a zero. At any possible singularity of the cyclic sum, reindex so E_0 is singular. Coprimality and $N \geq 4$ imply that $E_{\pm 1}, E_{\pm 2}, E_{\pm 3}$ are finite and nonzero there. Suppose E_0 has a pole of residue a_0 , and put $e_j = E(z + jh)$, so $e_{-j} = -e_j$. The only pole terms are $\mathcal{R}_2, \mathcal{R}_1, \mathcal{R}_0^{-1}, \mathcal{R}_{-1}^{-1}$. Their coefficients of $a_0/(u - z)$, before the factor $1/2$, are

$$\frac{e_1}{e_2e_3}, \quad -\frac{1}{e_2}, \quad \frac{1}{e_2}, \quad -\frac{e_1}{e_2e_3}.$$

They cancel. The reciprocal partners vanish rather than have a pole; no term has a higher-order principal part. If E_0 has a zero, replace E by $H = 1/E$, which interchanges the reciprocal terms and leaves their sum unchanged. The same calculation applies. All singularities are consequently removable. The sum is an entire elliptic function and is constant.

For $N = 3$, shifts in the quartet collide, so that simple-pole argument must not be used. Instead cyclic indexing gives $T_2 = \sum_i n_i \cdot n_{i+1}$. The internal-angle cosine of Q is $-n_i \cdot n_{i+1}$, making this the negative of the invariant in [3, Theorem 1]. For concentric circles the normals advance uniformly and the result is elementary. Reversal and fixed repeated traversals give the last assertions directly. $\square$

3 A projectivity preserving the full nested configuration

Translate the inner center to 0, and put the outer center at $(d, 0)$, with $d > 0$ and $R > d + r$. Set

$$q = R^2 - d^2, \quad T(x, y) = \frac{(x, y)}{2dx + q}. \quad (10)$$

On the outer circle $|X|^2 = 2dx + q$, so T agrees there with unit vertex inversion. On the interior it is a projectivity, not pointwise inversion. Its homogeneous matrix has rows $(1, 0, 0)$, $(0, 1, 0)$, $(2d, 0, q)$ and determinant $q > 0$. Throughout the closed outer disk,

$$2dx + q \geq 2d(d - R) + q = (R - d)^2 > 0. \quad (11)$$

Thus no vertex, edge, contact, or disk crosses its horizon.

The inverse is $X = qU/(1 - 2du)$. Substitution maps the outer circle to the circle with center $(-d/q, 0)$ and radius R/q , and maps the inner circle to

$$q^2(u^2 + v^2) = r^2(1 - 2du)^2. \quad (12)$$

Since $q > 2dr + r^2$, the quantity $\Delta = q^2 - 4r^2d^2$ is positive. Completing the square shows that (12) is a noncircular ellipse with center, squared semiaxes, and foci

$$C_{\mathcal{E}} = \left(-\frac{2r^2d}{\Delta}, 0\right), \quad A^2 = \frac{r^2q^2}{\Delta^2}, \quad B^2 = \frac{r^2}{\Delta}, \quad F_1 = 0, \quad F_2 = \left(-\frac{4r^2d}{\Delta}, 0\right). \quad (13)$$

In particular $A^2 - B^2 = 4r^4d^2/\Delta^2$.

The positive denominator makes T an orientation-preserving homeomorphism of the outer disk onto the transformed disk. Lines and tangencies are preserved, and the two compact disks stay strictly nested. For additional specificity, a tangent $n \cdot X = r$ maps to

$$(qn_x + 2dr)u + qn_yv = r,$$

with contact $T(rn)$. Actual segment contacts are preserved: if the original contact has parameter $s \in (0, 1)$ between endpoints P, Q of positive denominators g_P, g_Q , its image parameter is

$$s' = \frac{sg_Q}{(1 - s)g_P + sg_Q} \in (0, 1).$$

The construction therefore preserves strict convexity and tangency to segments, not just their supporting-line extensions.

4 An algebraic convex five-cycle

Take $R = 1$, $d = 1/5$, so $q = 24/25$, and use t, r from Theorem 1. Equation (1) has exactly one root in the stated interval: $g(1/10) = 141/40 > 0$, $g(1/5) = -9/5 < 0$, and $g'(t) = 75t^2 - 100t - 40 < -37$ there. Hence

$$\frac{18}{25} < r < \frac{39}{50} < \frac{4}{5} = R - d. \quad (14)$$

Let $L = \sqrt{1 - t^2} > 0$, and define $\theta, \psi \in (0, \pi/2)$ by

$$\cos \theta = \frac{5r}{6}, \quad \sin \theta = \frac{5rL}{4(1 + t)}, \quad \cos \psi = t, \quad \sin \psi = L. \quad (15)$$

The identity

$$(5t + 2)g(t) = 125t^4 - 200t^3 - 300t^2 - 40t + 16$$

implies $\cos^2 \theta + \sin^2 \theta = 1$. Directly from (15),

$$\frac{4}{5} \sin \theta L = r(1 - t). \quad (16)$$

The positive signs exclude an extraneous squared-equation branch. Also $\cos \theta > 3/5 > t$, so $0 < \theta < \psi < \pi/2$.

Define

$$\begin{aligned} P_0 &= (6/5, 0), \\ P_1 &= \left(-\frac{4}{5} + \frac{25r^2}{18}, \frac{25r^2 L}{12(1+t)} \right), \\ P_2 &= (-r, 2tL), \quad P_3 = (-r, -2tL), \quad P_4 = (P_{1,x}, -P_{1,y}). \end{aligned} \quad (17)$$

Relative to the outer center $(1/5, 0)$, these are points on the unit circle with arguments $0, 2\theta, 2\psi, 2\pi - 2\psi, 2\pi - 2\theta$ in strictly increasing order. They are five distinct vertices of a strictly convex pentagon of winding one and least period five.

