

A Focal Inversion Area Ratio for Four-Periodic Elliptic Billiards

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Abstract

The experimental invariant list of Reznik, Garcia and Koiller prints the value 2 for the focal-inverse area of the outer tangent polygon divided by that of a four-periodic elliptic billiard orbit. We give a rational counterexample to this value and a self-contained area calculation showing that the correct constant is $1/2$. The result concerns vertexwise inversion of both polygons about the same focus and signed shoelace area. It applies to primitive four-periodic orbits with a confocal elliptic caustic. This is a reciprocal correction to a tabulated constant, not a refutation of the underlying constancy or a claim about the corresponding general-period conjecture.

1 The printed assertion and an exact counterexample

Let an ellipse have center O , semiaxes $a > b > 0$ and a focus F . For an ordered polygon $W = (W_1, \dots, W_N)$, define

$$\text{area}(W) = \frac{1}{2} \sum_{i=1}^N \det(W_i, W_{i+1}), \quad W_{N+1} = W_1.$$

This is the signed-area convention of [1, §2, equation (1)]. The outer tangent polygon P' of an orbit P has vertices at the intersections of the boundary tangents at consecutive orbit vertices. Inversion in the unit circle centered on F acts on vertices by

$$\mathcal{I}_F(X) = F + \frac{X - F}{|X - F|^2}.$$

In Section 3.9 and Table 9 of arXiv:2004.12497v11, the row $k_{806,b}$ prints $A_F^{\dagger}/A_F^{\dagger} = 2$ for $N = 4$, where $A_F^{\dagger} = \text{area}(\mathcal{I}_F(P'))$ and $A_F^{\dagger} = \text{area}(\mathcal{I}_F(P))$. The prime is applied before inversion: the outer polygon of an already inverted polygon is not meant here.

Consider

$$\frac{x^2}{25} + \frac{y^2}{16} = 1, \quad F = (3, 0), \quad P = ((5, 0), (0, 4), (-5, 0), (0, -4)).$$

At each axis vertex the reflection law changes just the corresponding velocity component; all edges have length $\sqrt{41}$. They are tangent to the strictly interior confocal ellipse

$$\frac{x^2}{625/41} + \frac{y^2}{256/41} = 1.$$

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For example, the edge $4x+5y=20$ is tangent at $(125/41, 64/41)$, since $16(625/41)+25(256/41)=400$. The other contacts follow by axis symmetry. Both ellipses have squared focal distance 9, and the four distinct vertices give a genuine orbit of least period four.

Its outer polygon is $P' = ((5, 4), (-5, 4), (-5, -4), (5, -4))$. Omitting the common translation by F , the inverse vertices are

$$\begin{aligned}\mathcal{I}_F(P) - F &= \left(\left(\frac{1}{2}, 0 \right), \left(-\frac{3}{25}, \frac{4}{25} \right), \left(-\frac{1}{8}, 0 \right), \left(-\frac{3}{25}, -\frac{4}{25} \right) \right), \\ \mathcal{I}_F(P') - F &= \left(\left(\frac{1}{10}, \frac{1}{5} \right), \left(-\frac{1}{10}, \frac{1}{20} \right), \left(-\frac{1}{10}, -\frac{1}{20} \right), \left(\frac{1}{10}, -\frac{1}{5} \right) \right).\end{aligned}$$

Shoelace gives

$$A_F^\dagger = \frac{1}{10}, \quad A_F'^\dagger = \frac{1}{20}, \quad \frac{A_F'^\dagger}{A_F^\dagger} = \frac{1}{2} \neq 2.$$

Every inversion is defined and both areas are positive. Unsigned areas therefore give the same counterexample. We next show that this value is not exceptional.

2 The corrected area identity

Theorem 1. *Let P be a billiard orbit of least period four inside an ellipse with semiaxes $a > b > 0$, tangent to a strictly nested confocal elliptic caustic. Let P' be its outer tangent polygon in the corresponding cyclic order. For either focus F , both inverse areas are nonzero and*

$$\frac{\text{area}(\mathcal{I}_F(P'))}{\text{area}(\mathcal{I}_F(P))} = \frac{1}{2}.$$

The quotient is unchanged if the same positive inversion radius is used for both polygons.

2.1 A rectangular support frame

We use the classical central symmetry of even-period billiards with an elliptic caustic, as recalled in the proof of [2, proof of Theorem 5.2]. The primitive four-periodic orbit has vertices $P, Q, -P, -Q$ in positive order. It is a convex parallelogram; in general it is not a rhombus.

Let u and v be the unit directions of the sides from P to Q and from Q to $-P$. The following directions are $-u, -v$. The reflection law makes the outward normals at P and Q parallel to $-(u+v)$ and $u-v$, respectively. These nonzero vectors are perpendicular, because $(u+v) \cdot (u-v) = |u|^2 - |v|^2 = 0$. Hence consecutive boundary tangents are perpendicular and P' is a rectangle.

Choose positively oriented orthonormal coordinates along these two outward normals. In this frame write $F = (f, g)$. The ellipse is $X^\top D^{-1} X = 1$, where

$$D = b^2 I + F F^\top, \quad a^2 = b^2 + f^2 + g^2.$$

Set

$$h^2 = b^2 + f^2, \quad k^2 = b^2 + g^2, \quad H = a^2 b^2, \quad h, k > 0. \quad (1)$$

The support points in the two coordinate directions are

$$P = (h, fg/h), \quad Q = (fg/k, k), \quad (2)$$

and the outer vertices in positive order are

$$X = (h, k), \quad Y = (-h, k), \quad -X, \quad -Y.$$

We will repeatedly use

$$h^2 k^2 - f^2 g^2 = H, \quad h^2 + g^2 = k^2 + f^2 = a^2. \quad (3)$$

The focus is strictly inside the ellipse and the rectangle, so all inversion denominators are positive.

