

Focal Inversion Area Products in Elliptic Billiards

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Abstract

For an elliptic billiard with a fixed nondegenerate confocal elliptic caustic, invert the orbit vertices in a unit circle centered at either focus. When the billiard has least period divisible by four, the signed area of the orbit times the signed area of its focal inversion is a positive constant throughout the family, identical for the two foci. The proof treats every admissible coprime star winding by complexifying the Jacobi parameter and cancelling all possible poles. In particular, the apparent higher-order poles of the inverted area reduce to simple poles by focal isotropic tangency and reversal symmetry. Two exact six-periodic orbits in the same family show that a repeated list of length twelve does not satisfy the same assertion. The result addresses the genuine-period interpretation of source invariant $k_{804,a}$; the known four-periodic case is credited, and no absolute-priority or unqualified resolution of the source's repetition convention is claimed.

1 The geometric objects and the theorem

Fix the confocal ellipses

$$E : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad C : \frac{x^2}{a^2 - \lambda} + \frac{y^2}{b^2 - \lambda} = 1, \quad a > b > 0, \quad 0 < \lambda < b^2.$$

Put $c = \sqrt{a^2 - b^2}$ and $F_e = (ec, 0)$ for $e \in \{1, -1\}$. For an ordered polygon $W = (W_0, \dots, W_{n-1})$, define its signed area by the cyclic shoelace sum

$$S(W) = \frac{1}{2} \sum_{j=0}^{n-1} \det(W_j, W_{j+1}).$$

This is the algebraic area in trajectory order, including for star polygons, not a sum of bounded-face areas. Unit inversion at F_e is

$$I_{F_e}(X) = F_e + \frac{X - F_e}{(X - F_e) \cdot (X - F_e)}.$$

For real points the denominator is the Euclidean squared distance. In the complex continuation, the dot product is bilinear, not Hermitian. The translation by F_e does not change a polygon's area and will be omitted in formulas for the inverse vertices.

The original arXiv version of Reznik, Garcia and Koiller lists $S(P)S(I_{F_e}(P))$ as invariant $k_{804,a}$ for a period divisible by four [1, Table 9, p. 11]. It concerns inversion of the *original* orbit vertices, not an outer tangent, pedal, or polar polygon. The final journal version [2] removes this row and uses k_{804} for a different expression. Garcia and Reznik retain the area product as $k_{805,a}$ and prove its simple four-periodic value 4 in [3, Proposition 4.9].

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Theorem 1. *Suppose a billiard trajectory on E tangent to C has genuine least period $n = 4r$. Then*

$$S(P)S(I_{F_e}(P))$$

is a positive constant as its initial phase varies, and the constant is identical for the two foci. Every admissible coprime star winding is included. For d traversals of such a primitive orbit, both areas are multiplied by d , and their product by d^2 .

Reversal changes the signs of both areas and preserves their product. The caustic and winding are fixed throughout each family. Hyperbolic and degenerate caustics are not asserted. We distinguish least period from the length of a repeated list; Section 6.1 shows why this matters. The circular case is treated separately below.

2 The actual confocal billiard parameter

Set

$$a_0 = \sqrt{a^2 - \lambda}, \quad b_0 = \sqrt{b^2 - \lambda}, \quad k = c/a_0, \quad k' = b_0/a_0, \quad K = K(k), \quad K' = K(k').$$

All Jacobi functions have modulus $k \in (0, 1)$. Choose the unique $\delta \in (0, K)$ for which $\text{cn } \delta = b_0/b$. Confocality gives

$$a = \frac{a_0 \text{dn } \delta}{\text{cn } \delta}, \quad b = \frac{b_0}{\text{cn } \delta}, \quad \text{sn}^2 \delta = \frac{\lambda}{b^2}, \quad \text{dn}^2 \delta = \frac{a^2(b^2 - \lambda)}{b^2(a^2 - \lambda)}.$$

The standard confocal parameter [4] is

$$P(u) = (-a \text{sn } u, b \text{cn } u), \quad P_j = P(u + jh), \quad h = 2\delta. \quad (1)$$

