

Cosine Sums of Focal-Inversive Elliptic Billiards

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12 October 2026

Abstract

Fix an ellipse and a nondegenerate confocal elliptic caustic admitting a periodic billiard family. We prove that inversion of the vertices about either focus preserves a phase-independent sum of internal-angle cosines, provided the genuine least period is not four. The proof includes every admissible coprime star winding. A meromorphic continuation of the inverted-edge unit direction has an explicit eight-point divisor; the only possible double-pole collision occurs at genuine period four. Reflection symmetry removes all remaining simple poles of the cyclic cosine trace. The two focal constants coincide. An exact four-periodic pair also explains why the hypothesis cannot be replaced by the length of an arbitrary repeated vertex list. The theorem addresses the elliptic-caustic, genuine-period formulation of the focal-inversive cosine conjecture. Hyperbolic caustics are not asserted, and known special cases and exceptions are credited.

1 The conjecture and its scope

Let

$$E : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad C : \frac{x^2}{a^2 - \lambda} + \frac{y^2}{b^2 - \lambda} = 1, \quad a > b > 0, \quad 0 < \lambda < b^2.$$

Write $c = \sqrt{a^2 - b^2}$ and $F_{\pm} = (\pm c, 0)$. Unit inversion at F sends a vertex P_i to

$$Q_i = F + \frac{P_i - F}{|P_i - F|^2}.$$

The angle cosine at Q_i means

$$\frac{(Q_{i-1} - Q_i) \cdot (Q_{i+1} - Q_i)}{|Q_{i-1} - Q_i| |Q_{i+1} - Q_i|}. \quad (1)$$

This convention also applies to star polygons in trajectory order; no choice of bounded faces is involved.

The conjectured constancy of the sum of (1) appears as k_{803} in Table 9 of the arXiv version of Reznik, Garcia and Koiller [1], and as k_{804} in their journal version [2]. The original setting consists of two confocal ellipses. Roitman, Garcia and Reznik formulate the focal-inversive assertion as Conjecture 1 in [3]; their proved bicentric cosine invariant and focal-inversive perimeter invariant are different statements. The triangular case is already known; see [3, Proposition 1 and Appendix E.1]. The elliptic-function method used below is classical and is not claimed as new.

Theorem 1. *Suppose the billiard on E with caustic C has genuine least period $n \geq 3$ and fixed coprime winding. If $n \neq 4$, the sum of the angle cosines of its focal-inversive polygon is constant throughout the family. The result includes all admissible star windings, and the constant is identical for F_+ and F_- . Any positive inversion radius gives the same sum.*

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Repeated traversal of an allowed primitive orbit multiplies its sum by the repetition count. Least period, rather than the length of a possibly repeated list, is essential. We give an exact boundary check in Section 6. Hyperbolic and degenerate caustics are not included. The known hyperbolic type-II six-periodic exception is [4, Counter-example 4.19]; it is not a new counterexample here.

2 The billiard step and its chord direction

Put

$$a_0 = \sqrt{a^2 - \lambda}, \quad b_0 = \sqrt{b^2 - \lambda}, \quad k = \frac{c}{a_0}, \quad k' = \frac{b_0}{a_0}, \quad K = K(k), \quad K' = K(k').$$

All Jacobi functions below have modulus $k \in (0, 1)$. Choose $\delta \in (0, K)$ by $\text{cn } \delta = b_0/b$ and set $h = 2\delta$. Then

$$a = \frac{a_0 \text{dn } \delta}{\text{cn } \delta}, \quad b = \frac{b_0}{\text{cn } \delta}, \quad P(u) = (-a \text{sn } u, b \text{cn } u), \quad P_j = P(u + jh). \quad (2)$$

This is the canonical billiard parameter [5, Theorem 2]. For completeness, the tangent to C at $(-a_0 \text{sn } v, b_0 \text{cn } v)$ is

$$-\frac{\text{sn } v}{a_0}X + \frac{\text{cn } v}{b_0}Y = 1.$$

The addition formulas [6, §22.8] show that the tangents at $v = u \pm \delta$ meet at $P(u)$, so consecutive vertices share the tangent at $u + \delta$. The real parameter runs once around E in $4K$. After reversing orientation if necessary, primitivity gives

$$h = \frac{4\tau K}{n}, \quad \gcd(\tau, n) = 1, \quad 0 < \tau < n/2. \quad (3)$$

Consequently $h = K$ holds precisely for genuine period four.

Set $D(v) = 1 - k^2 \text{sn}^2 v \text{sn}^2 \delta$. The chord from $P(v - \delta)$ to $P(v + \delta)$ is

$$-\frac{2a \text{sn } \delta \text{dn } v}{D(v)}(\text{cn } v, k' \text{sn } v). \quad (4)$$

The two addition formulas give this identity directly, using $b \text{dn } \delta / a = k'$. On the real axis, $D(v) > 0$, $\text{dn } v > 0$, and $\text{cn}^2 v + k'^2 \text{sn}^2 v = \text{dn}^2 v$. Thus the *positive* chord length and its complex unit direction are

$$L(v) = \frac{2a \text{sn } \delta \text{dn}^2 v}{D(v)}, \quad e(v) = -\frac{\text{cn } v + ik' \text{sn } v}{\text{dn } v}. \quad (5)$$

In particular $e^{-1} = -(\text{cn } v - ik' \text{sn } v) / \text{dn } v$ meromorphically.

