

Focal Inversion Area Ratios in Elliptic Billiards

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Abstract

Fix an elliptic billiard and a nondegenerate confocal elliptic caustic. For every primitive periodic family whose least period is congruent to two modulo four, the signed area of the orbit divided by the signed area of its unit focal inversion is a positive phase invariant, identical for the two foci. The argument includes every admissible coprime star winding. On a quotient of the Jacobi parameter torus the two area functions have the same two possible simple poles and the same imaginary antiperiodicity; cancellation of one residue therefore forces proportionality. We prove the requisite pole-order reduction and real nonvanishing explicitly. Exact six-periodic controls reproduce the known value $32/27$, while a doubled-triangle example excludes an interpretation based only on the length of an arbitrarily repeated vertex list. Circles and repetitions of admitted primitive orbits are treated separately. No assertion for hyperbolic caustics or absolute-priority claim is made.

1 Statement and conventions

Fix the confocal ellipses

$$E : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad C : \frac{x^2}{a^2 - \lambda} + \frac{y^2}{b^2 - \lambda} = 1, \quad a > b > 0, \quad 0 < \lambda < b^2.$$

Put $c = \sqrt{a^2 - b^2}$ and $F_e = (ec, 0)$, where $e \in \{1, -1\}$. For an ordered polygon $W = (W_0, \dots, W_{n-1})$, use signed area

$$S(W) = \frac{1}{2} \sum_{j=0}^{n-1} \det(W_j, W_{j+1}), \quad W_n = W_0.$$

For stars this is the shoelace area in trajectory order, not a sum of bounded-face areas. Unit inversion at a focus is

$$I_{F_e}(X) = F_e + \frac{X - F_e}{(X - F_e) \cdot (X - F_e)}.$$

The dot product is Euclidean on the real plane and bilinear, not Hermitian, in the complex continuation. Translating all inverse vertices by $-F_e$ does not affect their area.

The area ratio is row k_{805} in the original arXiv source [1, Table 9, p. 11]. Its geometric context is a pair of confocal ellipses. The present statement specifies least period and retains that elliptic-caustic context. The simple six-periodic formula is already proved by Garcia and Reznik [2, Proposition 4.16]; it is not a new special case here.

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Theorem 1. *Suppose a billiard family on E tangent to C has genuine least period $n \geq 6$, with $n \equiv 2 \pmod{4}$. Then*

$$\frac{S(P)}{S(I_{F_e}(P))}$$

is a positive constant throughout the family, identical for both foci. Every admissible coprime star winding is included. Any positive integer number of traversals of an admitted primitive orbit has the same ratio.

The caustic and winding are fixed while the initial phase varies. Reversal negates both signed areas and preserves the ratio. Neither hyperbolic nor degenerate caustics are included. The circular case is handled directly in Section 5. Least period cannot be replaced by the length of an arbitrary repeated list, as the exact example in Section 5.2 shows.

2 The confocal parameter and the trace torus

Write

$$a_0 = \sqrt{a^2 - \lambda}, \quad b_0 = \sqrt{b^2 - \lambda}, \quad k = c/a_0, \quad k' = b_0/a_0, \quad K = K(k), \quad K' = K(k').$$

All Jacobi functions below have modulus $k \in (0, 1)$. Choose $\delta \in (0, K)$ with $\text{cn } \delta = b_0/b$. Then

$$a = \frac{a_0 \text{dn } \delta}{\text{cn } \delta}, \quad b = \frac{b_0}{\text{cn } \delta}, \quad \text{sn}^2 \delta = \frac{\lambda}{b^2}.$$

The classical confocal parametrization [3] is

$$P(u) = (-a \text{sn } u, b \text{cn } u), \quad P_j = P(u + jh), \quad h = 2\delta. \quad (1)$$

For completeness, the caustic point $T(v) = (-a_0 \text{sn } v, b_0 \text{cn } v)$ has tangent

$$-\text{sn } v \frac{X}{a_0} + \text{cn } v \frac{Y}{b_0} = 1.$$

The addition formulas [4, Section 22.8] show that the tangents at $v = u \pm \delta$ intersect at $P(u)$. Therefore $P(u)$ and $P(u + 2\delta)$ share the caustic tangent at $T(u + \delta)$. The consistent forward tangent branch is the reflected billiard map. This identifies the actual billiard shift; it does not assume a constant increment in eccentric anomaly.