For a direct identification, the caustic point $T(v) = (-a_0 \text{sn } v, b_0 \text{cn } v)$ has tangent

$$-\text{sn } v \frac{X}{a_0} + \text{cn } v \frac{Y}{b_0} = 1.$$

The Jacobi addition formulas [5, Section 22.8] show that the tangents at $v = u \pm \delta$ meet at $P(u)$. Thus $P(u)$ and $P(u + 2\delta)$ share the caustic tangent at $T(u + \delta)$; the consistent forward tangent branch is the reflected billiard map. The real parameter traverses E once modulo $4K$. After reversing orientation if needed, primitivity means

$$h = \frac{4\tau K}{n}, \quad \gcd(\tau, n) = 1, \quad 0 < \tau < n/2. \quad (2)$$

This includes stars and does not suppose that eccentric anomaly advances uniformly. As $n = 4r$, τ is odd, and

$$P(u + nh/2) = -P(u). \quad (3)$$

3 The complete complex singularity sets

Work on the compact torus with periods $4K$ and $4iK'$, and put $p = iK'$ and $Z = K + iK'$. Define

$$Q_e(u) = \frac{P(u) - F_e}{(a + e \text{csn } u)^2}, \quad (4)$$

$$A(u) = \frac{1}{2} \sum_j \det(P(u + jh), P(u + (j + 1)h)),$$

$$B_e(u) = \frac{1}{2} \sum_j \det(Q_e(u + jh), Q_e(u + (j + 1)h)).$$

These are meromorphic functions on the stated torus. The denominator identity

$$(P(u) - F_e) \cdot (P(u) - F_e) = (a + e c \operatorname{sn} u)^2 \quad (5)$$

follows from $\operatorname{cn}^2 u = 1 - \operatorname{sn}^2 u$ and $c^2 = a^2 - b^2$. For real u , $a + e c \operatorname{sn} u \geq a - c > 0$, so every inverse vertex is finite.

The standard shift identities [5, Section 22.4] give

$$\begin{aligned} P(p + w) &= \left(-\frac{a}{k \operatorname{sn} w}, -\frac{ib \operatorname{dn} w}{k \operatorname{sn} w} \right), \\ P(Z + w) &= \left(-\frac{a \operatorname{dn} w}{k \operatorname{cn} w}, -\frac{ibk'}{k \operatorname{cn} w} \right). \end{aligned} \quad (6)$$

Consequently

$$P(p - w) = -P(p + w), \quad P(Z - w) = P(Z + w). \quad (7)$$

Also $P(u + 2K) = -P(u)$, while $P(u + 2iK')$ is the reflection of $P(u)$ in the x -axis. In particular $2iK'$ is not a common period: the areas change sign under that shift. The larger common period $4iK'$ is used throughout.

The only poles of P are the simple poles

$$p + 2\ell K + 2miK', \quad \ell, m \in \{0, 1\}, \quad (8)$$

modulo the chosen lattice. At each such pole Q_e is holomorphic and vanishes, since its numerator has order at most one and its denominator exactly order two, with nonzero leading coefficient.

The remaining possible poles of Q_e are the roots of $a + e c \operatorname{sn} u$. For $e = 1$ they are precisely

$$Z + 2K \pm \delta, \quad Z + 2K + 2iK' \pm \delta; \quad (9)$$

for $e = -1$, omit $2K$. Indeed (6) gives $\operatorname{sn}(Z + w) = \operatorname{dn} w / (k \operatorname{cn} w)$ and $a/c = \operatorname{dn} \delta / (k \operatorname{cn} \delta)$. At each listed root,

$$\operatorname{cn}^2 u = -\frac{b^2}{c^2}, \quad \operatorname{dn}^2 u = -\frac{\lambda}{a_0^2},$$

so the derivative $e c \operatorname{cn} u \operatorname{dn} u$ is nonzero. The four roots are distinct because $0 < \delta < K$. They exhaust the zeros: sn has four simple poles on this torus, hence degree four. Thus Q_e has at most double poles at (9) and no other poles.

The possible poles of A and B_e are the respective sets (8) and (9), translated by $-jh$. They are disjoint when $4 \mid n$. A coincidence would imply

$$K \pm \delta + jh = 2\ell K, \quad n + 2\tau(2j \pm 1) = 2\ell n,$$

which is impossible modulo four since τ is odd. Therefore A is holomorphic at every possible pole of B_e , and conversely. No two consecutive original vertices have poles simultaneously: $0 < h < 2K$, whereas real-separated poles of P differ by $2K$ modulo $4K$. Each summand of A , and thus A , has at most a simple pole. Nonadjacent simultaneous poles do not increase that order.

