

A Six-Periodic Zero of Focal Outer-Antipedal Area

Alper Ferudun*

2 October 2026

Abstract

For a periodic elliptic billiard, form the polygon of intersections of consecutive boundary tangents, then its antipedal with respect to either focus. Classical central symmetry implies equality of the two signed antipedal areas for effective even periods. Their quotient is therefore 1 where defined, but it need not be defined everywhere. We give an explicit convex billiard of least period six on an ellipse of aspect ratio $1 + \sqrt{3}$ for which both areas vanish, while all antipedal vertices are finite, all edges are nonzero and the vertices are not collinear. A direct coordinate calculation gives the signed area of an axial six-periodic family and isolates the zero exactly. This is a denominator obstruction, not a counterexample to equality on the quotient's natural domain. The earlier original-polygon antipedal zero at aspect ratio 2 is distinguished and credited. No absolute priority claim is made.

1 The outer polygon and the area quotient

Let E and E_c be the confocal ellipses

$$E : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad E_c : \frac{x^2}{a^2 - \lambda} + \frac{y^2}{b^2 - \lambda} = 1, \quad a > b > 0, \quad 0 < \lambda < b^2.$$

Write $F_{\pm} = (\pm c, 0)$, $c = \sqrt{a^2 - b^2}$. For an ordered billiard orbit $P = (P_i)$ on E tangent to E_c , let

$$V_i = T(P_i) \cap T(P_{i+1}),$$

where $T(P_i)$ is the tangent to E . Thus V is the *outer* polygon, denoted by P' in [1, 2].

The antipedal of V with respect to F is the ordered polygon $Q(F) = (Q_i(F))$ defined by consecutive intersections of the lines

$$L_i(F) = \{X : (V_i - F) \cdot (X - V_i) = 0\}, \quad Q_i(F) = L_i(F) \cap L_{i+1}(F). \quad (1)$$

We use supporting lines; no choice of oriented half-rays is involved. For any ordered cyclic polygon W , its signed area is

$$\mathcal{A}(W) = \frac{1}{2} \sum_i \det(W_i, W_{i+1}). \quad (2)$$

This is the signed convention in [1, Section 2]; it is not the sum of absolute lobe areas of a self-intersecting polygon.

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: [AlperTheKing](https://github.com/AlperTheKing).

Table 7 of the cited preprint version lists

$$\frac{\mathcal{A}(Q(F_+))}{\mathcal{A}(Q(F_-))} = 1 \quad \text{for even } N \quad (3)$$

as k_{608} . This row is absent from the final journal version [2]. The journal's outer *pedal* quotient $k_{604,b}$ is a different construction, not a renaming of k_{608} . The version change alone supplies neither a proof nor a correction.

Here an *effective even period* means that the least period of the orbit is even. Repetitions of such an orbit are allowed. We do not infer central symmetry merely by writing an odd-period orbit twice.

Theorem 1. *For every effective even-period billiard between E and E_c , the two focal outer antipedals have finite vertices and equal signed areas. Consequently (3) holds wherever its denominator is nonzero. There nevertheless exists a convex billiard of least period six with*

$$a = 1 + \sqrt{3}, \quad b = 1, \quad \lambda = 4 - 2\sqrt{3}$$

for which both areas are zero. Every antipedal edge is nonzero and the antipedal vertices are not all collinear.

The equality part is an elementary consequence of classical billiard symmetry. The purpose of the explicit calculation is to separate that equality from denominator nonvanishing. It does not yield two unequal defined area ratios.

2 Regularity and classical symmetry

Lemma 2. *For a genuine periodic orbit with the stated nondegenerate elliptic caustic, all consecutive outer tangent intersections and all focal outer-antipedal intersections are finite.*

Proof. A chord tangent to E_c cannot be a diameter of E , since a diameter passes through the interior of E_c . Thus two consecutive boundary tangents are not parallel. Consecutive outer vertices are distinct: otherwise three distinct real tangents to an ellipse would be concurrent at a finite point, whereas there are at most two real tangents from such a point. The points V_i, V_{i+1} lie on $T(P_{i+1})$. If $V_i - F$ and $V_{i+1} - F$ were parallel, then F would lie on this boundary tangent. This is impossible because either focus is strictly inside E . Thus the two equations defining $Q_i(F)$ have nonzero determinant. The reasoning also applies to genuine star orbits. $\square$

For completeness, Stachel's Jacobi parametrization [3, Theorem 2] writes a primitive orbit of period n and winding number τ as

$$P_i = \left(-a \operatorname{sn} \left(u_0 + \frac{4i\tau K}{n} \right), b \operatorname{cn} \left(u_0 + \frac{4i\tau K}{n} \right) \right), \quad \gcd(n, \tau) = 1. \quad (4)$$

The modulus belongs to the caustic, and K is its complete elliptic integral. For even n , the integer τ is odd. The shift by $n/2$ in (4) adds $2\tau K$; both Jacobi functions change sign. Hence

$$P_{i+n/2} = -P_i, \quad V_{i+n/2} = -V_i.$$

Equivalently, this is the classical central-symmetry statement of [4, Corollary 4.2(i)], with its odd-winding hypothesis. Substitution in (1) gives

$$Q_{i+n/2}(F_-) = -Q_i(F_+). \quad (5)$$

A half-turn preserves signed area, as does a cyclic shift. Thus the areas are equal, including for self-intersecting antipedals. Repeated traversals multiply both areas by the same repetition count. The circular case is separate and immediate: the two foci coincide, so the two identically defined polygons coincide. No hyperbolic or degenerate caustic extension is used here.