The real parameter traverses the boundary once modulo $4K$. After reversal if necessary, primitivity gives

$$h = \frac{4\tau K}{n}, \quad \gcd(\tau, n) = 1, \quad 0 < \tau < n/2. \quad (2)$$

Put $n = 2m$ with m odd. Then τ is odd, and

$$P(u + nh/2) = -P(u). \quad (3)$$

Define the meromorphic vector and area functions

$$\begin{aligned} Q_e(u) &= \frac{P(u) - F_e}{(a + e \text{csn } u)^2}, \\ A(u) &= \frac{1}{2} \sum_j \det(P(u + jh), P(u + (j+1)h)), \\ B_e(u) &= \frac{1}{2} \sum_j \det(Q_e(u + jh), Q_e(u + (j+1)h)). \end{aligned} \quad (4)$$

The exact focal-distance identity is

$$(P(u) - F_e) \cdot (P(u) - F_e) = (a + ec \operatorname{sn} u)^2. \quad (5)$$

It follows from $\operatorname{cn}^2 u = 1 - \operatorname{sn}^2 u$ and $c^2 = a^2 - b^2$. In particular, $a + ec \operatorname{sn} u \geq a - c > 0$ for real u , so all real inverse vertices are finite.

Both vectors have periods $4K, 4iK'$, and cyclic relabeling gives period h for each area function. Bézout's identity applied to (2) gives the smaller common real period

$$L = \frac{4K}{n} = \frac{2K}{m}. \quad (6)$$

Under $u \mapsto u + 2iK'$, $\operatorname{sn} u$ is unchanged and $\operatorname{cn} u$ changes sign [4, Section 22.4]. Each vector is reflected in the major axis, which fixes each focus. Therefore

$$A(u + 2iK') = -A(u), \quad B_e(u + 2iK') = -B_e(u). \quad (7)$$

We work on the compact quotient torus

$$\mathcal{T} = \mathbb{C}/(L\mathbb{Z} + 4iK'\mathbb{Z}).$$

The shift $2iK'$ is an antiperiod, not an actual period of the areas.

3 All possible poles and their orders

Let $p = iK'$. The original vector has only simple poles at

$$p + 2\ell K + 2riK', \quad \ell, r \in \mathbb{Z}.$$

Two consecutive original vertices cannot be singular together: $0 < h < 2K$, while two real-separated Jacobi poles differ by $2K$ modulo $4K$. Thus each original edge determinant and A have at most simple poles. On $\mathcal{T}$, the real shifts by $2K = mL$ and $jh = j\tau L$ disappear. Consequently the only possible poles of A are

$$p, \quad p + 2iK'. \quad (8)$$

At a pole of P , the numerator of Q_e has order at most one, whereas its denominator has exactly order two with nonzero leading coefficient. Hence Q_e is holomorphic there and vanishes. Its remaining possible poles are the zeros of $g_e = a + ec \operatorname{sn} u$. On the larger torus with periods $4K, 4iK'$, these are precisely

$$Z_e \pm \delta, \quad Z_e + 2iK' \pm \delta, \quad Z_{-1} = K + iK', \quad Z_1 = 3K + iK'. \quad (9)$$

Indeed the standard shift formulas give

$$P(K + iK' + w) = \left(-\frac{a \operatorname{dn} w}{k \operatorname{cn} w}, -\frac{ibk'}{k \operatorname{cn} w} \right). \quad (10)$$

Together with $a/c = \operatorname{dn} \delta / (k \operatorname{cn} \delta)$, this identifies the roots. At every root,

$$\operatorname{cn}^2 u = -\frac{b^2}{c^2}, \quad \operatorname{dn}^2 u = -\frac{\lambda}{a_0^2}.$$

Thus $g'_e = ec \operatorname{cn} u \operatorname{dn} u \neq 0$. The four displayed roots are distinct and exhaust the zeros, because sn has four simple poles on the larger torus and hence degree four. The vector Q_e has at most double poles at these roots and no others.

Lemma 2. *Every possible pole of the inverse area B_e is at most simple.*

Proof. Consider $u_* = Z_e - \delta$; all other singular phases are translates of this one. Exactly two consecutive inverse vertices are singular, at arguments $Z_e - \delta$ and $Z_e + \delta$, because $h = 2\delta$. On a horizontal period line these are the only two roots for the fixed focus, and the orbit arguments are distinct modulo $4K$. Formula (10) is even in w , and translation by $2K$ negates P . Thus $P(Z_e - w) = P(Z_e + w)$ for either center, so the two original values coincide at a finite point P_* .

