

Midpoint Polygons and Four Inversive Area Invariants of N -Periodics in the Elliptic Billiard

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Abstract

Reznik, Garcia and Koiller listed many experimentally found invariants of the families of N -periodic billiard trajectories in an ellipse. Let A and A' be the signed areas of the trajectory and of its outer polygon, and $A^\otimes$ and $A'^\ominus$ those of the polygons obtained by inverting the vertices of the trajectory in the caustic and those of the outer polygon in the billiard ellipse. Their rows k_{807} and k_{809} state that $AA^\otimes$ and $A'A'^\ominus$ are constant along the family for even N , and k_{808} and k_{810} that $A/A^\otimes$ and $A'/A'^\ominus$ are constant for odd N (records AMR-050-0049 to AMR-050-0052 of the UnsolvedMath collection). We prove all four, for every ellipse with semi-axes $a > b > 0$, every confocal elliptic caustic and every least period N of the stated parity, for convex and for self-intersecting trajectories. The inverted polygons are midpoint polygons, so their areas involve the area of the polygon of diagonals. In Stachel's canonical parametrisation by Jacobi elliptic functions, and by Stachel's conjugate-billiard relation, all areas are expressed by two elliptic functions F , G of the phase. For even N , $F(u)F(u + \Delta)$ and $F(u)G(u + \Delta)$ are constant by Liouville's theorem, the poles of one factor meeting zeros of the other that are forced by a folding symmetry. For odd N , $G = \gamma F$, and $k_{808} = k_{810} = 4\kappa/(2 + \gamma)$ with $\kappa = ab/(a''b'')$, where a'' , b'' are the semi-axes of the caustic and γ is an explicit algebraic function with $2 + \gamma > 0$. For $N = 4$ the constants are $4a^4b^4/(a^2 + b^2)^2$ and $4a^2b^2$. The argument also gives the row k_{110} , that AA'' is constant for even N , A'' being the area of the inner polygon (record AMR-050-0004); this row is not new, as it follows at once from Chavez-Caliz's theorem on AA' and Stachel's relation $A' = \kappa^2 A''$. The cases $N = 3$ of k_{808} and k_{810} follow from a theorem of Akopyan, Schwartz and Tabachnikov and Stachel's relation, and the cases $N = 4, 6$ of k_{807} and k_{809} from Chavez-Caliz's theorem. This is an unrefereed note.

1 Introduction

1.1 The rows

Let $\mathcal{E}$ be the ellipse $x^2/a^2 + y^2/b^2 = 1$ with $a > b > 0$ and centre O , and put $c = \sqrt{a^2 - b^2}$. A billiard trajectory in $\mathcal{E}$ all of whose chords touch a confocal ellipse $\mathcal{C}$, its *caustic*, is called an N -periodic if it closes after N reflections and not earlier. By Poncelet's porism all trajectories with the caustic $\mathcal{C}$ are then N -periodic; they form a one-parameter *family*. Reznik, Garcia and Koiller [RGK21] list many quantities which they found experimentally to be constant along such families. For an N -periodic P they use the *outer polygon* P' , whose vertices are the intersections of the tangents to $\mathcal{E}$ at consecutive vertices of P , and the *inner polygon* P'' , whose vertices are the points where the sides of P touch $\mathcal{C}$; A , A' and A'' are the signed areas of P , P' and P'' . In the section of [RGK21, v11] on inversive objects, $P^\otimes$ is the polygon of the inversions of the vertices of P in the caustic, and $P'^\ominus$ the polygon of the

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inversions of the vertices of P' in the billiard ellipse, with signed areas $A^\otimes$ and $A'^\ominus$; the inversion of a point in an ellipse is described there, with a reference to [GSO16], as the midpoint of the chord joining the points of contact of the two tangents from the point. The precise definitions are in Section 2. This note treats the five rows of Table 1.

record	row	quantity	N
AMR-050-0049	k_{807}	$AA^\otimes$	even
AMR-050-0050	k_{808}	$A/A^\otimes$	odd
AMR-050-0051	k_{809}	$A'A'^\ominus$	even
AMR-050-0052	k_{810}	$A'/A'^\ominus$	odd
AMR-050-0004	k_{110}	AA''	even

Table 1: The rows of [RGK21, v11] treated here: corpus record [ula26], row code, quantity, and the N for which the source states that the quantity is constant along the family. The source gives no value and no proof for any of them.

The corpus records [ula26] ask to prove that the quantity of the row is constant as P varies through the family. The four inversive rows k_{807} – k_{810} are rows of the arXiv version v11; the journal version [RGK21] does not contain them, and it lists k_{110} without value or proof, as v11 does.

1.2 Results

Throughout, $a > b > 0$, the caustic is a confocal *ellipse* with semi-axes $a'' > b'' > 0$, the trajectories have *least* period N , and all areas are signed. The vertices of every member go τ times around O , where $1 \leq \tau < N/2$ and $\gcd(\tau, N) = 1$ (Lemma 3.3); the source's tables concern $\tau = 1$, and our results hold for every τ . Put $\lambda = a^2 - a''^2 = b^2 - b''^2 \in (0, b^2)$ and $\kappa = ab/(a''b'')$. Our main result, Theorem 7.1, is the following.

- (a) If N is even, there are constants c_1, c_2 such that at every member $AA^\otimes = \kappa^{-1}(\frac{1}{2}c_1 + \frac{1}{4}c_2)$, $A'A'^\ominus = \kappa(\frac{1}{2}c_1 + \frac{1}{4}c_2)$ and $AA'' = c_1/\kappa$. For $N = 4$, $AA^\otimes = 4a^4b^4/(a^2 + b^2)^2$ and $A'A'^\ominus = 4a^2b^2$.
- (b) If N is odd, then at every member $A'^\ominus = \frac{1}{4}(2 + \gamma)A$ and $A^\otimes = \frac{1}{4\kappa}(2 + \gamma)A$, where

$$\gamma = \frac{2a^2b^2(a^2b^2 - (a^2 + b^2)\lambda)}{a^4b^4 - c^4\lambda^2}, \quad 2 + \gamma > 0.$$

Hence $A/A^\otimes = A'/A'^\ominus = 4\kappa/(2 + \gamma)$ at every member with $A \neq 0$, in particular at every member of a convex family.

Two observations reduce the rows to two functions. First, the inversion of a point in a conic is the midpoint of its chord of contact, so $P^\otimes$ and $P'^\ominus$ are the midpoint polygons of P'' and of P , whose areas involve the area of the polygon of diagonals (Lemma 4.1). Second, in Stachel's canonical parametrisation $P(u) = (-a \operatorname{sn} u, b \operatorname{cn} u)$ the vertices of a member are $P(u + 2l\Delta)$, and the inner and outer polygons are the images under the axial scalings $\sigma^{\mp 1}$, $\sigma = \operatorname{diag}(a/a'', b/b'')$, of the member with phase $u + \Delta$, Stachel's conjugate billiard (Proposition 3.4). So all areas are expressed by the area $F(u)$ and the area $G(u)$ of the polygon of diagonals of the member with phase u , elliptic functions with at most simple poles. For even N (Theorem A) the poles of one factor of $F(u)F(u + \Delta)$ and of $F(u)G(u + \Delta)$ lie on a coset of singular phases on which the other factor vanishes, by a folding symmetry (Lemma 5.3), and Liouville's theorem concludes. For odd N (Theorem B), F and G have the same poles, and a comparison of residues gives $G = \gamma F$.

The same argument shows that AA'' is constant for even N (row k_{110}). This is not new: by Stachel's relation $A' = \kappa^2 A''$ it is equivalent to Chavez-Caliz's theorem that AA' is constant for even N (Proposition 7.4). Section 8 records consequences for other rows.

1.3 Credit and limits

The rows and the observation that they hold are due to Reznik, Garcia and Koiller [RGK21]. The parametrisation by Jacobi elliptic functions is Stachel's canonical parametrisation [Sta22a, Theorem 2], and the description of the inner and outer polygons by the member with phase $u + \Delta$ is Stachel's conjugate billiard [Sta21, Subsection 3.2, Definition 3.10, Lemma 3.11]; we re-derive both. The method (complexify the phase, locate the poles at degenerate polygons, cancel them by a symmetry, conclude by Liouville's theorem) is that of Chavez-Caliz's proof that AA' is constant for even N [CC21, Theorem 6]; it was used for sibling rows in earlier notes of the author [Fer26f, Fer26a], and we do not claim it as new. The row k_{110} follows at once from [CC21, Theorem 3] and Stachel's relation, and Garcia and Reznik [GR22] gave closed forms of it for $N = 4$ and $N = 6$. The cases $N = 3$ of k_{808} , k_{810} follow from the theorem of Akopyan, Schwartz and Tabachnikov that A'/A is constant for odd N [AST22, Theorem 3] (with Stachel's relation for k_{808}), and the cases $N = 4, 6$ of k_{807} , k_{809} from Chavez-Caliz's theorem (Remark 7.3). New, to our knowledge, are the rows $k_{807}-k_{810}$ for every N of the stated parity and the formula for γ . No printed or deposited proof of $k_{807}-k_{810}$ for general N was found in the sources we searched ("Scope and priority" in Section 9); a search that finds nothing is not a proof of novelty, and we claim no priority.