4 Reduction of the inverse-area pole order

Fix a possible singular phase $u_* = Z_e - \delta$, where Z_e is the appropriate center from (9). Exactly two inverse vertices are singular, with parameters $Z_e - \delta$ and $Z_e + \delta$. They are consecutive because $2\delta = h$, and have the same original value P_* by (7). There are no further singular vertices: on the

horizontal period line a fixed focus has just these two roots, and the orbit parameters are distinct modulo $4K$. This assertion also holds for $n = 4$; the antipodal vertices correspond to the other focus and are regular for the fixed inversion.

Let $V = P - F_e$ and $g = a + e \operatorname{csn}$. At a root u_0 , write $V = V_0 + V_1 w + O(w^2)$ and $g = g_1 w + g_2 w^2 + O(w^3)$, where $w = u - u_0$ and $g_1 \neq 0$. Identity (5) implies

$$V_0 \cdot V_0 = 0, \quad V_0 \cdot V_1 = 0.$$

Here $V_0 \neq 0$: its coordinates are $(eb^2/c, b \operatorname{cn} u_0)$. In complex dimension two the perpendicular space to a nonzero isotropic vector is its own line. Hence V_1 is parallel to V_0 . Both negative Laurent coefficients of $Q_e = V/g^2$ are consequently parallel to V_0 :

$$\frac{V_0}{g_1^2} w^{-2}, \quad \left(\frac{V_1}{g_1^2} - \frac{2g_2 V_0}{g_1^3} \right) w^{-1}.$$

The two singular vertices share the same isotropic direction $V_0 = P_* - F_e$. Their determinant therefore has no terms of orders four or three. Every other determinant has at most one singular vertex. Thus B_e has pole order at most two at u_* .

The cyclic index reversal $j \mapsto 1 - j$ and the evenness in (7) give

$$Q_e(u_* - \epsilon + jh) = Q_e(u_* + \epsilon + (1 - j)h).$$

Reversing polygon order negates signed area, so

$$B_e(u_* - \epsilon) = -B_e(u_* + \epsilon). \quad (10)$$

An odd Laurent series has no term of order two. Therefore every possible pole of B_e is at most simple. The identical reversal applied to P makes A odd about u_* . Since A is holomorphic there, $A(u_*) = 0$. This zero cancels the possible simple pole of B_e in AB_e , at every translated singular phase.

5 Cancellation at the original-area poles

Central symmetry (3) implies

$$B_e(u) = B_{-e}(u). \quad (11)$$

Indeed relative inversion at opposite foci satisfies

$$\frac{-X - F_e}{(-X - F_e) \cdot (-X - F_e)} = -\frac{X - F_{-e}}{(X - F_{-e}) \cdot (X - F_{-e})}.$$

A half-grid relabeling and simultaneous negation of every inverse vertex leave its area unchanged.

At p , the oddness in (7), followed by $j \mapsto -j$, gives

$$B_e(p - \epsilon) = -B_{-e}(p + \epsilon) = -B_e(p + \epsilon).$$

Since B_e is holomorphic at p , $B_e(p) = 0$. Cyclic shifts, the $2K$ shift and reflection after a $2iK'$ shift give the same zero at all possible poles (8) of A and all their $-jh$ translates. The poles of A are at most simple, so AB_e is holomorphic there as well.

All possible poles have now been exhausted. Thus AB_e is a holomorphic function on a compact torus and is constant. This is actual cancellation of all principal parts, not an inference from translation invariance of a pole set. For a real forward trajectory, O and both foci lie strictly inside the elliptic caustic, to the left of each directed chord. Hence

$$\det(P_j, P_{j+1}) > 0, \quad \det(Q_{e,j}, Q_{e,j+1}) = \frac{\det(P_j - F_e, P_{j+1} - F_e)}{|P_j - F_e|^2 |P_{j+1} - F_e|^2} > 0.$$

The constant is positive, also for a star winding. Equation (11) gives equality for the two foci and proves Theorem 1. No proportionality to an outer-area expression or separate Poncelet area-product theorem is assumed.

The constant may be evaluated by setting $u = 0$ in the finite sums. For instance, an exact eight-periodic fixture with $a^2 = 40$, $b^2 = 15$, $\lambda = 24/7$ gives $133/5$. This finite value is not used in the general proof.

6 Repetitions, circles, and a repeated-length obstruction

Repeating an admitted least-period orbit d times multiplies both areas by d , hence their product by d^2 . For concentric circular billiards of radius a , inversion at their common center scales vertices by a^{-2} . A regular star of period n and winding τ has area $na^2 \sin(2\pi\tau/n)/2$, and therefore area product $n^2 \sin^2(2\pi\tau/n)/4$. This is a direct circular extension; the elliptic argument does not take a limit of degenerating tori.