At either root u_0 , put $w = u - u_0$ and write

$$V = P - F_e = V_0 + V_1 w + O(w^2), \quad g_e = g_1 w + g_2 w^2 + O(w^3), \quad g_1 \neq 0.$$

Equation (5) implies $V_0 \cdot V_0 = V_0 \cdot V_1 = 0$. The vector $V_0 = (eb^2/c, b \operatorname{cn} u_0)$ is nonzero. In complex dimension two the orthogonal complement of a nonzero isotropic vector is its own line; hence V_1 is parallel to V_0 . Both negative Laurent coefficients of V/g_e^2 are on this same line:

$$\frac{V_0}{g_1^2} w^{-2}, \quad \left(\frac{V_1}{g_1^2} - \frac{2g_2 V_0}{g_1^3} \right) w^{-1}.$$

The two singular vertices share $V_0 = P_* - F_e$. Their determinant has no terms of orders four or three. Every remaining determinant has at most one singular vertex, so B_e has order at most two.

The index reversal $j \mapsto 1 - j$ and the evenness about Z_e give

$$Q_e(u_* - \epsilon + jh) = Q_e(u_* + \epsilon + (1 - j)h).$$

Reversing cyclic order negates signed area. Thus $B_e(u_* - \epsilon) = -B_e(u_* + \epsilon)$, and the Laurent series is odd. Its order-two coefficient vanishes, proving the lemma. This argument still applies if other original vertices are singular at u_* : their inverse vertices are holomorphic zeros. Disjointness of original and inverse pole sets has not been assumed. $\square$

The possible inverse-area poles are the points in (9) translated by $-jh$. The two roots $Z_e \pm \delta$ differ by h , so they give one location per imaginary row on the quotient torus. Since m and τ are odd,

$$K - \delta = \frac{(m - \tau)K}{m} = \frac{m - \tau}{2} L \in L\mathbb{Z}. \quad (11)$$

Adding $2K$ is also an integer real-period shift. Therefore all possible poles of B_e reduce to the same two points (8) as those of A , and Lemma 2 bounds their order by one.

4 Proportionality and real nonvanishing

Lemma 3. *On $\mathcal{T}$, the meromorphic functions with at most simple poles at $p, p + 2iK'$ and satisfying $f(u + 2iK') = -f(u)$ form a vector space of dimension at most one.*

Proof. Let f be a nonzero such function. Its residue at p is nonzero: otherwise that pole is removable, and antiperiodicity makes the second pole removable too. The resulting entire function on a compact torus would be constant, then zero by antiperiodicity, a contradiction. For another such function g , set

$$r = \frac{\operatorname{Res}_{u=p} g(u)}{\operatorname{Res}_{u=p} f(u)}.$$

Then $g - rf$ has no pole at p . Antiperiodicity cancels its principal part at $p + 2iK'$ as well. It is entire and hence constant on the compact torus; antiperiodicity makes that constant zero. Thus $g = rf$. $\square$

For the real forward orientation, every directed chord has the elliptic caustic to its left. This also holds for the admitted star windings: the step $0 < h < 2K$ advances less than a half-turn on the boundary. The center and both foci lie strictly inside the caustic. Consequently every summand below is positive:

$$\det(P_j, P_{j+1}) > 0, \quad \det(Q_{e,j}, Q_{e,j+1}) = \frac{\det(P_j - F_e, P_{j+1} - F_e)}{|P_j - F_e|^2 |P_{j+1} - F_e|^2} > 0.$$

In particular $A > 0$ and $B_e > 0$ at every real phase; A is not identically zero. Lemma 3 applies and gives $B_e = r_e A$ for a constant $r_e > 0$. Hence $A/B_e = 1/r_e$ is defined and positive everywhere on the real family. This proves constancy without assuming anything about unknown complex zeros of B_e .

Finally, (3) and relative inversion give $Q_e(u + nh/2) = -Q_{-e}(u)$. Cyclic relabeling and simultaneous negation leave area unchanged, so $B_e = B_{-e}$. The constants are equal for the two foci, proving Theorem 1.

5 Extensions, exact values, and the period convention

If a polygon is traversed d times, each shoelace sum is multiplied by d ; its defined area ratio is unchanged. Thus arbitrary repetitions of an admitted primitive family are covered, even when the repeated list length is not congruent to two modulo four. For a circular billiard of radius a , the two foci coincide with its center and unit inversion scales vertices by a^{-2} . Therefore

$$S(I_O(P)) = a^{-4} S(P), \quad \frac{S(P)}{S(I_O(P))} = a^4$$

whenever $S(P) \neq 0$. Regular stars with $0 < \tau < n/2$ have positive signed area. This is a direct circular calculation, not a limiting argument on a degenerating elliptic torus.