2 Definitions and conventions

Conics. For vectors of $\mathbb{R}^2$ or $\mathbb{C}^2$ we write $X \cdot Y = X_1Y_1 + X_2Y_2$ and $X \times Y = X_1Y_2 - X_2Y_1$; both are bilinear. Let $M = \text{diag}(m_1, m_2)$ with $m_1, m_2 > 0$ and $\Gamma = \{X : X \cdot MX = 1\}$. The tangent of Γ at $T \in \Gamma$ is $\{X : X \cdot MT = 1\}$, and the polar of a point Y is $\{X : X \cdot MY = 1\}$. A line $\{X : X \cdot n = 1\}$ is tangent to Γ if and only if $n \cdot M^{-1}n = 1$, and then it touches Γ at $M^{-1}n$. We put $M_{\mathcal{E}} = \text{diag}(a^{-2}, b^{-2})$, so that $\mathcal{E} = \{X \cdot M_{\mathcal{E}}X = 1\}$. The confocal ellipses inside $\mathcal{E}$ are the ellipses

$$\mathcal{C} = \{X \cdot M_{\mathcal{C}}X = 1\}, \quad M_{\mathcal{C}} = \text{diag}(a''^{-2}, b''^{-2}), \quad a''^2 = a^2 - \lambda, \quad b''^2 = b^2 - \lambda, \quad 0 < \lambda < b^2.$$

The *inversion* in Γ is the map $\iota_{\Gamma}(Y) = Y/(Y \cdot MY)$ for $Y \neq 0$. The next lemma shows that it agrees with both descriptions in use: the radial one, and the midpoint of the chord of contact used by the source.

Lemma 2.1 (inversion in an ellipse). *Let $Y \in \mathbb{R}^2$ with $Y \cdot MY > 1$.*

- (a) *The two tangents from Y to Γ touch Γ at the two points $T_{\pm}$ where the polar of Y meets Γ .*
- (b) *The midpoint of T_+T_- is $\iota_{\Gamma}(Y)$.*
- (c) *$\iota_{\Gamma}(Y)$ lies on the ray OY , and $|OY| \cdot |O\iota_{\Gamma}(Y)| = |OQ|^2$, where $Q = Y/\sqrt{Y \cdot MY}$ is the point of Γ on this ray.*

Proof. (a) For $T \in \Gamma$, the point Y lies on the tangent at T if and only if $Y \cdot MT = 1$, that is, $T \cdot MY = 1$. (b) Let $t = 1/(Y \cdot MY) \in (0, 1)$ and let $d \neq 0$ with $d \cdot MY = 0$. The polar of Y consists of the points $tY + sd$, $s \in \mathbb{R}$, and such a point lies on Γ if and only if $t^2 Y \cdot MY + s^2 d \cdot Md = 1$, that is, $s^2 = (1 - t)/(d \cdot Md) > 0$. The two solutions $\pm s_0$ give $T_{\pm} = tY \pm s_0 d$, whose midpoint is $tY = \iota_{\Gamma}(Y)$. (c) $|OY| \cdot |O\iota_{\Gamma}(Y)| = t|Y|^2 = |Y|^2/(Y \cdot MY) = |OQ|^2$. $\square$

Trajectories. A *billiard trajectory* in $\mathcal{E}$ is a sequence $(P_l)_{l \in \mathbb{Z}}$ of points of $\mathcal{E}$ with $P_{l+1} \neq P_l$ such that at every P_l the normal of $\mathcal{E}$ bisects the angle between the chords $P_l P_{l-1}$ and $P_l P_{l+1}$ (the reflection law). Its *caustic* is $\mathcal{C}$ if every line $P_l P_{l+1}$ touches $\mathcal{C}$. From a point $X \in \mathcal{E}$ there are two distinct tangents to $\mathcal{C}$, and the reflection in the normal of $\mathcal{E}$ at X exchanges them, the classical reflection property of

confocal conics [Tab05], [Sta21, Lemma 2.1]; neither of them is fixed by the reflection (otherwise both would be fixed, and the only lines through X fixed by the reflection are the normal and the tangent of $\mathcal{E}$ at X , while the tangent does not meet $\mathcal{C}$). So the two sides of a trajectory with caustic $\mathcal{C}$ at P_l are the two tangents from P_l to $\mathcal{C}$. A trajectory is *periodic* with period N if $P_{l+N} = P_l$ for all l ; the least such $N \geq 1$ is its *least period*. Following the source, an N -*periodic* is a trajectory with caustic $\mathcal{C}$ and least period N , viewed as the polygon $P = (P_0, \dots, P_{N-1})$ with indices modulo N (the source numbers the vertices from 1 to N ; only the cyclic order matters below). For a fixed caustic, the *family* is the set of all these polygons. By Lemma 3.3, if one trajectory with a given caustic is periodic, then all are, with the same least period, and their vertices go around O the same number τ of times. The convex N -periodics are those with $\tau = 1$; we call those with $\tau \geq 2$ *star N -periodics*.

Areas. For a polygon $W = (W_l)_{l \in \mathbb{Z}/N}$ we write

$$S(W) = \frac{1}{2} \sum_{l \in \mathbb{Z}/N} W_l \times W_{l+1}, \quad S_2(W) = \frac{1}{2} \sum_{l \in \mathbb{Z}/N} W_l \times W_{l+2}.$$

$S(W)$ is the signed area of the source, “all polygon areas are signed” [RGK21, v11]. $S_2(W)$ is the signed area of the polygon of diagonals $W_0 W_2 W_4 \dots$: for odd N a single polygon through all vertices, for even N the sum of the signed areas of $W_0 W_2 \dots W_{N-2}$ and $W_1 W_3 \dots W_{N-1}$. A cyclic relabelling changes neither S nor S_2 , the reversal of the cyclic order changes the sign of both, and a linear map L multiplies both by $\det L$.

The five areas. Let P be an N -periodic with caustic $\mathcal{C}$. Its *outer polygon* P' has as vertex P'_l the intersection of the tangents of $\mathcal{E}$ at P_l and P_{l+1} , and its *inner polygon* P'' has as vertex P''_l the point where the line $P_l P_{l+1}$ touches $\mathcal{C}$ (by Lemma 3.3 these points are finite). Following [RGK21, v11], $P^\otimes$ has the vertices $P_l^\otimes = \iota_{\mathcal{C}}(P_l)$, the inversions of the vertices of P in the caustic, and $P'^\ominus$ has the vertices $P_l'^\ominus = \iota_{\mathcal{E}}(P'_l)$, the inversions of the vertices of P' in the billiard ellipse. Then

$$A = S(P), \quad A' = S(P'), \quad A'' = S(P''), \quad A^\otimes = S(P^\otimes), \quad A'^\ominus = S(P'^\ominus).$$

This is the assignment of the source’s list of symbols, of the footnote to its table of inversive invariants and of the corpus records [ula26]; one sentence of the source’s text attaches the symbols $\otimes, \ominus$ to the two polygons the other way round, a harmless slip. If the order of the vertices of P is reversed, all five polygons are reversed (up to a cyclic relabelling), so all five areas change sign, and the products and ratios of the rows do not change.

3 Stachel’s parametrisation and the conjugate billiard

Jacobi functions. For a modulus $0 < k < 1$ let $k' = \sqrt{1 - k^2}$, let K and K' be the complete elliptic integrals of the first kind of moduli k and k' , and let $\text{sn}, \text{cn}, \text{dn}$ be the Jacobi elliptic functions of modulus k . We use the following standard facts [DLM26, §§22.2–22.8]. (J1) $\text{sn}, \text{cn}, \text{dn}$ are meromorphic on $\mathbb{C}$, and their poles are the points of $\Pi = iK' + 2K\mathbb{Z} + 2iK'\mathbb{Z}$, all simple. (J2) $\text{sn}^2 u + \text{cn}^2 u = 1$ and $\text{dn}^2 u + k^2 \text{sn}^2 u = 1$. (J3) $\text{sn}(u + 2K) = -\text{sn} u$, $\text{cn}(u + 2K) = -\text{cn} u$, $\text{sn}(u + 2iK') = \text{sn} u$, $\text{cn}(u + 2iK') = -\text{cn} u$; sn is odd, cn and dn are even. (J4) $\text{sn}(u + iK') = 1/(k \text{sn} u)$, $\text{cn}(u + iK') = -i \text{dn} u / (k \text{sn} u)$ and $\text{sn}(K - u) = \text{cn} u / \text{dn} u$; near $u = 0$, $\text{sn}(iK' + u) = 1/(ku) + O(u)$ and $\text{cn}(iK' + u) = -i/(ku) + O(u)$. (J5) With $D = 1 - k^2 \text{sn}^2 u \text{sn}^2 v$ and $D_2 = 1 - k^2 \text{sn}^4 u$,

$$\begin{aligned} \text{sn}(u \pm v) &= \frac{\text{sn} u \text{cn} v \text{dn} v \pm \text{sn} v \text{cn} u \text{dn} u}{D}, & \text{cn}(u \pm v) &= \frac{\text{cn} u \text{cn} v \mp \text{sn} u \text{sn} v \text{dn} u \text{dn} v}{D}, \\ \text{sn} 2u &= \frac{2 \text{sn} u \text{cn} u \text{dn} u}{D_2}, & \text{cn} 2u &= \frac{\text{cn}^2 u - \text{sn}^2 u \text{dn}^2 u}{D_2}, & \text{dn} 2u &= \frac{\text{dn}^2 u - k^2 \text{sn}^2 u \text{cn}^2 u}{D_2}. \end{aligned}$$

(J6) For real u , $(\operatorname{sn} u, \operatorname{cn} u) = (\sin \varphi, \cos \varphi)$ with $u = \int_0^\varphi (1 - k^2 \sin^2 \theta)^{-1/2} d\theta$, so φ increases with u , by 2π when u increases by $4K$, and $\varphi = \pi/2$ at $u = K$; $\operatorname{dn} u \geq k' > 0$; and $\operatorname{sn} u = 0$ if and only if $\operatorname{dn} u = 1$, if and only if $u \in 2K\mathbb{Z}$.

