

A Telescoping Formula for Evolute Areas in Elliptic Billiards

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Abstract

For a billiard orbit of least period $N > 4$ between a noncircular ellipse and a fixed confocal elliptic caustic, we give closed formulas for the signed area of its caustic-contact polygon divided by the signed area of that polygon's discrete evolute, and for the corresponding outer tangent polygon ratio. These are the invariants k_{703} and k_{702} in the list of Reznik, Garcia and Koiller. A rational formula for each consecutive circumcenter reduces the first area relation to an explicit polynomial divisibility certificate and a cyclic telescoping sum. A conic-specific polarity calculation gives the second ratio. The argument also covers star trajectories. Least period is essential: two rational four-cycles on the same confocal pair have different defined ratios, and their eight-term repetitions invalidate a broader list-length interpretation. We state this source-scope distinction explicitly rather than extend the formula through its exceptional parameter.

1 The invariant and its scope

Reznik, Garcia and Koiller [1, §3.8, Table 8] list area-ratio invariants involving the discrete evolutes of polygons associated with elliptic billiards. The inner polygon has vertices at the contacts of the billiard chords with their confocal caustic. The outer polygon has vertices at the intersections of the tangents to the billiard boundary at consecutive bounce points. We establish explicit formulas for the inner and outer ratios, respectively k_{703} and k_{702} .

We retain two conventions from the source. The caustic is an *ellipse*, not a hyperbola, and all areas are signed shoelace areas [1, §2, equation (1)]. The evolute vertices are ordered intersections of successive perpendicular bisectors, equivalently consecutive-triple circumcenters. This discrete construction is also studied in [3]; it is not the differential evolute of a smooth curve. In particular, the area of a self-intersecting evolute does not mean the unsigned area of a union of regions.

Our N is the *least period* of the physical billiard orbit, not the length of a list formed by repeating a shorter orbit. This is consistent with the coprime period and turning-number convention in Stachel's canonical parametrization [2, Theorem 2]. The inspected Table 8 does not itself spell out least period. The distinction cannot be suppressed: Section 7 gives a failure of the literal repeated-four-cycle extension. Neither a convexity restriction on the orbit polygon nor a winding-number-one restriction is needed for the theorem below.

The proof uses classical conic tangency, circumcenter equations, polarity, Vieta's formulas and shoelace algebra. Its load-bearing calculation is the displayed polynomial certificate, not finite sampling of periodic families. Exact algebra and geometric regression checks accompany the derivation. This is a self-audited, unrefereed preprint prepared with AI assistance; it does not claim independent human review, proof-assistant formalization, an exhaustive historical search, or absolute priority.

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2 Statement and contact geometry

Let the billiard boundary and its strictly nested confocal caustic be

$$\frac{X^2}{u} + \frac{Y^2}{v} = 1, \quad \frac{X^2}{A} + \frac{Y^2}{B} = 1, \quad u > v > 0, \quad 0 < \lambda < v, \quad A = u - \lambda, \quad B = v - \lambda. \quad (1)$$

Write

$$\alpha = \sqrt{A}, \quad \beta = \sqrt{B}, \quad \Delta = u - v = A - B, \quad d = \lambda \left(\frac{1}{v} - \frac{1}{u} \right), \quad e = -1 + \lambda \left(\frac{1}{u} + \frac{1}{v} \right). \quad (2)$$

and put $\nu = d^2 + 2e - 1$. For an ordered polygon $W = (W_0, \dots, W_{m-1})$, use cyclic indices and define

$$\text{area}(W) = \frac{1}{2} \sum_{j=0}^{m-1} \det(W_j, W_{j+1}).$$

The vertex $\text{Ev}(W)_j$ is the Euclidean circumcenter of W_{j-1}, W_j, W_{j+1} , when these three points are noncollinear. Compared with intersections of the bisectors of the sides $W_j W_{j+1}$ and $W_{j+1} W_{j+2}$, this is only a cyclic shift.