The first side P_0P_1 has outward unit normal $(\cos \theta, \sin \theta)$ and support $(6/5)\cos \theta = r$ from the inner center. The next side has normal $(\cos(\theta + \psi), \sin(\theta + \psi))$ and support

$$(1 + d)\cos \theta \cos \psi + (1 - d)\sin \theta \sin \psi = rt + \frac{4}{5} \sin \theta L = r.$$

The middle side is $x = -r$, and reflection supplies the other two tangencies. Every contact lies strictly inside the outer disk, hence inside the corresponding actual chord segment. This proves an exact primitive convex five-cycle for the fixed circle pair.

Poncelet's porism, equivalently the fixed-step Jacobi uniformization (7), now gives the whole five-periodic family. Indeed closure of the displayed convex cycle forces $5h = 2K$ in that uniformization, and the same translation closes every initial phase. Applying T maps the entire family to a fixed nested circle/ellipse family. By (11) it stays convex, primitive, and tangent on actual edge segments. The transformed pair is precisely (2)–(3). Lemma 2 makes its sum of squared sides constant, since $T(P_i) = P_i/|P_i|^2$ on the original outer circle.

5 The constant and the excluded center-or-focus conditions

We evaluate the constant without an opaque symbolic-computation step. The original pentagon's five outward-normal arguments are

$$\theta, \quad \theta + \psi, \quad \pi, \quad 2\pi - \theta - \psi, \quad 2\pi - \theta.$$

Consequently the trace in (6) is

$$T_2 = 2 \cos(2\theta + \psi) + \cos(2\theta + 2\psi) - 2 \cos \theta.$$

Using (15) gives

$$\begin{aligned} \cos(2\theta + \psi) &= r^2 \left(\frac{125t}{36} - \frac{25}{12} \right) - t, \\ \cos(2\theta + 2\psi) &= \frac{25r^2}{18} (5t^2 - 3t - 1) - 2t^2 + 1. \end{aligned}$$

It follows that

$$\mathcal{S}(T(P)) = \frac{4 + 5r/3 + 2t + 2t^2}{2r^2} - \frac{125t^2 + 50t - 100}{36}.$$

Substituting $r = 4/5 - 2t^2$ and expanding yields the exact factorization

$$2r^2 \left(\mathcal{S}(T(P)) - \frac{175}{18} + \frac{625t}{48} \right) = -\frac{(20t^3 - 27t^2 + 2t + 8)g(t)}{18} = 0.$$

This proves (4) for the displayed member and therefore for its entire Poncelet family.

For $d > 0$, the circumcenter $(-d/q, 0)$ differs from $F_1 = 0$. By (13), it could equal the ellipse center only if $\Delta = 2r^2q$, or the other focus only if $\Delta = 4r^2q$. Here (14) gives

$$\begin{aligned} \Delta - 2r^2q &= \frac{576}{625} - \frac{52}{25}r^2 < 0, \\ \Delta - 4r^2q &= \frac{576}{625} - 4r^2 < 0. \end{aligned}$$

Neither coincidence occurs. All claims of Theorem 1 are proved. In particular the necessary center-or-focus condition in the fixed- n conjecture of [1] fails for this strictly nested ellipse and convex primitive pentagonal family.

6 Verification and the billiard-source distinction

The accompanying standalone program `check_projective_pentagon.py` reproduces the exact member using arithmetic modulo (1) and the positive radical L . The cubic is irreducible over $\mathbb{Q}$: its residues at $0, \dots, 6$ modulo 7 are $1, 6, 5, 1, 4, 3, 1$, so it has no root modulo 7. The program uses rational root bisection and integer-square-root bounds, not floating point or a computer-algebra dependency. It checks both original and transformed incidences, actual segment tangencies, the non-coincidence of centers and foci, and the displayed exact value. The finite certificate supplements the proof; constancy over the whole family rests on the analytic argument in Section 2.

The investigation originated with invariant k_{811} in [5, Section 3.9, Table 9, Figure 8]. That source calls the side lengths those of a focal “dual,” describes a dual as the inverse of a pedal, but its figure caption calls for vertex inversion of a bicentric polar about its incircle center and claims a resulting circle centered there. These operations are distinct. The pedal reciprocal $D_i = C + n_i/r$ lies on that centered circle; the vertex inverse I_i does not. In fact Section 3 gives its circle center $-d/q$ after translating C to zero. Equation (5) identifies I as the midpoint polygon of D , not D itself.

Both precisely defined squared-side sums are constant. For D ,

$$\mathcal{S}(D) = \frac{2}{r^2} \left(N - \sum_i n_i \cdot n_{i+1} \right) = \frac{2}{r^2} \left(N + \sum_i \cos \beta_i \right),$$

an immediate corollary of the bicentric cosine theorem [3, Theorem 1]. For I the different trace is proved above. Neither interpretation is identified by fiat with every occurrence of the source’s ambiguous terminology. An unqualified resolution of that frozen source record is not claimed. The pentagonal counterexample instead addresses the separately stated 2026 criterion; it is not a claim to have solved an additional record of an open-problem dataset.

This is an AI-assisted, self-audited, unrefereed preprint. It has not undergone independent human review or proof-assistant verification. The analytic invariance principle is credited to the published results above; a bounded literature search does not establish absolute priority.

References

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