The parametrisation. Fix a caustic $\mathcal{C}$, that is, $\lambda \in (0, b^2)$, and put $k = c/a''$. Since $a''^2 - c^2 = b''^2 > 0$, we have $0 < k < 1$ and $k' = b''/a''$. Let

$$P(u) = (-a \operatorname{sn} u, b \operatorname{cn} u), \quad Q(u) = (-a'' \operatorname{sn} u, b'' \operatorname{cn} u), \quad \sigma = \operatorname{diag}\left(\frac{a}{a''}, \frac{b}{b''}\right), \quad R(u) = \sigma P(u),$$

so that $Q = \sigma^{-1}P$ and $\det \sigma = \kappa = ab/(a''b'')$. By (J2) and (J6), for real u the point $P(u)$ lies on $\mathcal{E}$ and $Q(u)$ on $\mathcal{C}$, and $u \mapsto P(u)$ is a $4K$ -periodic parametrisation of $\mathcal{E}$, injective on $[0, 4K)$, which runs counter-clockwise.

Lemma 3.1 (the half step). *There is a unique $\Delta \in (0, K)$ with $\operatorname{sn}^2 \Delta = \lambda/b^2$, and*

$$b'' = b \operatorname{cn} \Delta, \quad a'' = \frac{a \operatorname{cn} \Delta}{\operatorname{dn} \Delta}, \quad \kappa = \frac{\operatorname{dn} \Delta}{\operatorname{cn}^2 \Delta}, \quad \operatorname{sn}(K - \Delta) = \frac{a''}{a}.$$

Proof. Existence and uniqueness follow from (J6), since $0 < \lambda/b^2 < 1$. Then $\operatorname{cn}^2 \Delta = 1 - \lambda/b^2 = b''^2/b^2$, and, since $a''^2 b^2 - c^2 \lambda = (a^2 - \lambda)b^2 - (a^2 - b^2)\lambda = a^2 b''^2$,

$$\operatorname{dn}^2 \Delta = 1 - \frac{c^2 \lambda}{a''^2 b^2} = \frac{a^2 b''^2}{a''^2 b^2} = \frac{a^2 \operatorname{cn}^2 \Delta}{a''^2}.$$

As $\operatorname{cn} \Delta, \operatorname{dn} \Delta > 0$, this gives b'', a'' and $\kappa = ab/(a''b'')$. Finally $\operatorname{sn}(K - \Delta) = \operatorname{cn} \Delta / \operatorname{dn} \Delta = a''/a$ by (J4). $\square$

Lemma 3.2 (chords). *For every $u \in \mathbb{C}$ at which the expressions are finite,*

$$P(u - \Delta) \cdot M_{\mathcal{C}} Q(u) = P(u + \Delta) \cdot M_{\mathcal{C}} Q(u) = 1, \quad M_{\mathcal{E}}^{-1} M_{\mathcal{C}} Q(u) = R(u).$$

Hence, for real u with $P(u - \Delta) \neq P(u + \Delta)$, the line through $P(u - \Delta)$ and $P(u + \Delta)$ is the tangent of $\mathcal{C}$ at $Q(u)$, and the tangents of $\mathcal{E}$ at these two points meet at $R(u)$.

Proof. Write $s, \bar{c}, d$ for $\operatorname{sn} u, \operatorname{cn} u, \operatorname{dn} u$, and s_0, c_0, d_0 for $\operatorname{sn} \Delta, \operatorname{cn} \Delta, \operatorname{dn} \Delta$, and $D = 1 - k^2 s^2 s_0^2$. We have $M_{\mathcal{C}} Q(u) = (-s/a'', \bar{c}/b'')$, and $a/a'' = d_0/c_0$, $b/b'' = 1/c_0$ by Lemma 3.1. By (J5),

$$P(u \pm \Delta) \cdot M_{\mathcal{C}} Q(u) = \frac{d_0 s (s c_0 d_0 \pm s_0 \bar{c} d) + \bar{c} (\bar{c} c_0 \mp s s_0 d d_0)}{c_0 D} = \frac{s^2 d_0^2 + \bar{c}^2}{D} = \frac{s^2 (1 - k^2 s_0^2) + 1 - s^2}{D} = 1.$$

Next, $M_{\mathcal{E}}^{-1} M_{\mathcal{C}} Q(u) = (-a^2 s/a'', b^2 \bar{c}/b'') = \sigma P(u)$. For real u , $Q(u) \in \mathcal{C}$, so $\{X : X \cdot M_{\mathcal{C}} Q(u) = 1\}$ is the tangent of $\mathcal{C}$ at $Q(u)$, and it contains the two points $P(u \pm \Delta)$ of $\mathcal{E}$. The tangent of $\mathcal{E}$ at $P(u \pm \Delta)$ is $\{X : X \cdot M_{\mathcal{E}} P(u \pm \Delta) = 1\}$, and it contains $R(u)$ because $R(u) \cdot M_{\mathcal{E}} P(u \pm \Delta) = M_{\mathcal{C}} Q(u) \cdot P(u \pm \Delta) = 1$. The two tangents are distinct, since the points are. $\square$

Lemma 3.3 (the family). *Let $N \geq 3$ and $1 \leq \tau < N/2$ with $\gcd(\tau, N) = 1$.*

- (a) *There is $\lambda \in (0, b^2)$ with $\Delta = 2\tau K/N$. As $\lambda \rightarrow b^2$ one has $K \rightarrow \infty$ and $K - \Delta \rightarrow \operatorname{arsinh}(c/b)$.*
- (b) *Let $\Delta = 2\tau K/N$ and $h = 2\Delta = 4\tau K/N$. For every real u the points $P_l = P(u + lh)$, $l \in \mathbb{Z}$, form a billiard trajectory with caustic $\mathcal{C}$ and least period N , whose vertices go τ times around O . Its side $P_l P_{l+1}$ touches $\mathcal{C}$ at $Q(u + lh + \Delta)$, and the tangents of $\mathcal{E}$ at P_l and P_{l+1} meet at $R(u + lh + \Delta)$.*

(c) Conversely, for any $\lambda \in (0, b^2)$, every billiard trajectory with caustic $\mathcal{C}$ is, after reversing the order of its vertices if necessary, of the form $P_l = P(u + 2l\Delta)$ with a real u . If it is periodic with least period N , then $N \geq 3$ and $\Delta = 2\tau K/N$, where τ is the number of turns of its vertices around O , $1 \leq \tau < N/2$ and $\gcd(\tau, N) = 1$.

Proof. (a) The ratio Δ/K depends continuously on $\lambda \in (0, b^2)$. As $\lambda \rightarrow 0$, $k \rightarrow c/a < 1$, so $K \geq \pi/2$ while $\Delta \leq \arcsin(\sqrt{\lambda}/b)/k' \rightarrow 0$ by (J6); hence $\Delta/K \rightarrow 0$. As $\lambda \rightarrow b^2$, $k \rightarrow 1$ and $K \rightarrow \infty$. By Lemma 3.1, $t = K - \Delta \in (0, K)$ satisfies $\operatorname{sn} t = a''/a$, so by (J6)

$$t = \int_0^{\theta_0} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}, \quad \sin \theta_0 = \frac{a''}{a} < 1.$$

The integrand is at most $1/\cos \theta$ and increases to it as $k \rightarrow 1$, while $\theta_0 \rightarrow \arcsin(c/a) < \pi/2$ because $a'' \rightarrow c$. Hence t stays bounded, and by dominated convergence $t \rightarrow \int_0^{\arcsin(c/a)} d\theta / \cos \theta = \operatorname{artanh}(c/a) = \operatorname{arsinh}(c/b)$. So $\Delta/K = 1 - t/K \rightarrow 1$, and by the intermediate value theorem Δ/K takes the value $2\tau/N \in (0, 1)$.

(b) Since $0 < h < 2K$ and $0 < 2h < 4K$, the points P_{l-1} , P_l , P_{l+1} are pairwise distinct. By Lemma 3.2 at $u + lh + \Delta$, the line $P_l P_{l+1}$ touches $\mathcal{C}$ at $Q(u + lh + \Delta)$, and the tangents of $\mathcal{E}$ at P_l and P_{l+1} meet at $R(u + lh + \Delta)$. So the two sides at P_l are two distinct tangents from P_l to $\mathcal{C}$, that is, both tangents, and the reflection property of Section 2 gives the reflection law at P_l . Next, $P_{l+N} = P(u + lh + 4\tau K) = P_l$. If $P_{l+N'} = P_l$ for all l , then $N'h \in 4K\mathbb{Z}$ by injectivity, that is, $N \mid N'\tau$, hence $N \mid N'$. When l runs from 0 to N the parameter increases by $Nh = 4\tau K$, so the vertices go τ times around O .