Theorem 1. *Let a physical billiard orbit for (1) have least period $N > 4$. Let Q be its caustic-contact polygon in trajectory order, and let $\mathcal{T}$ be its outer boundary-tangent polygon. Their consecutive circumcenters exist and both evolute areas are nonzero. For signed areas,*

$$\frac{\text{area}(Q)}{\text{area}(\text{Ev}(Q))} = \frac{8AB(1 - d^2)}{\Delta^2 \nu}, \quad (3)$$

$$\frac{\text{area}(\mathcal{T})}{\text{area}(\text{Ev}(\mathcal{T}))} = \frac{8AB(1 - d^2)}{\Delta^2 \nu e^2}. \quad (4)$$

In particular, both quotients are independent of the initial phase within the fixed-confocal-pair family. Star trajectories are included.

2.1 A symmetric biquadratic

Parametrize the caustic by unit complex parameters,

$$q(z) = (\alpha \operatorname{Re} z, \beta \operatorname{Im} z), \quad |z| = 1.$$

The tangent at $q(z)$ has equation $q(z)^T \operatorname{diag}(A, B)^{-1} p = 1$. Suppose consecutive contacts have parameters $x = \exp(i(m - h))$ and $y = \exp(i(m + h))$. After scaling by $\operatorname{diag}(1/\alpha, 1/\beta)$, their tangent intersection is $(\cos m / \cos h, \sin m / \cos h)$. Its membership in the outer ellipse gives

$$\cos^2 h = \frac{A}{u} \cos^2 m + \frac{B}{v} \sin^2 m.$$

Clearing the unit-circle coordinates yields

$$F(x, y) = (1 - dx^2)y^2 + 2exy + x^2 - d = 0. \quad (5)$$

Antipodal caustic tangents are parallel and cannot meet at a finite bounce vertex, so there is no omitted physical case at $\cos h = 0$. Equation (5) is symmetric in x, y .

Lemma 2. *The physical parameters satisfy*

$$0 < d < 1, \quad 1 + e - d = \frac{2\lambda}{u} > 0, \quad 1 - e - d = 2 \left(1 - \frac{\lambda}{v}\right) > 0, \quad |e| < 1 - d. \quad (6)$$

For a contact z , its preceding and following contacts are distinct roots z_-, z_+ of $F(z, w)$ and obey

$$z_- + z_+ = \frac{-2ez}{1 - dz^2}, \quad z_- z_+ = \frac{z^2 - d}{1 - dz^2}. \quad (7)$$

Every three consecutive contacts are distinct and noncollinear.

Proof. The identities and inequalities follow directly from (2). For $|z| = 1$, they imply

$$\begin{aligned} \frac{F(z, z)}{z^2} &= 2 + 2e - d(z^2 + z^{-2}) \geq 2(1 + e - d) > 0, \\ \frac{F(z, -z)}{z^2} &= 2 - 2e - d(z^2 + z^{-2}) \geq 2(1 - e - d) > 0. \end{aligned}$$

Thus adjacent contacts cannot be equal or antipodal. The quadratic discriminant of $F(z, w)$, divided by z^2 , is

$$4[2d \cos(2 \arg z) + e^2 - 1 - d^2] < 0,$$

because $e^2 < (1 - d)^2$. Its roots are distinct, and its leading coefficient $1 - dz^2$ is nonzero.

Geometrically, the tangent at a caustic contact cuts the outer ellipse at two different endpoints. The other caustic tangents through those endpoints differ: otherwise one line would meet the current tangent in two points. The confocal billiard reflection property interchanges the two caustic tangent lines at a bounce, rather than retracing its incoming line. Hence the preceding and following contacts are exactly these two roots; Vieta gives (7). The three contacts are distinct, and a line meets a nondegenerate ellipse in at most two points. They are therefore noncollinear. $\square$