2.2 Inverting the orbit

As $h > |f|$ and $k > |g|$, equations (1)–(3) give the following four inverse vertices relative to F :

$$\begin{aligned}\mathcal{I}_F(P) - F &= H^{-1}(h^2(h+f), -hg(h+f)), \\ \mathcal{I}_F(Q) - F &= H^{-1}(-fk(k+g), k^2(k+g)), \\ \mathcal{I}_F(-P) - F &= H^{-1}(-h^2(h-f), -hg(h-f)), \\ \mathcal{I}_F(-Q) - F &= H^{-1}(-fk(k-g), -k^2(k-g)).\end{aligned}$$

For example, $P - F = (h-f)(1, -g/h)$ has squared length $a^2(h-f)^2/h^2$, and $(h-f)(h+f) = b^2$ gives the first formula. For any four ordered vertices W_1, W_2, W_3, W_4 ,

$$\text{area}(W) = \frac{1}{2} \det(W_3 - W_1, W_4 - W_2).$$

The inverse diagonals are $(-2h^3, 2hfg)/H$ and $(2kfg, -2k^3)/H$. Consequently,

$$A_F^\dagger = \frac{2hk(h^2k^2 - f^2g^2)}{H^2} = \frac{2hk}{H} > 0. \quad (4)$$

This agrees with the already published orbit area-product formula [3, Proposition 4.9]; that prior formula is not a separate new claim of this note.

2.3 Inverting the outer rectangle

We first give a four-term identity, independent of billiards.

Lemma 2. *Suppose $|X|^2 = |Y|^2 = R^2$, and none of $X, Y, -X, -Y$ equals F . With $C = \det(X, Y)$,*

$$\text{area}(\mathcal{I}_F(X, Y, -X, -Y)) = \frac{2C(R^4 - |F|^4)}{|X - F|^2|Y - F|^2 - |X + F|^2|Y + F|^2}.$$

Proof. Put $B = R^2 + |F|^2$, $s = 2F \cdot X$, $t = 2F \cdot Y$, $p = \det(F, X)$ and $q = \det(F, Y)$. The four squared distances are $B - s, B - t, B + s, B + t$. The four cross-product numerators are $C + p - q, C + p + q, C - p + q, C - p - q$. On putting their half-sum over the product of the four squared distances, its numerator is $2(CB^2 + B(tp - sq))$. The identity $tp - sq = -2|F|^2C$ gives the result. $\square$

For our rectangle $C = 2hk$, $R^2 = a^2 + b^2$ and $R^4 - |F|^4 = 4a^2b^2 = 4H$. Its squared-distance product is

$$\begin{aligned}\prod_{W \in \{X, Y, -X, -Y\}} |W - F|^2 &= 16 \left[(a^4 - h^2f^2 - k^2g^2)^2 - 4h^2k^2f^2g^2 \right] \\ &= 16 \left[(H + 2f^2g^2)^2 - 4(H + f^2g^2)f^2g^2 \right] \\ &= 16H^2.\end{aligned}$$

Lemma 2 therefore yields

$$A_F^\dagger = \frac{2(2hk)(4H)}{16H^2} = \frac{hk}{H} > 0. \quad (5)$$

Equations (4) and (5) prove Theorem 1. The other focus replaces (f, g) by $(-f, -g)$ and leaves these area formulas unchanged. Reversing the orbit order changes both signs. Inversion of radius $r > 0$ multiplies both areas by r^4 , without changing their quotient.

3 Interpretation and verification

The numerical value in the printed row is reciprocal to the correct one: $A_F^\dagger/A_F'^\dagger = 2$, whereas the displayed outer-over-original quotient equals $1/2$. Thus the constant-value assertion as written is false, but its constancy survives. We make no claim here about $k_{806,a}$ for general periods. We also exclude hyperbolic-caustic self-intersecting four-periodics: their two focal inverses can have zero area [3, Corollary 3.9], so extending the quotient to them would require interpreting $0/0$.

The rational example includes exact checks of incidence, reflection, caustic tangency, outer tangents and signed areas. A second rational checker tests the support-frame formulas for many non-axis choices. These checks supplement the displayed general algebra; they do not replace it. The original numerical invariant and the known orbit area product remain credited to their sources. A bounded search of current primary literature did not locate a reciprocal correction; this does not certify absolute priority. This self-audited, unrefereed note was prepared with AI assistance and has no independent human review or proof-assistant formalization.

References

- [1] D. Reznik, R. Garcia and J. Koiller, *Eighty New Invariants in the Elliptic Billiard*, arXiv:2004.12497v11 (2020), Section 3.9 and Table 9. [arXiv source](#).
- [2] H. Stachel, *On the Motion of Billiards in Ellipses*, European Journal of Mathematics **8** (2022), 1602–1622. [doi:10.1007/s40879-021-00524-2](#).
- [3] R. Garcia and D. Reznik, *Exploring Self-Intersected N -Periodics in the Elliptic Billiard*, Annales Mathematicae et Informaticae (2022). [doi:10.33039/ami.2022.02.001](#).