(c) Let (P_l) be a trajectory with caustic $\mathcal{C}$, $P_0 = P(u)$ with u real, and $h = 2\Delta \in (0, 2K)$; then $P(u - h)$, $P(u)$, $P(u + h)$ are pairwise distinct, and by Lemma 3.2 at $u \mp \Delta$ the lines through $P(u)$ and $P(u \mp h)$ are the two tangents from P_0 to $\mathcal{C}$. After reversing the order of the vertices if necessary, $P_0 P_1$ is the second one and $P_0 P_{-1}$ the first, so $P_{\pm 1} = P(u \pm h)$. If $P_{l-1} = P(u + lh - h)$ and $P_l = P(u + lh)$, then P_{l+1} lies on the other tangent from P_l , the line through $P(u + lh)$ and $P(u + lh + h)$, so $P_{l+1} = P(u + lh + h)$; the same holds backwards. If the trajectory has least period N , then $Nh = 4\tau K$ with an integer $\tau \geq 1$, the number of turns of the vertices; $h < 2K$ gives $\tau < N/2$, hence $N \geq 3$, and least period N gives $\gcd(\tau, N) = 1$ as in (b). $\square$

So a family of N -periodics is given by N , τ and a caustic with $\Delta = 2\tau K/N$; we fix these from now on, put $h = 2\Delta$, and call

$$P^{[u]} = (P(u + lh))_{l \in \mathbb{Z}/N}$$

the *member with phase u* . By Lemma 3.3(c) every member of the family is a $P^{[u]}$ with real u , up to the order of its vertices, which does not affect the rows (Section 2).

Proposition 3.4 (the inner and outer polygons; Stachel). *For every real u , the inner and outer polygons of $P^{[u]}$ are*

$$P'' = \sigma^{-1} P^{[u+\Delta]}, \quad P' = \sigma P^{[u+\Delta]},$$

vertex by vertex: $P_l'' = \sigma^{-1} P(u + \Delta + lh)$ and $P_l' = \sigma P(u + \Delta + lh)$. In particular $P' = \sigma^2 P''$.

Proof. This is Lemma 3.3(b), since $Q = \sigma^{-1} P$ and $R = \sigma P$. $\square$

In Stachel's terms, $P^{[u+\Delta]}$ is the billiard *conjugate* to $P^{[u]}$: the axial scaling σ (denoted α in [Sta21]) maps the points of contact of the sides to the vertices of the conjugate billiard, and these to the poles of the sides, which are the vertices of the outer polygon [Sta21, Subsection 3.2, Definition 3.10, Lemma 3.11]; in canonical coordinates the conjugate billiard is the shift by Δ [Sta21, Subsection 3.2]. The parametrisation itself is [Sta22a, Theorem 2], where $P(u)$ is written $(-a_e \operatorname{sn} \tilde{u}, b_e \operatorname{cn} \tilde{u})$ and $\Delta = \Delta \tilde{u}$; Lemmas 3.2 and 3.3 make this note independent of both citations.

4 The areas

Lemma 4.1 (midpoint polygons). *Let $N \geq 3$, $W = (W_l)_{l \in \mathbb{Z}/N}$ and $M_l = \frac{1}{2}(W_l + W_{l+1})$. Then $S(M) = \frac{1}{2}S(W) + \frac{1}{4}S_2(W)$.*

Proof. $4M_l \times M_{l+1} = (W_l + W_{l+1}) \times (W_{l+1} + W_{l+2}) = W_l \times W_{l+1} + W_l \times W_{l+2} + W_{l+1} \times W_{l+2}$; summing over l gives $8S(M) = 4S(W) + 2S_2(W)$. $\square$

Lemma 4.2 (the inverted polygons). *For every N -periodic P and every l ,*

$$P_l^\otimes = \frac{1}{2}(P_{l-1}'' + P_l''), \quad P_l'^\ominus = \frac{1}{2}(P_l + P_{l+1}).$$

So $P'^\ominus$ is the midpoint polygon of P , and $P^\otimes$ is the midpoint polygon of P'' relabelled by one step; hence $A'^\ominus = \frac{1}{2}A + \frac{1}{4}S_2(P)$ and $A^\otimes = \frac{1}{2}A'' + \frac{1}{4}S_2(P'')$.

Proof. The point P_l lies outside $\mathcal{C}$, because $P_l \cdot M_{\mathcal{C}}P_l > P_l \cdot M_{\mathcal{E}}P_l = 1$. The two tangents from P_l to $\mathcal{C}$ are the sides $P_{l-1}P_l$ and P_lP_{l+1} (Section 2), which touch $\mathcal{C}$ at P_{l-1}'' and P_l'' ; Lemma 2.1(b) gives the first formula. The point P_l' lies on the tangents of $\mathcal{E}$ at P_l and P_{l+1} and not on $\mathcal{E}$ (it would be the point of contact of both), so it lies outside $\mathcal{E}$, and the two tangents from it touch $\mathcal{E}$ at P_l and P_{l+1} ; Lemma 2.1(b) gives the second formula. The areas follow from Lemma 4.1, as a cyclic relabelling does not change areas. $\square$

For $u \in \mathbb{C}$ we define, with the vertices $P(u + lh)$ of $P^{[u]}$,

$$F(u) = S(P^{[u]}) = \frac{1}{2} \sum_{l=0}^{N-1} P(u + lh) \times P(u + (l+1)h),$$

$$G(u) = S_2(P^{[u]}) = \frac{1}{2} \sum_{l=0}^{N-1} P(u + lh) \times P(u + (l+2)h).$$

The sums are cyclic because $P(u + Nh) = P(u + 4\tau K) = P(u)$ by (J3), and $P^{[u+h]}$ is $P^{[u]}$ relabelled by one step; so F and G have the period h .

Lemma 4.3 (the five areas). *At the member with real phase u ,*

$$A = F(u), \quad A' = \kappa F(u + \Delta), \quad A'' = \frac{F(u + \Delta)}{\kappa},$$

$$A'^\ominus = \frac{F(u)}{2} + \frac{G(u)}{4}, \quad A^\otimes = \frac{1}{\kappa} \left(\frac{F(u + \Delta)}{2} + \frac{G(u + \Delta)}{4} \right).$$

Proof. By Proposition 3.4, P' and P'' are the images of $P^{[u+\Delta]}$ under σ and σ^{-1} , with $\det \sigma = \kappa$; so $A' = \kappa F(u + \Delta)$, $A'' = F(u + \Delta)/\kappa$ and $S_2(P'') = G(u + \Delta)/\kappa$. Lemma 4.2 gives the last two formulas. $\square$

Corollary 4.4 (reductions). *At the member with real phase u , writing $X(v)$ for the value of a quantity X at the member with phase v :*

- (a) $A'/A'' = \kappa^2$, and $AA'' = \kappa^{-2}AA'$;
- (b) $(A'A'^\ominus)(u) = \kappa^2(AA^\otimes)(u + \Delta)$ and $(A'/A'^\ominus)(u) = (A/A^\otimes)(u + \Delta)$.

In particular k_{110} and the row k_{106} (AA' constant) are constant on the same families, and so are k_{807} and k_{809} , and k_{808} and k_{810} .

Proof. (a) is immediate from Lemma 4.3. (b) By Lemma 4.3 and the period $h = 2\Delta$ of F and G , $(AA^\otimes)(u + \Delta) = \kappa^{-1}F(u + \Delta)(\frac{1}{2}F(u) + \frac{1}{4}G(u)) = \kappa^{-2}(A'A'^\ominus)(u)$, and the ratios are treated in the same way. $\square$

5 The area functions on the complex plane

Throughout this section and the next, N , τ and the caustic are fixed as at the end of Section 3, and $R_x = \text{diag}(1, -1)$, $R_y = \text{diag}(-1, 1)$.

- Lemma 5.1** (symmetries). (a) $P(u + 2K) = -P(u)$, $P(u + 2iK') = R_x P(u)$, $P(-u) = R_y P(u)$, and therefore $P(2iK' - u) = -P(u)$.
- (b) F and G are meromorphic on $\mathbb{C}$, and for $X \in \{F, G\}$: $X(u + h) = X(u)$, $X(u + 2K) = X(u)$, $X(u + 2iK') = -X(u)$ and $X(-u) = X(u)$.
- (c) $h\mathbb{Z} + 2K\mathbb{Z} = \frac{2K}{N} \gcd(2\tau, N)\mathbb{Z}$. For odd N this group is $\frac{2K}{N}\mathbb{Z}$, and it contains Δ ; for even N it is $\frac{4K}{N}\mathbb{Z}$, and it does not contain Δ .