3 The rational circumcenter map

Lemma 3. *For three distinct unit parameters z_1, z_2, z_3 , set*

$$W = \frac{z_1 + z_2 + z_3 + (z_1 z_2 z_3)^{-1}}{2}.$$

Their actual Euclidean circumcenter is

$$C_x = \frac{\Delta}{2\alpha} \operatorname{Re} W, \quad C_y = -\frac{\Delta}{2\beta} \operatorname{Im} W. \quad (8)$$

For the consecutive triple about a parameter z , its center parameter W equals

$$R(z) = \frac{1}{2} \left[z - \frac{2ez}{1 - dz^2} + \frac{1 - dz^2}{z(z^2 - d)} \right]. \quad (9)$$

The real linear map $W \mapsto C$ has determinant $-\Delta^2/(4\alpha\beta)$.

Proof. Let C be the unique circumcenter, whose existence follows from noncollinearity. Put $s = 2\alpha C_x/\Delta$ and $t = 2\beta C_y/\Delta$. Restricting its circle equation to the caustic gives $\cos^2 \theta - s \cos \theta - t \sin \theta + \kappa = 0$ for a real constant κ . Multiplication by $4z^2$ gives the monic quartic

$$z^4 - 2(s - it)z^3 + (2 + 4\kappa)z^2 - 2(s + it)z + 1.$$

Its three specified roots are z_1, z_2, z_3 , so the fourth root, counted with multiplicity, is $(z_1 z_2 z_3)^{-1}$. Comparing the cubic coefficient gives $W = s - it$, proving (8). This argument starts with the actual circle; it does not assume a converse coefficient-matching criterion.

For a consecutive triple z_-, z, z_+ , substitute (7) into the formula for W to obtain (9). Finally the diagonal real map in (8) has the stated negative determinant. Both $z^2 - d$ and $1 - dz^2$ are nonzero on the unit circle by $0 < d < 1$. $\square$

The rational function R has real coefficients. Consequently $\overline{R(z)} = R(1/z)$ for physical unit parameters. This lets us express evolute shoelace terms without introducing real and imaginary coordinates separately. The negative determinant in Lemma 3 must be retained; replacing it by its absolute value would change the signed invariant.

4 A compact polynomial certificate

Assume for now that $e \neq 0$. Define

$$D = d^2 - 1, \quad \nu = D + 2e, \quad c = \frac{\nu}{2D}, \quad L(t) = (t - d)(1 - dt). \quad (10)$$

The center parameter and a rational primitive can be written as

$$R(z) = \frac{-dz^6 + (1 + 2d^2 - 2e)z^4 + d(2e - 3)z^2 + 1}{2zL(z^2)}, \quad (11)$$

$$H(z) = -\frac{\nu(z^4 - 1)[D(z^4 + 1) - 2d(D - 2e^2)z^2]}{8eDz^2L(z^2)}. \quad (12)$$

The first expression is obtained from (9) by collecting fractions. Introduce two one-variable polynomials

$$U(r) = D\nu(r^4 + 1 + 2d^2r^2) - 4e^2D^2r^2 + 4de[D + e(d^2 + 1)](r^3 + r), \quad (13)$$

$$V(r) = dD\nu(r^2 + 1) + 2e(d^4 - 1 + 4d^2e)r. \quad (14)$$

Proposition 4. *For the rational discrepancy*

$$E(x, y) = R(1/x)R(y) - R(x)R(1/y) - c(y/x - x/y) - H(y) + H(x),$$

and the clearing factor

$$T(x, y) = 8eDx^2y^2L(x^2)L(y^2),$$

there is the exact polynomial identity

$$TE = -(x^2 - y^2)F(x, y)[U(xy) - V(xy)(x^2 + y^2)]. \quad (15)$$

Thus, on every physical contact edge with $e \neq 0$,

$$R(1/x)R(y) - R(x)R(1/y) - c(y/x - x/y) = H(y) - H(x). \quad (16)$$

Proof. Substitute (11)–(12), multiply by T , and collect the polynomial terms. Their factorization is exactly the right side of (15), with (5) and (13)–(14). This displayed identity is a finite ordinary-multiplication certificate over the polynomial ring in x, y, d, e ; no numerical orbit is used. It also supplies an explicit quotient by F , rather than an unspecified zero remainder.