3 A meromorphic formula for the inverted angles

Fix $F = F_-$ and omit the harmless translation by F after inversion. Define

$$\begin{aligned} g(u) &= a - c \text{sn } u, \\ z(u) &= -a \text{sn } u + c + ib \text{cn } u, \quad \tilde{z}(u) = -a \text{sn } u + c - ib \text{cn } u, \\ m(u) &= z(u)/g(u). \end{aligned}$$

The identity $z\tilde{z} = g^2$ follows from $c^2 = a^2 - b^2$ and $\text{sn}^2 u + \text{cn}^2 u = 1$. For real u , $g(u) \geq a - c > 0$ is the Euclidean distance from $P(u)$ to F . Therefore $|m(u)| = 1$ and the inverse vertex is $q(u) = z(u)/g(u)^2 = 1/\tilde{z}(u)$.

The inverted chord has positive length $L(v)/(g(v - \delta)g(v + \delta))$. On the real axis,

$$q_+ - q_- = -\frac{\overline{z_+ - z_-}}{\overline{z_+} \overline{z_-}}.$$

Dividing by that length yields its unit direction

$$R(v) = -e(v)^{-1}m(v - \delta)m(v + \delta). \quad (6)$$

Equation (6) defines the meromorphic continuation; complex conjugation is not used off the real axis. It follows that

$$C(u) = -\frac{1}{2} \left(\frac{R(u - \delta)}{R(u + \delta)} + \frac{R(u + \delta)}{R(u - \delta)} \right), \quad S(u) = \sum_{j=0}^{n-1} C(u + jh) \quad (7)$$

agrees with the actual cosine and cosine sum on the real axis. The minus sign reverses the incoming forward edge at a vertex.

Work on the common compact torus

$$\mathbb{C}/\Lambda, \quad \Lambda = 4K\mathbb{Z} + 4iK'\mathbb{Z}.$$

Every displayed function is meromorphic on this torus. Also S is h -periodic, since $nh = 4\tau K$. The imaginary shift $2iK'$ is *not* used as a common period of all three Jacobi functions.

4 The complete divisor and the exceptional collision

Set $Z = K + iK'$ and $V = 2iK'$. The standard formulas [6, Table 22.4.3] are

$$\operatorname{sn}(Z + w) = \frac{\operatorname{dn} w}{k \operatorname{cn} w}, \quad \operatorname{cn}(Z + w) = -\frac{ik'}{k \operatorname{cn} w}, \quad \operatorname{dn}(Z + w) = \frac{ik' \operatorname{sn} w}{\operatorname{cn} w}. \quad (8)$$

Under V , the triple $(\operatorname{sn}, \operatorname{cn}, \operatorname{dn})$ becomes $(\operatorname{sn}, -\operatorname{cn}, -\operatorname{dn})$; under $2K$ it becomes $(-\operatorname{sn}, -\operatorname{cn}, \operatorname{dn})$.

The zeros of g are exactly $Z \pm \delta$ and $Z \pm \delta + V$, modulo Λ . They follow from (8) and $a/c = \operatorname{dn} \delta / (k \operatorname{cn} \delta)$. They are simple: at a root, $\operatorname{cn}^2 u = -b^2/c^2$ and $\operatorname{dn}^2 u = -\lambda/a_0^2$ are nonzero. They exhaust the roots because sn has four simple poles on the chosen torus. At $Z \pm \delta$,

$$\operatorname{sn} u = a/c, \quad \operatorname{cn} u = -ib/c, \quad z(u) = 0, \quad \tilde{z}(u) = -2b^2/c \neq 0.$$

Hence z has order two and m has order one there. The V shift exchanges z and $\tilde{z}$, giving poles of m at the other two roots. At a common Jacobi pole m has a finite nonzero limit, since its leading coefficients involve $a + b$ or $a - b$.

Likewise $(\operatorname{cn} + ik' \operatorname{sn})(\operatorname{cn} - ik' \operatorname{sn}) = \operatorname{dn}^2$ determines the divisor of e . At Z , dn has a simple zero, its numerator in e has a double zero, and the other factor is nonzero. The shifts give the other three points. At a common Jacobi pole the ratio is finite and nonzero, since $0 < k' < 1$. The complete lists are therefore

	simple zeros	simple poles
m	$Z \pm \delta$	$Z \pm \delta + V$
e	$Z, Z + 2K$	$Z + V, Z + 2K + V$

Combining these lists in (6) gives

$$\begin{aligned} R \text{ zeros : } & Z - h, Z, Z + h, Z + 2K + V, \\ R \text{ poles : } & Z - h + V, Z + V, Z + h + V, Z + 2K. \end{aligned} \quad (9)$$

Every order is one, including at Z , where two zeros of the shifted m factors cancel one pole of e^{-1} . The bounds $0 < h < 2K$ make each list distinct and keep the zero and pole lists disjoint.