Proof. (a) is (J3); then $P(2iK' - u) = R_x P(-u) = R_x R_y P(u) = -P(u)$. (b) F and G are polynomials in the coordinates of the vertices, which are meromorphic. The period h was noted above. The vertex sequence of $P^{[u+2K]}$ is $(-P(u + lh))_l$, and the terms of F , G are quadratic. That of $P^{[u+2iK']}$ is $(R_x P(u + lh))_l$, and $\det R_x = -1$. That of $P^{[-u]}$ is $(P(-u + lh))_l = (R_y P(u - lh))_l$: the reflection R_y of $P^{[u]}$ traversed in the opposite order, and both operations change the signs of S and S_2 . (c) $h\mathbb{Z} + 2K\mathbb{Z} = \frac{2K}{N}(2\tau\mathbb{Z} + N\mathbb{Z})$. For odd N , $\gcd(2\tau, N) = \gcd(\tau, N) = 1$, and $\Delta = \tau \cdot \frac{2K}{N}$. For even N , τ is odd, $\gcd(2\tau, N) = 2$, and $\Delta = \tau \cdot \frac{2K}{N}$ is an odd multiple of $\frac{2K}{N}$. $\square$

Let $\Lambda_0 = h\mathbb{Z} + 2K\mathbb{Z} + 2iK'\mathbb{Z}$ and

$$S_0 = iK' + \Lambda_0, \quad S_1 = S_0 - \Delta.$$

Both are cosets of Λ_0 , and $S_0 + \Delta = S_1 + h = S_1$. We call their points the singular phases.

- Lemma 5.2** (singular phases). (a) Some vertex of $P^{[u]}$ is infinite if and only if $u \in S_0$; F and G are holomorphic on $\mathbb{C} \setminus S_0$.

(b) If N is odd, then $S_0 = S_1$; if N is even, then $S_0 \cap S_1 = \emptyset$.

(c) Let $u \in S_0$. The indices l for which $P(u + lh)$ is infinite form one residue class modulo N if N is odd, and two classes $l_0, l_0 + \frac{N}{2}$ if N is even. F and G have at most simple poles at u . For $N = 4$, G vanishes identically.

Proof. (a) By (J1) the vertex $P(u + lh)$ is infinite if and only if $u + lh \in \Pi$, and $\bigcup_l (\Pi - lh) = S_0$. Elsewhere F and G are polynomials in holomorphic functions. (b) $S_0 = S_1$ if and only if $\Delta \in \Lambda_0$; as Δ is real, this holds if and only if $\Delta \in h\mathbb{Z} + 2K\mathbb{Z}$, that is, by Lemma 5.1(c), if and only if N is odd; otherwise the two cosets are disjoint. (c) Let $P(u + l_0 h)$ be infinite. Then $P(u + lh)$ is infinite if and only if $(l - l_0)h \in 2K\mathbb{Z} + 2iK'\mathbb{Z}$, that is, $(l - l_0)h \in 2K\mathbb{Z}$, that is, $N \mid 2\tau(l - l_0)$, that is, $N \mid 2(l - l_0)$. In particular two vertices with indices differing by 1 are never both infinite ($N \geq 3$), and two with indices differing by 2 only if $N = 4$. The terms of $2F$ containing an infinite vertex P_l are $P_{l-1} \times P_l + P_l \times P_{l+1} = P_l \times (P_{l+1} - P_{l-1})$, where P_l has a simple pole by (J1) and $P_{l\pm 1}$ are holomorphic at u ; so F has at most a simple pole. The same holds for G , with $P_l \times (P_{l+2} - P_{l-2})$, if $N \neq 4$. If $N = 4$, then $h = K$ and $P_{l+2} = P(u + lh + 2K) = -P_l$, so every term $P_l \times P_{l+2}$ of G vanishes. $\square$

Lemma 5.3 (folding). Let $p \in S_0 \cup S_1$ and $X \in \{F, G\}$. Then $X(p - \varepsilon) = -X(p + \varepsilon)$ for all ε at which both sides are finite. Hence the Laurent expansion of X at p contains only odd powers of ε , and if X is holomorphic at p , then $X(p) = 0$.

Proof. Write $p = p_0 + \mu$ with $p_0 \in \{iK', iK' - \Delta\}$ and $\mu = mh + 2nK + 2qiK' \in \Lambda_0$. By Lemma 5.1(b), $X(p \pm \varepsilon) = (-1)^q X(p_0 \pm \varepsilon)$, so we may assume $p = p_0$. If $p_0 = iK'$, then $2iK' - (p_0 + \varepsilon + lh) = p_0 - \varepsilon - lh$. If $p_0 = iK' - \Delta$, then $2iK' - p_0 = iK' + \Delta = p_0 + h$, so $2iK' - (p_0 + \varepsilon + lh) = p_0 - \varepsilon + (1 - l)h$. In both cases Lemma 5.1(a) gives, with $j_0 = 0$ or $j_0 = 1$,

$$P(p_0 - \varepsilon + (j_0 - l)h) = -P(p_0 + \varepsilon + lh) \quad (l \in \mathbb{Z}) :$$

the member with phase $p_0 - \varepsilon$ is the image of the member with phase $p_0 + \varepsilon$ under the half-turn $-I$, traversed in the opposite order. The half-turn preserves S and S_2 , and the reversal changes their signs. $\square$

6 Two theorems on elliptic functions

Theorem A (even N). *Let $N \geq 4$ be even. Then $F(u)F(u + \Delta)$ and $F(u)G(u + \Delta)$ are constant functions of $u \in \mathbb{C}$; call their values c_1 and c_2 . Also $G(u)F(u + \Delta) = c_2$.*

Proof. Let $\Phi(u) = F(u)F(u + \Delta)$. By Lemma 5.1(b), Φ has the periods h and $2iK'$ (each factor changes sign under $u \mapsto u + 2iK'$), which span a lattice. By Lemma 5.2, F has at most simple poles, all in S_0 , and $F(\cdot + \Delta)$ has at most simple poles, all in $S_0 - \Delta = S_1$; and $S_0 \cap S_1 = \emptyset$ since N is even. Let $p \in S_0$. Then $F(\cdot + \Delta)$ is holomorphic at p , and since $p + \Delta \in S_0 + \Delta = S_1$, Lemma 5.3 gives $F(p + \Delta) = 0$; so the at most simple pole of F at p is cancelled, and Φ is holomorphic at p . Let $p \in S_1$. Then F is holomorphic at p and $F(p) = 0$ by Lemma 5.3, while $F(\cdot + \Delta)$ has at most a simple pole at p ; again Φ is holomorphic at p . At all other points both factors are holomorphic. So Φ is an entire doubly periodic function, hence constant by Liouville's theorem. The same argument applies to $F(u)G(u + \Delta)$, because G , like F , has at most simple poles, all in S_0 , and is odd about every point of $S_0 \cup S_1$. Finally, replacing u by $u + \Delta$ and using the period $h = 2\Delta$ of G gives $F(u + \Delta)G(u) = c_2$. $\square$

Theorem B (odd N). *Let $N \geq 3$ be odd. Then $F(u + \Delta) = F(u)$, $G(u + \Delta) = G(u)$ and $G = \gamma F$ on $\mathbb{C}$, where*

$$\gamma = \frac{\operatorname{sn} 2h (1 + \operatorname{dn} h)}{\operatorname{sn} h (1 + \operatorname{dn} 2h)}, \quad h = \frac{4\tau K}{N}.$$

At iK' , F has a simple pole with residue

$$\rho = \frac{iab(\operatorname{dn} h - 1)}{k^2 \operatorname{sn} h} \neq 0,$$

and G has the residue $iab(\operatorname{dn} 2h - 1)/(k^2 \operatorname{sn} 2h)$.

Proof. The period Δ is Lemma 5.1(c). The lattice $\Lambda'' = \frac{2K}{N}\mathbb{Z} + 4iK'\mathbb{Z}$ consists of periods of F and G (by Lemma 5.1(b),(c)), and the possible poles of F and G lie in $S_0 = iK' + \frac{2K}{N}\mathbb{Z} + 2iK'\mathbb{Z}$, which modulo Λ'' consists of the two points iK' and $3iK'$; all poles are at most simple (Lemma 5.2).

At the phase iK' the only infinite vertex is P_0 (Lemma 5.2(c)), and near $\varepsilon = 0$, by (J4),

$$P(iK' + \varepsilon) = -\frac{1}{k\varepsilon}(a, ib) + O(\varepsilon), \quad P(iK' \pm h) = \mp \frac{1}{k \operatorname{sn} h}(a, ib \operatorname{dn} h).$$

The terms of $2F$ containing P_0 are $P_0 \times (P_1 - P_{N-1})$, and $P_1 - P_{N-1} = P(iK' + h) - P(iK' - h) + O(\varepsilon)$. Hence the singular part of $2F$ at $\varepsilon = 0$ is

$$-\frac{1}{k\varepsilon}(a, ib) \times \left(-\frac{2a}{k \operatorname{sn} h}, -\frac{2ib \operatorname{dn} h}{k \operatorname{sn} h} \right) = \frac{2iab(\operatorname{dn} h - 1)}{k^2 \varepsilon \operatorname{sn} h},$$

so $\text{res}_{iK'} F = \rho$. Here $0 < h < 2K$ (as $\tau < N/2$), so $\text{sn } h \neq 0$ and $\text{dn } h \neq 1$ by (J6), and $\rho \neq 0$. The same computation with $P_{\pm 2}$ instead of $P_{\pm 1}$, that is, with $2h$ instead of h , gives the residue of G ; it applies because $2h = 8\tau K/N \notin 2K\mathbb{Z}$ for odd $N \geq 3$.