At unit physical parameters, $x, y \neq 0$, $D \neq 0$ and $L(x^2)L(y^2) \neq 0$. With $e \neq 0$, the clearing factor is nonzero. Equation (5) therefore reduces (15) to (16). $\square$

The primitive satisfies $H(1/z) = -H(z)$, directly from (12). That symmetry is compatible with conjugation on the unit circle. More importantly, the e in its denominator is a substantive singularity. It will not be removed by substituting the formal $e \rightarrow 0$ limit into the coefficient c .

5 Cyclic area summation and exceptional periods

Let $z_0, \dots, z_{N-1}$ be the closed contact sequence, and let $W_j = R(z_j)$. The complex shoelace expressions are

$$\text{area}(Q) = \frac{\alpha\beta}{4i} \sum_j \left(\frac{z_{j+1}}{z_j} - \frac{z_j}{z_{j+1}} \right), \quad (17)$$

$$\text{area}(W) = \frac{1}{4i} \sum_j [R(1/z_j)R(z_{j+1}) - R(z_j)R(1/z_{j+1})]. \quad (18)$$

Summing (16) cancels each term of H once with each sign. Therefore

$$\text{area}(W) = \frac{c}{\alpha\beta} \text{area}(Q), \quad \text{area}(\text{Ev}(Q)) = -\frac{\Delta^2 c}{4AB} \text{area}(Q). \quad (19)$$

When the latter area is nonzero, this gives (3). The argument is a closed cyclic sum: it does not depend on convexity or winding number. Traversal reversal changes both signs. Repeating a shorter closed sequence with $e \neq 0$ multiplies the two areas by the same integer.

5.1 The contact area does not vanish for a physical trajectory

An orientation-preserving axial scaling makes the caustic the unit circle. Every bounce vertex is external to that circle. Rotate such a vertex to $p = (r, 0)$, $r > 1$, and put $s = \sqrt{1 - r^{-2}}$. Its two tangent contacts are $q_- = (r^{-1}, -s)$ and $q_+ = (r^{-1}, s)$. For the incoming ray $q_- \rightarrow p$ and outgoing ray $p \rightarrow q_+$,

$$\det(p - q_-, -q_-) = rs > 0, \quad \det(q_+ - p, -p) = rs > 0.$$

Thus the caustic stays on the same oriented side of the incoming and outgoing segments. The reversed traversal has both signs negative. The billiard's use of the other tangent preserves this choice at every vertex.

Before rotation, the two contact angles are $\phi \pm \arccos(1/r)$. Each positively oriented successive increment is consequently $2\arccos(1/r) \in (0, \pi)$; all increments are negative in the reversed orientation. Returning to the caustic ellipse multiplies cross-products by $\alpha\beta > 0$. Every summand of $2\text{area}(Q)$ therefore has one strict sign. In particular, $\text{area}(Q) \neq 0$, even if the polygon is a star and winds more than once.

5.2 Exceptional period parameters

Lemma 5. *If $e = 0$, the least contact period is four. If $\nu = 0$, it is three. The bounce and contact sequences have the same least period.*

Proof. At $e = 0$, (7) gives $z_{j+1} = -z_{j-1}$, so $z_{j+2} = -z_j$ and $z_{j+4} = z_j$. Equal or antipodal consecutive contacts and a repeated quadratic root were excluded in Lemma 2. The least period is therefore exactly four.