3 A genuine six-periodic family

Normalize $b = 1$ and take any $a > 1$. Put

$$d = \sqrt{2a+1}, \quad D = d^2, \quad c = \sqrt{a^2-1}, \quad x = \frac{a^2}{a+1}, \quad y = \frac{d}{a+1}, \quad \lambda = \frac{a^2}{(a+1)^2}. \quad (6)$$

Consider the ordered polygon

$$P = ((a, 0), (x, y), (-x, y), (-a, 0), (-x, -y), (x, -y)). \quad (7)$$

All six points lie on E by direct substitution. Since $0 < x < a$ and $y > 0$, they are distinct and in convex order. The oblique edges have length 1, while the horizontal edges have length $2x$.

At $(a, 0)$, reflection changes only the horizontal velocity component. At (x, y) , the incoming and outgoing unit velocities and an ellipse normal are

$$v_{\text{in}} = \frac{(-a, d)}{a+1}, \quad v_{\text{out}} = (-1, 0), \quad n = \frac{(1, d)}{a+1}.$$

As $n \cdot n = 2/(a+1)$ and $v_{\text{in}} \cdot n = 1/(a+1)$,

$$v_{\text{in}} - 2 \frac{v_{\text{in}} \cdot n}{n \cdot n} n = v_{\text{in}} - n = v_{\text{out}}.$$

Coordinate reflections supply the remaining vertices. Thus (7) satisfies the directed billiard law, and its least period is six.

We also check the caustic directly. Since $0 < \lambda < 1$, it is a nondegenerate confocal ellipse. The horizontal edges are tangent because $y^2 = 1 - \lambda$. An oblique sideline is $dX + aY = ad$; its tangency condition is

$$d^2(a^2 - \lambda) + a^2(1 - \lambda) = a^2 d^2.$$

This follows from $d^2 + a^2 = (a+1)^2$ and (6). The other oblique sidelines are its coordinate reflections. Every caustic contact lies inside the corresponding chord: the caustic is strictly inside E . No limiting or degenerate orbit has been inserted.

4 Explicit focal outer-antipedal area

The first three boundary tangents are $X = a$, $X + dY = a + 1$ and $-X + dY = a + 1$. The six consecutive intersections are

$$V = ((a, 1/d), (0, (a+1)/d), (-a, 1/d), (-a, -1/d), (0, -(a+1)/d), (a, -1/d)). \quad (8)$$

For the right focus $F_+ = (c, 0)$, set

$$\begin{aligned} X_+ &= \frac{(a+1)(a-c)}{D}, & X_- &= \frac{(a+1)(a+c)}{D}, \\ U &= 2 + a - a^2, & Y_{\pm} &= \frac{U \pm ac}{d}, \\ L &= -a - \frac{a-c}{D}, & R &= a + \frac{a+c}{D}. \end{aligned} \quad (9)$$

Solving (1) gives the antipedal

$$Q(F_+) = ((X_+, Y_+), (-X_-, Y_-), (L, 0), (-X_-, -Y_-), (X_+, -Y_+), (R, 0)). \quad (10)$$

These expressions can be checked by substituting into each pair of line equations. For the first pair, the determinant is $a(a+1-c)/d$; $c^2 = a^2 - 1$ and $D = 2a + 1$ reduce its first coordinate to X_+ . The axial intersections initially have the forms

$$L = -a - \frac{1}{D(a+c)}, \quad R = a + \frac{1}{D(a-c)},$$

which agree with (9) because $(a-c)(a+c) = 1$. Reflection in the horizontal axis supplies the lower half.

For explicit regularity, the six determinants of consecutive physical normals $V_i - F_+$ are

$$\frac{1}{d}(a(a+1-c), a(a+1+c), 2(a+c), a(a+1+c), a(a+1-c), 2(a-c)). \quad (11)$$

All are strictly positive since $0 < c < a$; changing the focus sign permutes the factors. This calculation involves no division by a signed area.

Proposition 3. *For the axial family (7), either focal antipedal of the outer polygon has signed area*

$$\mathcal{A}(Q(F_+)) = \mathcal{A}(Q(F_-)) = \frac{4a(a+1)(2+2a-a^2)}{(2a+1)^{3/2}}. \quad (12)$$

Proof. Grouping the horizontally reflected vertices in the shoelace sum (2) gives

$$\mathcal{A}(Q(F_+)) = Y_-(X_+ - L) + Y_+(X_- + R).$$

The two differences are

$$X_+ - L = a + \frac{(a+2)(a-c)}{D}, \quad X_- + R = a + \frac{(a+2)(a+c)}{D}.$$

Substitution and $U + c^2 = a + 1$ yield

$$\begin{aligned} \mathcal{A}(Q(F_+)) &= \frac{2a}{d} \left(U + \frac{(a+2)(U+c^2)}{D} \right) \\ &= \frac{2a}{d} \left(U + \frac{(a+2)(a+1)}{D} \right) \\ &= \frac{4a(a+1)(2+2a-a^2)}{d^3}. \end{aligned}$$

The other focus has the same area by (5), or directly by replacing c by $-c$ in the calculation. $\square$

After restoring scale b , replace a in (12) by a/b and multiply the expression by b^2 . The formula describes the axial phase while the ellipse parameter varies. It is not a claim that an individual antipedal area is invariant through every phase.