5.1 A known six-periodic value

Garcia and Reznik [2, Proposition 4.16] prove, for simple six-periodics and unit inversion, the value

$$\frac{S(P)}{S(I_F(P))} = \frac{4a^3b^4}{(2a-b)(a+b)^2}.$$

For $a = 2$, $b = 1$, $\lambda = 4/9$, this gives $32/27$. Two exact phases in this family are

$$\begin{aligned} H &= ((2, 0), (4/3, \sqrt{5}/3), (-4/3, \sqrt{5}/3), \\ &\quad (-2, 0), (-4/3, -\sqrt{5}/3), (4/3, -\sqrt{5}/3)), \\ V &= ((0, 1), (-4\sqrt{2}/3, 1/3), (-4\sqrt{2}/3, -1/3), \\ &\quad (0, -1), (4\sqrt{2}/3, -1/3), (4\sqrt{2}/3, 1/3)). \end{aligned}$$

The common caustic has squared semiaxes $32/9, 5/9$. For example, the slanted sides $\sqrt{5}x + 2y = 2\sqrt{5}$ of H and $x - 2\sqrt{2}y = -2\sqrt{2}$ of V have respective squared caustic supports 20 and 8, the squares of their right-hand sides. The horizontal or vertical sides are tangent at caustic vertices; axis reflections give the remaining tangencies. Direct reflection checks give genuine primitive billiards, and inversion at either focus gives

	$S(P)$	$S(I_F(P))$	$S(P)/S(I_F(P))$
H	$20\sqrt{5}/9$	$15\sqrt{5}/8$	$32/27$
V	$32\sqrt{2}/9$	$3\sqrt{2}$	$32/27$

These are reproducibility controls for a known result, not the proof of the all-period theorem.

5.2 A doubled-triangle obstruction

Consider the ellipse with squared semiaxes 21, 16, the confocal caustic parameter $\lambda = 336/25$, and the focus $F = (\sqrt{5}, 0)$. Let

$$T = ((\sqrt{21}, 0), (-3\sqrt{21}/5, 16/5), (-3\sqrt{21}/5, -16/5)).$$

Its vertices lie on the ellipse. The caustic has squared semiaxes $189/25, 64/25$. Its vertical side is tangent at the left caustic vertex. A slanted side is $2x + \sqrt{21}y = 2\sqrt{21}$, whose squared caustic support is

$$4\frac{189}{25} + 21\frac{64}{25} = 84 = (2\sqrt{21})^2.$$

At the upper nonaxial vertex, the incoming unit velocity is $v = (-\sqrt{21}/5, 2/5)$ and a scaled outward normal is $N = (-\sqrt{21}/7, 1)$. Then $v - 2(v \cdot N)N/(N \cdot N) = (0, -1)$, the outgoing unit velocity. At the axial vertex the horizontal component reverses, and reflection in the major axis verifies the last collision. All contacts are internal. Thus T is a primitive triangular billiard, as is its central negative $-T$ in the same fixed family.

For this fixed focus, exact inversion and shoelace evaluation give

$$\begin{aligned} S(T) &= S(-T) = 128\sqrt{21}/25, \\ S(I_F(T)) &= \frac{\sqrt{21}(147 - \sqrt{105})}{12096}, \\ S(I_F(-T)) &= \frac{\sqrt{21}(147 + \sqrt{105})}{12096}. \end{aligned}$$

Both inverse areas are positive, and therefore

$$\frac{S(T)}{S(I_F(T))} = \frac{10584 + 72\sqrt{105}}{25}, \quad \frac{S(-T)}{S(I_F(-T))} = \frac{10584 - 72\sqrt{105}}{25}.$$

Traversing each triangle twice yields lists of length six with the same unequal ratios. Hence a condition on arbitrary listed length $N \equiv 2 \pmod{4}$ is insufficient. This does not refute Theorem 1; its hypothesis is genuine least period.

6 Attribution and reproducibility

The simple-six formula is prior work [2]. The confocal parameter [3], Jacobi identities [4], and residue cancellation on a compact torus are classical. The argument above proves proportionality through exhaustive pole control rather than assuming a transfer from a different focal-inversion invariant. A bounded primary-source search did not identify a general theorem for this exact area ratio, but does not certify novelty, priority, or completeness of the literature search. The broader frozen problem record does not specify every caustic and repetition convention; no unqualified resolution of all its interpretations is claimed.

The accompanying exact arithmetic controls verify ellipse incidence, unit-velocity reflection, common-caustic tangency, internal contacts, positive focal denominators, the displayed area ratios, and their repetition behavior. Finite numerical checks at higher periods are diagnostics, not proofs or certified error bounds. Neither replaces the all-period argument in Sections 2–4.

AI assistance was used in deriving the argument, checking algebra, reviewing sources, and preparing this manuscript. The author is responsible for its claims and presentation. This is a self-audited, unrefereed preprint; no independent human review, proof-assistant formalization, or absolute-priority certification is claimed.

References

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