At $\nu = 0$ we have $2e = 1 - d^2$, in particular $e > 0$. Given an edge $F(x, y) = 0$, its next contact is

$$z = -\frac{2ey}{1 - dy^2} - x.$$

Vieta gives $F(y, z) = 0$. Under this specialization, direct expansion also gives the exact identity

$$\begin{aligned} F(z, x) &= \frac{F(x, y)J(x, y)}{(dy^2 - 1)^2}, \\ J(x, y) &= d^3xy - d^3y^2 + d^2x^2y^2 - dx^2 - dxy + 1. \end{aligned} \quad (20)$$

Hence $F(z, x) = 0$. The distinct roots about z are y and x , so the following contact is x . The three contacts are distinct by Lemma 2; the least period is exactly three.

Finally, consecutive contact tangents reconstruct each bounce as their unique intersection. Conversely, each bounce chord has a unique contact with the strictly convex caustic. A period of one ordered sequence is therefore a period of the other, giving equal least periods. $\square$

For a least-period $N > 4$ orbit, Lemma 5 supplies $e \neq 0$ and $\nu \neq 0$. All the denominators used in the certificate are nonzero; the center triples are noncollinear; and $\text{area}(Q) \neq 0$. Equation (19) then proves that $\text{area}(\text{Ev}(Q)) \neq 0$. This completes the inner part of Theorem 1 without an additional hidden generic-area assumption.

6 The outer polygon by conic polarity

Let $C = \text{diag}(u, v)$ and $E_c = \text{diag}(A, B)$. A billiard chord with caustic contact q has equation $q^T E_c^{-1} p = 1$. Its pole with respect to the outer ellipse is Mq , where

$$M = C E_c^{-1} = \text{diag}(u/A, v/B).$$

Indeed, each endpoint p satisfies $p^T C^{-1}(Mq) = 1$, which is its boundary tangent equation evaluated at Mq . Thus Mq is the intersection of the boundary tangents at the chord endpoints, precisely the source's outer vertex. In ordered polygon notation, $\mathcal{T} = MQ$.

For points q_i, q_j on the fixed caustic, subtraction of their conic equations gives

$$|Mq_j|^2 - |Mq_i|^2 = \gamma(|q_j|^2 - |q_i|^2), \quad \gamma = \frac{u^2/A - v^2/B}{A - B} = 1 - \frac{\lambda^2}{AB}. \quad (21)$$

If C_i is the circumcenter of a consecutive triple in Q , the linear equal-distance equations are

$$2C_i \cdot (q_j - q_k) = |q_j|^2 - |q_k|^2.$$

Using (21) shows that the actual center for the transformed triple is $\gamma M^{-1}C_i$. Here M is diagonal; the transformation preserves noncollinearity. This is a consequence of the common conic, *not* general affine covariance of Euclidean circumcenters.

Signed-area determinants now yield

$$\frac{\text{area}(\mathcal{T})}{\text{area}(\text{Ev}(\mathcal{T}))} = \frac{\det(M)^2}{\gamma^2} \frac{\text{area}(Q)}{\text{area}(\text{Ev}(Q))}.$$

The coefficient simplifies to

$$\frac{\gamma}{\det(M)} = \frac{AB - \lambda^2}{uv} = 1 - \lambda \left(\frac{1}{u} + \frac{1}{v} \right) = -e. \quad (22)$$

Consequently the outer quotient is the inner quotient divided by e^2 , proving (4). Since $e \neq 0$ for the stated least periods, $\gamma \neq 0$ and the outer evolute area is also nonzero. This completes Theorem 1.

7 Why the period convention matters

The telescope is not valid at $e = 0$. This is not merely a defect of its particular denominator: the actual four-cycle ratio can vary within the fixed-caustic family.