5 The zero and its interpretation

Take $a = 1 + \sqrt{3}$. Then $a^2 - 2a - 2 = 0$, $c^2 = 2a + 1$, $d = c$ and $\lambda = 4 - 2\sqrt{3} \in (0, 1)$. Formula (12) gives

$$\mathcal{A}(Q(F_+)) = \mathcal{A}(Q(F_-)) = 0.$$

The original polygon remains convex of least period six, and all the determinants (11) remain positive. The antipedal is not entirely collinear: Q_0 and Q_4 have equal positive first coordinate and opposite nonzero second coordinates, since

$$Y_+ = \frac{a(c-1)}{c} > 0,$$

whereas $Q_5 = (R, 0)$ has a different first coordinate. Indeed

$$R - X_+ = \frac{a(a+1) + (a+2)c}{D} > 0.$$

All adjacent vertices in (10) are distinct. For the first pair their first coordinates have opposite signs. A vertex on the left axis cannot equal its neighbors because $L + X_- = [-a(a+1) + (a+2)c]/D$ and, at the chosen parameter, $Y_- = -a(1+c)/c \neq 0$. The other pairs follow by reflection or by $Y_+ \neq 0$. This proves the remaining assertions of Theorem 1.

At this orbit, (3) is $0/0$, not a defined real number. There is no contradiction to equality of the signed areas or to ratio 1 on its natural domain. In particular, this example does not show that the common area vanishes at every phase of its Poncelet family. It also does not assert a minimum possible zero period.

6 Prior observations and reproducibility

Garcia and Reznik [5, Table 2, row 03] already list a six-periodic zero-area antipedal at aspect ratio 2. Their linked author animation [6] identifies the antipedal of the *original* billiard polygon with respect to the left focus. The reference polygon and aspect ratio both differ from our outer construction: (12) is nonzero at $a = 2$. The earlier zero-area phenomenon and classical central symmetry must therefore be credited; neither is claimed as a new discovery here. A bounded literature search does not certify priority for the explicit outer formula or its zero.

The accompanying exact certificate reconstructs the zero in

$$\mathbb{Q}[c]/(c^4 - 6c^2 - 3), \quad \frac{5}{2} < c < \frac{13}{5}, \quad a = \frac{c^2 - 1}{2}.$$

The selected positive root is unique: the polynomial is negative at $5/2$, positive at $13/5$ and strictly increasing there. Rational interval refinement verifies the needed strict signs. The checker recomputes the impact coordinates, unit-velocity reflections, confocal caustic tangencies, boundary-tangent intersections, both antipedals and both shoelace areas. It uses only Python's standard library and no floating-point tolerance. Separate symbolic and algebraic-number-field computations provide additional controls. These finite tests supplement, rather than replace, the analytic proof.

This unrefereed preprint was prepared with AI assistance in mathematical exploration, computation, literature search and drafting. The argument has been self-audited; no independent human review, proof-assistant verification or absolute-priority certification is claimed.

References

- [1] D. Reznik, R. Garcia and J. Koiller, *Eighty New Invariants of N-Periodics in the Elliptic Billiard*, arXiv:2004.12497v11 (2020), Sections 2, 3.5 and Table 7. <https://arxiv.org/abs/2004.12497v11>.
- [2] D. Reznik, R. Garcia and J. Koiller, *Fifty New Invariants of N-Periodics in the Elliptic Billiard*, Arnold Mathematical Journal **7** (2021), 341–355. <https://doi.org/10.1007/s40598-021-00174-y>.

- [3] H. Stachel, *On the Motion of Billiards in Ellipses*, arXiv:2105.03624v2 (2021), Theorem 2. <https://arxiv.org/abs/2105.03624v2>.
- [4] H. Stachel, *The geometry of billiards in ellipses and their poncelet grids*, Journal of Geometry **112**, 40 (2021). <https://doi.org/10.1007/s00022-021-00606-2>.
- [5] R. Garcia and D. Reznik, *Exploring self-intersected N -periodics in the elliptic billiard*, Annales Mathematicae et Informaticae **55** (2022), 49–75; Table 2, row 03. <https://doi.org/10.33039/ami.2022.02.001>.
- [6] D. Reznik, *Elliptic Billiard Capoeira: $N = 6$ Zero-Area Focus-Antipedals and Constant Perimeter Focus-Inversives*, author animation and description, 28 October 2020. https://www.youtube.com/watch?v=HMhZW_kWLGw.