Lemma 2. *If $h \neq K$, C has at most simple poles. Every possible pole lies at*

$$Z + \nu V \pm 3\delta \quad \text{or} \quad Z + 2K + \nu V \pm \delta, \quad \nu \in \{0, 1\}. \quad (10)$$

Proof. The two arguments of R in a ratio in (7) differ by h . A double pole would require a zero and a pole of R separated by h . On the horizontal row through Z , the zeros have relative positions $-h, 0, h$ and the only pole is at $2K$; on the other row their roles reverse. Thus a collision requires

$$2K - x \equiv \pm h \pmod{4K}, \quad x \in \{-h, 0, h\}.$$

Within $0 < h < 2K$, the only solution is $h = K$. Otherwise each ratio has poles of order at most one. Adjacent zeros, or adjacent poles, cancel in a ratio. Only the two ends of the three-point chain and the isolated fourth point can contribute, yielding (10). Further cancellations at special periods are harmless. $\square$

5 Cancellation of the cosine trace

Assume $n \neq 4$. By (3), $h \neq K$, so Lemma 2 applies. Translating (10) by $-jh$ shows that every possible pole of S is congruent, modulo h , to $Z_0 + h/2$, where

$$Z_0 \in \{Z, Z + 2K, Z + V, Z + 2K + V\}.$$

Here $3\delta = \delta + h$ and $-3\delta = \delta - 2h$. Even when several summands are singular at once, S has order at most one.

By (8), sn and cn are even and dn is odd about each Z_0 . Thus m is even and e is odd there. The two shifted m factors in (6) exchange under reflection, giving

$$R(Z_0 - w) = -R(Z_0 + w), \quad C(Z_0 - w) = C(Z_0 + w).$$

Cyclic reindexing $j \mapsto -j$ and $nh \in \Lambda$ then yield $S(Z_0 - w) = S(Z_0 + w)$. Because S is h -periodic, it is also even about every $Z_0 + h/2 + jh$. These are all possible singularities.

An even Laurent series has no term of order -1 . Since no pole has higher order, every possible singularity is removable. Therefore S is holomorphic on a compact torus and is constant, proving the result at F_- . Reflection in the y -axis is $u \mapsto -u$ in (2); it interchanges the foci and reverses the orbit. Angles are preserved, so the two constants coincide. Changing the inversion radius only scales the inverse polygon. This proves Theorem 1.

For concentric circles, the statement has a direct interpretation: a regular star family rotates, and inversion at the common center scales all vertices uniformly. No limiting elliptic torus is needed.

6 Why genuine period four must be excluded

Take $E : x^2/4 + y^2 = 1$, with caustic $x^2/(16/5) + y^2/(1/5) = 1$ and focus $F = (\sqrt{3}, 0)$. Two genuine four-periodic orbits in this same family are

$$H = ((2, 0), (0, 1), (-2, 0), (0, -1)), \\ V_0 = \frac{1}{\sqrt{5}}((4, 1), (-4, 1), (-4, -1), (4, -1)).$$

The sides $x \pm 2y = \pm 2$ of H obey the dual tangency identity $16/5 + 4/5 = 4$. The sides of V_0 lie at the caustic semiaxes. The reflection law is immediate at the axial vertices of H ; for V_0 , the normals $(x/4, y)$ bisect its horizontal and vertical edges.

Direct substitution into inversion and (1) gives

$$\sum \cos \angle I_F(H) = \frac{9}{5}, \quad \sum \cos \angle I_F(V_0) = 0.$$

These values are also certified by the accompanying exact-arithmetic script. Twice traversing each orbit produces lists of length eight with sums $18/5$ and 0 . Thus a statement based only on listed length would be false. The four-periodic exception was already recorded in [4, Counter-example 4.10]; this calculation is a boundary control, not a priority claim.

7 Reproducibility and status

The source package contains exact symbolic checks of the focal and chord identities and the four-periodic example. Exact rational bookkeeping additionally tests the divisor collision and translated symmetry centers for primitive steps through period 80. Independent geometric angle calculations at 70 decimal digits test 300 real phases across 100 elliptic families, including star windings, and complex tests check the divisor and removable poles. Normal and optimized Python runs agree. These finite controls are not a substitute for the universal meromorphic proof above.

A bounded primary-source search located no later proof of this exact general theorem; no exhaustive or absolute-priority certification is claimed. This preprint is self-audited and unrefereed, not independently reviewed or formally verified. AI-assisted tools supported literature search, calculation, proof auditing and manuscript preparation; the author remains responsible for the claims and final text.

References

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- [6] NIST Digital Library of Mathematical Functions, [§22.4](#), [§22.6](#) and [§22.8](#), consulted 12 October 2026.