For a central quadrilateral $(a, b, -a, -b)$, write $S = \det(a, b) \neq 0$ and $h = (|a|^2 - |b|^2)/2$, and let $\mathcal{J}$ be rotation by $\pi/2$. Its centers are

$$-\frac{h}{S}\mathcal{J}b, \quad -\frac{h}{S}\mathcal{J}a, \quad \frac{h}{S}\mathcal{J}b, \quad \frac{h}{S}\mathcal{J}a.$$

This follows directly from the equal-distance equations: the first center is perpendicular to b and has dot product h with a , and the next is perpendicular to a and has dot product $-h$ with b . Their signed area is $-2h^2/S$, while the quadrilateral's area is $2S$. Hence

$$\frac{\text{area}(\text{Ev}(Q))}{\text{area}(Q)} = -\frac{(|a|^2 - |b|^2)^2}{4 \det(a, b)^2}. \quad (23)$$

For entirely rational controls, fix

$$(u, v, \lambda) = (336, 105, 80), \quad (A, B) = (256, 25), \quad d = \frac{11}{21}, \quad e = 0.$$

Two contact cycles on this same pair are

$$Q_1 = ((16, 0), (0, 5), (-16, 0), (0, -5)),$$

$$Q_2 = \left(\left(\frac{64}{5}, 3 \right), \left(-\frac{192}{13}, \frac{25}{13} \right), \left(-\frac{64}{5}, -3 \right), \left(\frac{192}{13}, -\frac{25}{13} \right) \right).$$

Their consecutive caustic tangent intersections are, respectively,

$$(16, 5), (-16, 5), (-16, -5), (16, -5),$$

$$(-4, 10), \left(-\frac{128}{7}, -\frac{5}{7} \right), (4, -10), \left(\frac{128}{7}, \frac{5}{7} \right).$$

These lie on the outer ellipse, with contacts inside the corresponding chords. The confocal reflection property gives genuine four-periodic billiards; direct directional-reflection checks give the same conclusion. Formula (23) gives

$$\frac{\text{area}(Q_1)}{\text{area}(\text{Ev}(Q_1))} = -\frac{25600}{53361}, \quad \frac{\text{area}(Q_2)}{\text{area}(\text{Ev}(Q_2))} = -\frac{422500}{53361}.$$

Both evolute areas are nonzero, but the ratios differ. Repeating each contact cycle twice produces an eight-term closed list; each area doubles and its ratio stays unchanged. All local circumcenter triples still exist. Thus a version of k_{703} using only list length $N > 4$ and allowing primitive-four repetitions is false. These lists are not least-period-eight trajectories and are not counterexamples to Theorem 1.

For additional insight, at $e = 0$ a unit contact cycle is $(x, y, -x, -y)$. Its center parameters are a common complex-linear image of these four points, with factor

$$K = \frac{1 - (xy)^{-2}}{2}, \quad |K|^2 = \sin^2(\arg(xy)).$$

This phase-dependent area multiplier need not equal the formal $e \rightarrow 0$ value $c = 1/2$. The singular primitive cannot be replaced by continuity of that coefficient alone. At phases with $|a| = |b|$ in (23), the evolute collapses and the ratio itself is undefined.

8 Checks and conclusion

The proof is the local identity (15), its cyclic summation, and the source-matched geometric bridge. Exact symbolic checks expand the certificate, verify the period-three identity (20), and compare the rational center with direct circumcenters. Finite six- and eight-periodic examples are regression controls only. For example, the least-period-eight contact cycle

$$(16, 0), (64/5, 27/5), (0, 9), (-64/5, 27/5), \\ (-16, 0), (-64/5, -27/5), (0, -9), (64/5, -27/5).$$

for $(u, v, \lambda) = (280, 105, 24)$ has

$$\text{area}(Q) = \frac{2016}{5}, \quad \text{area}(\text{Ev}(Q)) = -\frac{8575}{48}, \quad \frac{\text{area}(Q)}{\text{area}(\text{Ev}(Q))} = -\frac{13824}{6125},$$

as predicted. No finite census is used to infer the all-period result.

The formulas apply to the source's signed areas and nondegenerate confocal elliptic caustics with least period $N > 4$. They do not assert an unsigned region-area identity, a hyperbolic-caustic extension, a circular-billiard ratio, or the repeated-four-cycle statement disproved above. The inner and outer assertions are one coupled result, with their coefficients related by (22). Historical priority is a separate question from the displayed proof and is not claimed here.

References

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