6.1 Two exact six-periodic phases

Consider the ellipse $x^2/4 + y^2 = 1$, its caustic with squared axes $32/9, 5/9$ ($\lambda = 4/9$), and the fixed focus $F = (\sqrt{3}, 0)$. The ordered hexagons

$$\begin{aligned} H &= ((2, 0), (4/3, \sqrt{5}/3), (-4/3, \sqrt{5}/3), \\ &\quad (-2, 0), (-4/3, -\sqrt{5}/3), (4/3, -\sqrt{5}/3)), \\ V &= ((0, 1), (-4\sqrt{2}/3, 1/3), (-4\sqrt{2}/3, -1/3), \\ &\quad (0, -1), (4\sqrt{2}/3, -1/3), (4\sqrt{2}/3, 1/3)) \end{aligned}$$

are genuine primitive billiards in the same fixed-caustic family. Incidence follows directly by substitution. A slanted side of H is $\sqrt{5}x + 2y = 2\sqrt{5}$; its squared support on the caustic is

$$5 \frac{32}{9} + 4 \frac{5}{9} = 20 = (2\sqrt{5})^2.$$

Its horizontal side $y = \sqrt{5}/3$ is tangent at the top caustic vertex. A slanted side of V is $x - 2\sqrt{2}y = -2\sqrt{2}$, with squared support

$$\frac{32}{9} + 8 \frac{5}{9} = 8 = (-2\sqrt{2})^2.$$

Its vertical side $x = -4\sqrt{2}/3$ is tangent at the left caustic vertex. Axis reflections verify the other sides; all contacts are interior.

At H 's vertex $(4/3, \sqrt{5}/3)$, the incoming unit velocity and an outward normal are

$$v = (-2/3, \sqrt{5}/3), \quad N = (1/3, \sqrt{5}/3).$$

The reflection $v - 2(v \cdot N)N / (N \cdot N)$ is $(-1, 0)$, the outgoing unit velocity. At V 's vertex $(-4\sqrt{2}/3, 1/3)$, the corresponding data are

$$v = (-2\sqrt{2}/3, -1/3), \quad N = (-\sqrt{2}/3, 1/3),$$

and reflection gives $(0, -1)$. At the axial vertices the normal component reverses; symmetry verifies the other collisions. The distinct six vertices and these directed reflections establish the least period six, rather than merely an arbitrary inscribed hexagon.

Direct focal inversion and shoelace evaluation give

	$S(P)$	$S(I_F(P))$	$S(P)S(I_F(P))$
H	$20\sqrt{5}/9$	$15\sqrt{5}/8$	$125/6$
V	$32\sqrt{2}/9$	$3\sqrt{2}$	$64/3$

Two traversals produce lists of length twelve but products $250/3$ and $256/3$. They differ even though the ellipse, caustic and focus are fixed. This refutes the unrestricted listed-length interpretation, not Theorem 1. A retained exact primitive-triangle calculation supplies an additional fourfold-repetition obstruction. Because the source does not explicitly settle the repetition convention, we do not claim an unqualified resolution of every reading of its record.

7 Attribution and reproducible controls

The four-periodic value 4 is prior work [3]. The canonical parameter [4], Jacobi shifts and addition formulas [5], and holomorphic constancy on a compact torus are classical. Chávez-Caliz's Poncelet area-product proof [6] supplies methodological precedent, not a theorem about the focal-inverted polygon. The present proof derives the necessary cancellation directly. A bounded primary-source search did not match a general proof of this exact $4r$ assertion, but does not establish absolute novelty, priority, or completeness of the literature.

The accompanying standard-library exact checkers retain polynomial identities for the focal square and isotropic tangent, original-ellipse incidence, unit-velocity reflection, common-caustic tangency, internal contacts, positive focal denominators, and the displayed area values. They include four-periodic phases, the eight-periodic fixture and the two repeated six-periodic controls. Numerical diagnostics are labeled as finite tests. Neither these tests nor their exact arithmetic replace the exhaustive complex singularity argument.

AI assistance was used in deriving the argument, checking algebra, reviewing sources and preparing this manuscript. The author is responsible for its claims and presentation. This is a self-audited, unrefereed preprint; no independent human review, proof-assistant formalization, or absolute-priority certification is claimed.

References

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