Let $\gamma = \text{res}_{iK'} G / \text{res}_{iK'} F$ and $H = G - \gamma F$. Then H is Λ'' -periodic, $H(u + 2iK') = -H(u)$, and H is holomorphic at iK' , as the residues cancel; by the anti-periodicity it is holomorphic at $3iK'$ as well, and hence on $\mathbb{C}$. So H is an entire doubly periodic function, hence a constant C , and $C = -C$ by the anti-periodicity; so $G = \gamma F$. Finally, for $v \in \{h, 2h\}$, $(\text{dn } v - 1)/\text{sn } v = (\text{dn}^2 v - 1)/(\text{sn } v(1 + \text{dn } v)) = -k^2 \text{sn } v / (1 + \text{dn } v)$ by (J2), with $1 + \text{dn } v \neq 0$ since $\text{dn } v > 0$ for real v ; this gives the stated form of γ . $\square$

For $N = 3$ we have $2h = 8K/3 \equiv -h \pmod{4K}$, hence $\text{sn } 2h = -\text{sn } h$, $\text{dn } 2h = \text{dn } h$ and $\gamma = -1$, in accordance with $S_2(W) = -S(W)$ for triangles.

Lemma 6.1 (the constant γ). *For odd N ,*

$$\gamma = \frac{2a^2b^2(a^2b^2 - (a^2 + b^2)\lambda)}{a^4b^4 - c^4\lambda^2}, \quad 2 + \gamma = \frac{2(2a^4b^4 - a^2b^2(a^2 + b^2)\lambda - c^4\lambda^2)}{a^4b^4 - c^4\lambda^2} > 0.$$

Proof. By (J2) and the duplication formulas (J5), for every w with $\text{sn } 2w \neq 0$,

$$1 - \text{dn } 2w = \frac{1 - k^2 \text{sn}^4 w - \text{dn}^2 w + k^2 \text{sn}^2 w \text{cn}^2 w}{1 - k^2 \text{sn}^4 w} = \frac{2k^2 \text{sn}^2 w \text{cn}^2 w}{1 - k^2 \text{sn}^4 w}, \quad \frac{1 - \text{dn } 2w}{\text{sn } 2w} = \frac{k^2 \text{sn } w \text{cn } w}{\text{dn } w}.$$

With $w = h$ and $w = \Delta$ (recall $h = 2\Delta$), the formula for γ as a quotient of residues becomes

$$\gamma = \frac{(1 - \text{dn } 2h)/\text{sn } 2h}{(1 - \text{dn } h)/\text{sn } h} = \frac{\text{sn } h \text{cn } h \text{dn } \Delta}{\text{dn } h \text{sn } \Delta \text{cn } \Delta} = \frac{2d_0^2(c_0^2 - s_0^2d_0^2)}{(1 - k^2s_0^4)(d_0^2 - k^2s_0^2c_0^2)},$$

where $s_0, c_0, d_0 = \text{sn } \Delta, \text{cn } \Delta, \text{dn } \Delta$ and the duplication formulas were used once more for $\text{sn } h, \text{cn } h, \text{dn } h$. By Lemma 3.1, $s_0^2 = \lambda/b^2$, $c_0^2 = b''^2/b^2$, $d_0^2 = a^2b''^2/(a''^2b^2)$ and $k^2 = c^2/a''^2$. Using $a''^2 = a^2 - \lambda$, $b''^2 = b^2 - \lambda$ and $c^2 = a^2 - b^2$, one finds

$$c_0^2 - s_0^2d_0^2 = \frac{b''^2(a^2b^2 - (a^2 + b^2)\lambda)}{a''^2b^4}, \quad d_0^2 - k^2s_0^2c_0^2 = \frac{b''^2(a^2b^2 - c^2\lambda)}{a''^2b^4}, \quad 1 - k^2s_0^4 = \frac{b''^2(a^2b^2 + c^2\lambda)}{a''^2b^4},$$

the last one from $a''^2b^4 - c^2\lambda^2 = (b^2 - \lambda)(a^2b^2 + c^2\lambda)$. Inserting these gives the formula for γ , and that for $2 + \gamma$ follows. For $0 < \lambda < b^2$ the denominator is positive, because $c^2\lambda < c^2b^2 < a^2b^2$. The numerator $q(\lambda) = 2a^4b^4 - a^2b^2(a^2 + b^2)\lambda - c^4\lambda^2$ is a concave function of λ with $q(0) = 2a^4b^4 > 0$ and $q(b^2) = b^4(2a^4 - a^4 - a^2b^2 - c^4) = b^6c^2 > 0$; so $q > 0$ on $[0, b^2]$. $\square$

For $N = 3$, $2 + \gamma = 1$ is a quadratic equation for λ , whose positive root

$$\lambda = \frac{a^2b^2(2\sqrt{a^4 - a^2b^2 + b^4} - a^2 - b^2)}{c^4}$$

is the caustic parameter of the 3-periodics. For convex families $2 + \gamma > 0$ also follows from $A > 0$ and $A'^\ominus = \frac{1}{4}(2 + \gamma)A > 0$ (Theorem 7.1).

7 The rows

Theorem 7.1. *Let $a > b > 0$, let $\mathcal{C}$ be a confocal ellipse inside $\mathcal{E}$ such that the billiard trajectories with caustic $\mathcal{C}$ have least period N , let τ be their number of turns around O (so $1 \leq \tau < N/2$ and $\gcd(\tau, N) = 1$), and let all areas be signed. Put $\kappa = ab/(a''b'')$ and $\lambda = a^2 - a''^2$.*

(a) If N is even, then at every member of the family

$$AA^\otimes = \frac{1}{\kappa} \left(\frac{c_1}{2} + \frac{c_2}{4} \right), \quad A'A'^\ominus = \kappa \left(\frac{c_1}{2} + \frac{c_2}{4} \right), \quad AA'' = \frac{c_1}{\kappa}, \quad AA' = \kappa c_1,$$

with the constants c_1, c_2 of Theorem A. So the rows k_{807} and k_{809} (and k_{110}, k_{106}) hold for every even N .

(b) If N is odd, then at every member

$$A'^\ominus = \frac{2+\gamma}{4} A, \quad A^\otimes = \frac{2+\gamma}{4\kappa} A,$$

with γ as in Theorem B and Lemma 6.1, and $2+\gamma > 0$. So $A^\otimes$ and $A'^\ominus$ vanish exactly where A vanishes, and at every member with $A \neq 0$

$$\frac{A}{A^\otimes} = \frac{A'}{A'^\ominus} = \frac{4\kappa}{2+\gamma} = \frac{2\kappa(a^4b^4 - c^4\lambda^2)}{2a^4b^4 - a^2b^2(a^2 + b^2)\lambda - c^4\lambda^2}.$$

(c) If $\tau = 1$, every member is a convex polygon and $A, A', A'', A^\otimes, A'^\ominus$ are positive at every member. So for the convex families of the source the rows k_{808} and k_{810} hold, with the value $4\kappa/(2+\gamma)$, at every member.

Proof. By Lemma 3.3(c) and Section 2, it suffices to consider the members $P^{[u]}$ with real u . (a) By Lemma 4.3 and Theorem A, $AA^\otimes = \kappa^{-1}F(u)(\frac{1}{2}F(u+\Delta) + \frac{1}{4}G(u+\Delta)) = \kappa^{-1}(\frac{1}{2}c_1 + \frac{1}{4}c_2)$, $A'A'^\ominus = \kappa F(u+\Delta)(\frac{1}{2}F(u) + \frac{1}{4}G(u)) = \kappa(\frac{1}{2}c_1 + \frac{1}{4}c_2)$, $AA'' = \kappa^{-1}F(u)F(u+\Delta)$ and $AA' = \kappa F(u)F(u+\Delta)$. (b) By Theorem B, $F(u+\Delta) = F(u)$ and $G = \gamma F$; Lemma 4.3 gives the two formulas, and Lemma 6.1 gives $2+\gamma > 0$ and the last expression. (c) For $\tau = 1$ we have $h = 4K/N$, so the vertices $P(u+lh)$, $0 \leq l < N$, lie on $\mathcal{E}$ in counter-clockwise order and go once around O ; hence $P^{[u]}$ is a convex polygon, positively oriented, and $A > 0$. The same holds for the member $P^{[u+\Delta]}$, so $A' > 0$ and $A'' > 0$ by Proposition 3.4; and the midpoint polygons $P'^\ominus$ of P and $P^\otimes$ of P'' (Lemma 4.2) are convex with the same orientation, so $A'^\ominus > 0$ and $A^\otimes > 0$. $\square$

Closed forms of c_1, c_2 are given here only for small N ; for $\tau = 1$, $c_1 > 0$ and $\frac{1}{2}c_1 + \frac{1}{4}c_2 > 0$ by part (c).

Corollary 7.2 (small N). (a) For $N = 4$: $\lambda = a^2b^2/(a^2 + b^2)$, $\kappa = (a^2 + b^2)/(ab)$, and at every member

$$AA^\otimes = \frac{4a^4b^4}{(a^2 + b^2)^2}, \quad A'A'^\ominus = 4a^2b^2, \quad AA'' = \frac{8a^4b^4}{(a^2 + b^2)^2}, \quad AA' = 8a^2b^2.$$

(b) For $N = 3$: $\gamma = -1$, and $A/A^\otimes = A'/A'^\ominus = 4\kappa$.

(c) For $N = 6$ (with $\tau = 1$): $G = F$, $c_2 = c_1$, $AA^\otimes = \frac{3}{4}AA''$ and $A'A'^\ominus = \frac{3}{4}AA'$.

Proof. (a) For $N = 4$ the member with phase 0 is the rhombus $P(0) = (0, b)$, $P(K) = (-a, 0)$, $P(2K) = (0, -b)$, $P(3K) = (a, 0)$. Its side through $(0, b)$ and $(-a, 0)$ is the line $-x/a + y/b = 1$, which touches $\mathcal{C}$ if and only if $a''^2/a^2 + b''^2/b^2 = 1$, that is, $\lambda = a^2b^2/(a^2 + b^2)$; then $a''^2 = a^4/(a^2 + b^2)$, $b''^2 = b^4/(a^2 + b^2)$ and $\kappa = (a^2 + b^2)/(ab)$. The rhombus has area $A = 2ab$; its outer polygon is the rectangle $[-a, a] \times [-b, b]$, of area $A' = 4ab$; the point of contact of the side $-x/a + y/b = 1$ with $\mathcal{C}$ is $(-a''^2/a, b''^2/b) = (-a^3, b^3)/(a^2 + b^2)$, so the inner polygon is the rectangle with the vertices $(\pm a^3, \pm b^3)/(a^2 + b^2)$, of area $A'' = 4a^3b^3/(a^2 + b^2)^2$. Every polygon W with $W_{l+2} = -W_l$ has

$S_2(W) = 0$, so by Lemma 4.2 $A'^\ominus = \frac{1}{2}A = ab$ and $A^\otimes = \frac{1}{2}A''$. This gives the four values at the rhombus, and by Theorem 7.1(a) they hold at every member. (b) was shown after Theorem B: $\gamma = -1$. (c) For even N the members are centrally symmetric, $P(u+lh+\frac{N}{2}h) = P(u+lh+2\tau K) = -P(u+lh)$ as τ is odd. For a hexagon with $W_{l+3} = -W_l$ one finds $S(W) = S_2(W) = W_0 \times W_1 + W_1 \times W_2 + W_0 \times W_2$. So $G = F$, hence $c_2 = c_1$, and Theorem 7.1(a) gives the two relations. $\square$

Remark 7.3 (what follows from printed results). For $N = 3$ the inverted polygons are medial triangles, $A'^\ominus = \frac{1}{4}A$ and $A^\otimes = \frac{1}{4}A''$, so $k_{810} = 4A'/A$ and $k_{808} = 4A/A'' = 4\kappa^2 A/A'$ are constant by [AST22, Theorem 3] (A'/A constant for odd N) and, for k_{808} , Stachel's relation. For $N = 4$ and $N = 6$ the relations $S_2 = 0$ and $S_2 = S$ reduce k_{807}, k_{809} to k_{110}, k_{106} , which follow from Chavez-Caliz's theorem (Proposition 7.4). For odd $N \geq 5$ and even $N \geq 8$ we found no such reduction in print: the rows need the area G and its relation to F (Theorems A and B).

Proposition 7.4 (the row k_{110} ; not new). *For every even N and every member, $AA'' = \kappa^{-2}AA'$ is constant. This follows from Chavez-Caliz's theorem [CC21, Theorem 3 = Theorem 6] together with Stachel's relation $A' = \kappa^2 A''$ [Sta21, Subsection 3.2], or from Chavez-Caliz's theorem applied to the inner polygons; Theorem 7.1(a) gives a third, self-contained proof.*

Proof. Chavez-Caliz's theorem: if two concentric ellipses γ_0, Γ in general position (transversal at every common point in $\mathbb{CP}^2$) admit a one-parameter family of Poncelet n -gons P inscribed in Γ and circumscribed about γ_0 , and Q is the polygon formed by the tangents to Γ at the vertices of P , then $A(P)A(Q)$ is constant for even n . (i) With $(\Gamma, \gamma_0) = (\mathcal{E}, \mathcal{C})$ we have $Q = P'$, so AA' is constant (row k_{106}), and $A' = \kappa^2 A''$ by Proposition 3.4. (ii) The side $P_{l-1}''P_l''$ is the chord of contact of the tangents from P_l to $\mathcal{C}$, the polar of P_l with respect to $\mathcal{C}$; as P_l runs on $\mathcal{E}$, these polars touch the polar reciprocal $\mathcal{E}^* = \{X \cdot M_{\mathcal{C}} M_{\mathcal{E}}^{-1} M_{\mathcal{C}} X = 1\}$ of $\mathcal{E}$, the ellipse with semi-axes $a''^2/a, b''^2/b$. So the inner polygons are Poncelet polygons for $(\Gamma, \gamma_0) = (\mathcal{C}, \mathcal{E}^*)$, and the tangents to $\mathcal{C}$ at their vertices form P (those at P_{l-1}'' and P_l'' meet at P_l); hence $A''A$ is constant. It remains to check general position. Two coaxial ellipses $x^2/p_1 + y^2/q_1 = 1, x^2/p_2 + y^2/q_2 = 1$ with $p_1q_2 \neq p_2q_1$ (here $a^2b''^2 \neq b^2a''^2$, that is, $\lambda c^2 \neq 0$, for both pairs) have no common point at infinity, and their common points are $(\pm x_0, \pm y_0)$ with (x_0^2, y_0^2) the solution of a linear system: $(a^2a''^2, -b^2b''^2)/c^2$ for $(\mathcal{E}, \mathcal{C})$ and $(a''^4, -b''^4)/c^2$ for $(\mathcal{C}, \mathcal{E}^*)$. So there are four distinct common points, and by Bézout's theorem each has intersection multiplicity 1. $\square$

Garcia and Reznik [GR22] computed closed forms of k_{106} and k_{110} for $N = 4$ and $N = 6$. Their $N = 6$ values $k_{106} = 4a^2b^2(2a+b)(a+2b)/(a+b)^2$ and $k_{110} = 4a^3b^3(2a+b)^2(a+2b)^2/(a+b)^6$ agree with our computations (Section 9), which do not prove them; with Corollary 7.2(c) they would give $k_{807} = 3a^3b^3(2a+b)^2(a+2b)^2/(a+b)^6$ and $k_{809} = 3a^2b^2(2a+b)(a+2b)/(a+b)^2$ for $N = 6$. For $N = 4$ the value of k_{110} printed in [GR22, Proposition 4.9] and in the arXiv versions v1–v3 of [GR22] is $2a^4b^4/(a^2+b^2)^2$, whereas Corollary 7.2(a) gives $8a^4b^4/(a^2+b^2)^2$.

8 Rows outside the records

The following consequences concern other rows of [RGK21, v11]. They are not among the records treated here, and the printed ones are not new.

Proposition 8.1. *Under the hypotheses of Theorem 7.1:*

- (a) $A'/A'' = \kappa^2 = [ab/(a''b'')]^2$ for every N (row k_{113}).
- (b) For odd N : $A' = \kappa A$ and $A = \kappa A''$ at every member, so $A'/A = A/A'' = \kappa$ (rows k_{103} and k_{109}) and $A'A''/A^2 = 1$ (row k_{112}) wherever $A \neq 0$.

- (c) For no N is $A'A''$ constant along the family (row k_{111} of $v11$, stated there for even N). At iK' the function F has a simple pole with residue ρ for odd N and 2ρ for even N , where $\rho = iab(\operatorname{dn} h - 1)/(k^2 \operatorname{sn} h) \neq 0$.

Proof. (a) is Corollary 4.4(a), and (b) follows from Lemma 4.3 and $F(u + \Delta) = F(u)$ (Theorem B). (c) By Lemma 4.3, $A'A'' = F(u + \Delta)^2$. For odd N the residue is Theorem B. For even N , at the phase iK' the infinite vertices are P_0 and $P_{N/2} = -P_0$ (Lemma 5.2(c); $P(v + 2\tau K) = -P(v)$ as τ is odd), with $P_{N/2\pm 1} = -P_{\pm 1}$; so the terms $P_{N/2} \times (P_{N/2+1} - P_{N/2-1}) = P_0 \times (P_1 - P_{N-1})$ contribute as much as $P_0 \times (P_1 - P_{N-1})$, and the computation in the proof of Theorem B gives the residue 2ρ ; again $0 < h < 2K$, so $\rho \neq 0$. So F is a non-constant meromorphic function, hence not constant on the real line, and neither is F^2 (if F^2 were constant on an interval, F would be constant there). As u runs through $\mathbb{R}$, so does $u + \Delta$. $\square$

Row k_{113} is stated in [RGK21, v11] with this value, credited to a private communication of Stachel, and is immediate from [Sta21, Subsection 3.2]. Row k_{103} is [AST22, Theorem 3] (the source also cites [CC21]); k_{109} then follows by (a), and the source credits k_{112} to a private communication of Akopyan. For $N = 4$ the row k_{111} was refuted in an earlier note of the author [Fer26e]. Theorem A, as $AA' = \kappa c_1$, re-proves [CC21, Theorem 6] for confocal pairs.

The hypotheses. *The caustic.* The rows concern elliptic caustics: the source's a'', b'' are the semi-axes of an ellipse, and k_{807} inverts in the caustic ellipse. Trajectories whose caustic is a confocal hyperbola (they exist for even N) are not treated; the verification runs computed some such families numerically (in some $A \equiv 0$, in others with $N = 6, 10$ the products $AA'', A'A'^\ominus$ were constant); we prove nothing about them. *The period.* We use least period N , as the source does; the results hold for every number of turns τ , while the source's tables concern $\tau = 1$; repetitions of shorter trajectories are not treated. *The areas.* All areas are signed; for $\tau = 1$ all five are positive (Theorem 7.1(c)), so the rows also hold for unsigned areas. For odd N and $\tau \geq 2$, at a member with $A = 0$ (if there is one) the ratios are $0/0$, while the identities of Theorem 7.1(b) hold. *The inversion.* The radial inversion in an ellipse and the midpoint of the chord of contact agree (Lemma 2.1), so there is no ambiguity in $A^\otimes$ and $A'^\ominus$; inverting the vertices of P in $\mathcal{E}$ instead of $\mathcal{C}$ would return P itself.

9 Computations and reproducibility

All results of this note are proved without computation; the computations test the statements, and test the reduction of Sections 3 and 4 against trajectories computed without it. The package has three parts.

The program of this note, check_note.py (mpmath at 50 digits and sympy; about 13 seconds), tests the statements as printed. (i) Trajectories by the reflection law alone (the caustic, found by bisection, enters only through the first chord): 56 families, $(a, b) \in \{(1, 0.5), (1, 0.8), (2, 1.2), (1.7, 0.3)\}$, $N = 3, \dots, 10$ with $\tau = 1$ and six star families with $N \leq 12$ per ellipse, seven members each; the derived polygons from their definitions, the inversions both as $Y/(Y \cdot MY)$ and as midpoints of chords of contact. The rows $k_{110}, k_{807}, k_{809}$ (even N) and k_{808}, k_{810} (odd N) are constant in every family (largest relative spread $3.1 \cdot 10^{-42}$, largest closure defect $3.4 \cdot 10^{-42}$), all five vary for the other parity (relative spread at least $3.0 \cdot 10^{-13}$), the side rows behave as in Proposition 8.1, and for $\tau = 1$ all areas are positive. (ii) c_1, c_2 from Jacobi functions at complex phases for even N ; $G = \gamma F$, the three expressions for γ and $4\kappa/(2 + \gamma)$ for odd N (largest relative deviation $2.2 \cdot 10^{-39}$). (iii) The whole family for $N = 4$ in exact arithmetic, with $P_0 = (-as, bt)$, $P_1 = (-at/d, -b^3s/(a^2d))$, $P_2 = -P_0$, $P_3 = -P_1$ modulo $t^2 = 1 - s^2$, $d^2 = 1 - (1 - b^4/a^4)s^2$: tangency, reflection law and the values of Corollary 7.2(a); also the algebra of Lemma 6.1 and of the proof of Proposition 7.4. (iv) On the

trajectories of (i): Lemma 3.1, $2N\Delta = 4\tau K$, Proposition 3.4 and Lemma 4.3 (largest relative deviation $2.0 \cdot 10^{-38}$). (v) Residues, oddness, the limit in Lemma 3.3(a), Lemma 6.1, Corollary 7.2(b),(c) and the values of [GR22] quoted in Section 7. All 194 checks pass. *The program with which the results were first obtained* (`original/verify_rows.py`; 40 digits, 669 checks, all passed) tests the rows with controls, the identities at complex phases and the formulas of Sections 4 and 6. *The programs of the three verification runs* are included with their recorded outputs, in the folders `verification_run_A/`, `verification_run_B/` and `independent_run_2/`; see the next paragraph.

Verification. The results were first obtained, with proofs of the lemmas and of Theorems A and B, and with the program in `original/`. Two independent verification runs, both AI-assisted, followed; each re-derived every proof, wrote its own programs, and re-ran the original program, whose recorded output both reproduced exactly. The first run compared the rows and definitions with the tex source of [RGK21, v11] and its journal version, read the cited works, computed trajectories by plane geometry alone at 60 digits (95 families), checked the analytic statements at complex phases and the algebra in exact arithmetic, and computed some families with hyperbolic caustics. The second run read the definitions independently, computed trajectories by the reflection law at 70 digits (245 families, all turning numbers for $N \leq 14$), the family for $N = 4$ in exact arithmetic, and $2 + \gamma$ for all odd $N \leq 51$ and all τ on a grid of moduli (always positive); it found the proof (ii) of Proposition 7.4 and repeated the literature search. Neither run found an error in a lemma or a theorem. Both reported that k_{110} follows from printed results and required that it be presented so; both required the residue for even N in Proposition 8.1(c) and corrections of citations; the first run required the limit argument in Lemma 3.3(a) (the first write-up had not noted that K diverges); and the second asked to state or prove the condition $2 + \gamma \neq 0$ for $\tau \geq 2$, left open in the first write-up. All corrections were made. When this note was written, Lemma 6.1, which removes that condition, the identity $\text{sn}(K - \Delta) = a''/a$ in Lemma 3.3(a) and the program `check_note.py` were added.

A third verification run, also AI-assisted, examined the text of this note, these additions first: it re-derived Lemma 6.1 from the quotient of the residues, with the positivity of $2 + \gamma$, and the identity and the limit in Lemma 3.3(a), then every other proof line by line; it compared `check_note.py` with Section 9, and the definitions and quoted statements with the tex source of [RGK21, v11], the arXiv tex of [Sta21, Sta22a, AST22, GR22] and the journal versions of [CC21, GR22]; and it repeated the literature search and the search of Zenodo, including the newest records of the two authors of the 2026 notes on sibling rows. Its three programs agree in every comparison: trajectories built from the tangents to the caustic at 60 digits, with the reflection law checked and the inversions computed by the radial definition (five ellipses, all turning numbers for $N \leq 13$; 140 families); the analytic statements at complex phases, residues by contour integrals, the limit in Lemma 3.3(a) at 250 digits, and γ for all 890 odd families with $N \leq 41$ and five moduli; and, in exact arithmetic, the algebra of Lemmas 2.1, 3.1, 3.2, 4.1 and 6.1 and Proposition 7.4, and Corollary 7.2(a) also at two members with irrational vertices. All recorded outputs of the package were reproduced exactly from the archive. It found no error in a lemma or a theorem; it required ten corrections of the text and of the package, which were made, among them that Proposition 8.1(b) is now stated by the identities $A' = \kappa A$ and $A = \kappa A''$ and that two sentences of Section 2 were made precise. No proof depends on a computation.

Scope and priority. The rows and the observation that they hold are due to Reznik, Garcia and Koiller [RGK21]; the parametrisation [Sta22a, Theorem 2] and the conjugate-billiard relation [Sta21, Subsection 3.2] are Stachel's, and the method is Chavez-Caliz's [CC21]. The row k_{110} is not new: it follows at once from [CC21, Theorem 3] and Stachel's relation (Proposition 7.4), and Garcia and Reznik gave closed forms of it for $N = 4, 6$ [GR22]. The cases $N = 3$ of k_{808} , k_{810} follow from [AST22, Theorem 3] (with Stachel's relation for k_{808}), and the cases $N = 4, 6$ of k_{807} , k_{809} from

[CC21] (Remark 7.3). The statements of Section 8 are printed or attributed in the source, except that k_{111} fails for every N (for $N = 4$ see [Fer26e]). To our knowledge the rows k_{807} – k_{810} for every N of the stated parity, the constancy of $F(u)G(u + \Delta)$ for even N and $G = \gamma F$ with Lemma 6.1 are not in print. Related refereed notes of the author are [Fer26c, Fer26d] (other rows of the same table), [Fer26b] (central symmetry) and [Fer26f, Fer26a] (the same method). No printed or deposited proof of k_{807} – k_{810} for general N was found in the sources searched: the source (arXiv v11 and journal), [GR22] (arXiv v1–v3 and journal), [CC21, Sta21, Sta22a, AST22, BT22], the survey [GR21], the works citing [RGK21] in OpenAlex (19) and Semantic Scholar (31) and those citing [CC21] in OpenAlex (23), arXiv queries on elliptic billiards, on areas, midpoints and inversions of Poncelet polygons and on the works of the proposers, of Stachel, Tabachnikov and Chavez-Caliz, zbMATH Open, Crossref, and Zenodo, including all records of the two authors of the 2026 notes on sibling rows. A search that finds nothing is not a proof of novelty, and we claim no priority.

What was read: [RGK21] in the tex source of arXiv v11 and in the journal version; [CC21] in arXiv v1 and in the journal; [Sta21] (Section 3) and [Sta22a] (Theorem 2) in arXiv v2, whose statement numbers we use; [AST22], [BT22] in arXiv v2; [GR22] in arXiv v1–v3 and in the journal. The proofs of the cited theorems were not checked. Not read: the chapter [Sta23] as published (an author’s preprint with this title, read by two verification runs, treats envelopes of diagonals and contains no statement on areas) and [Sta22b], of which only the abstract was read; what we say about [GSO16] is taken from [RGK21]. Bibliographic data were checked with Crossref and DataCite. The searches were made on 15 October 2026, with the public interfaces of the databases named and one general web search; the Zenodo search was repeated when this note was written, and all of them except Semantic Scholar, with one more web search, by the third verification run on 15–16 October 2026.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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