

Inner Steiner Pedal Ratios and Exceptional Zero-Area Families

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Abstract

For an odd-periodic billiard family between fixed confocal ellipses, consider the polygon of caustic contacts and its own Steiner curvature centroid. We prove that this centroid exists at every real phase and that its signed pedal area is a constant multiple of the signed contact polygon area. Thus the reciprocal ratio observed as k_{503} by Reznik, Garcia and Koiller is invariant on its defined domain. Its denominator, however, need not be nonzero. An explicit algebraic, strictly convex, least-period-five family has zero stationary pedal area at every phase; its parameter is the unique root in a stated rational interval of a degree-18 polynomial. Exact quadratic-field norms, regularity of both conjugate branches and rational Bernstein certificates justify this exception. The analytic proof includes odd star windings and odd repetitions; the circle case is treated directly. No absolute priority claim is made.

1 The contact polygon and the statement

Fix $\alpha > \beta > 0$ and $0 < \lambda < \beta$. The outer ellipse and its confocal elliptic caustic are

$$E : \frac{x^2}{\alpha} + \frac{y^2}{\beta} = 1, \quad E_c : \frac{x^2}{\alpha - \lambda} + \frac{y^2}{\beta - \lambda} = 1.$$

An ordered billiard polygon on E has all its sides tangent to E_c . Let $Q = (Q_i)$ be the ordered polygon of these contacts, called P'' in [1, 2]. For any ordered cyclic polygon V we use the signed shoelace area

$$\mathcal{A}(V) = \frac{1}{2} \sum_i \det(V_i, V_{i+1}).$$

This is not the sum of absolute face areas of a crossed star. Orient the traversal so its angular increments are positive. At each contact vertex let $\delta_i \in (0, \pi)$ be the turn between its consecutive edges, and use the internal-ray angle $\theta_i = \pi - \delta_i$. Its *own* Steiner curvature centroid is

$$C = \frac{\sum_i w_i Q_i}{\sum_i w_i}, \quad w_i = \sin(2\theta_i) = -\sin(2\delta_i). \quad (1)$$

The explicit ray convention also applies to stars. If $F_i(M)$ is the perpendicular foot of M on the supporting line $Q_i Q_{i+1}$, write

$$J = \mathcal{A}(Q), \quad B(u, M) = \mathcal{A}((F_i(M))_i), \quad B_C = B(u, C).$$

Here u denotes the billiard phase. These are the contact polygon's sidelines and angles, not those of the original or outer polygon.

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Theorem 1. *For every fixed nondegenerate confocal elliptic-caustic family of odd listed length, the centroid (1) is finite at every real phase. There is a real family-dependent constant R such that*

$$B_C(u) = RJ(u). \quad (2)$$

In the chosen orientation $J(u) > 0$. If $R \neq 0$, the ratio J/B_C is defined everywhere and constant. If $R = 0$, it is undefined everywhere in that family. Both cases occur: an explicit strictly convex family of least period five has $R = 0$.

All admitted coprime odd star windings and odd repeated traversals are included. Circular families have $R = \cos^2(\pi\tau/n) > 0$ for primitive winding τ and period n .

Table 6 of the original preprint [2] and the final article [1] records $k_{503} = A''/A_K''$ for odd N , with its value and proof marked unknown. The source uses signed areas and the contact polygon's own centroid. Although it does not explicitly exclude zero pedal area, the displayed quotient necessarily has a domain. The theorem proves its invariance on that entire domain and identifies an obstruction to an additional universal-definedness assertion. The obstruction is *not* two unequal finite ratios in one family and is not a degenerate original billiard. No hyperbolic or degenerate-caustic statement, neighboring k_{502} theorem, or resolution of an entire invariant table is asserted.

The fixed-polygon Steiner quadratic is classical; see [3, §2]. The Jacobi billiard parametrization is also established theory [4], and elliptic-function pole arguments are prior methodology for Poncelet areas [6]. The contact/dual correspondence in [5, §2.2] is an incidence statement, not a preservation of Euclidean perpendicular feet or angle-weighted centroids. We therefore prove the required metric cancellation directly.

2 The exact contact family and its metric

Initially take least period $n \geq 3$ odd and winding τ with $\gcd(n, \tau) = 1$ and $0 < \tau < n/2$. Put

$$\begin{aligned} A &= \sqrt{\alpha - \lambda}, \quad B = \sqrt{\beta - \lambda}, \quad k' = B/A, \quad k = \sqrt{1 - k'^2}, \\ K &= K(k), \quad K' = K(k'), \quad h = \frac{4\tau K}{n}, \quad \Delta = h/2. \end{aligned} \quad (3)$$

The canonical confocal billiard parametrization [4, Theorem 2] has $\text{sn}^2(\Delta, k) = \lambda/\beta$. After a common phase shift its contacts are

$$Q_i(u) = (-A \text{sn}(u + ih, k), B \text{cn}(u + ih, k)). \quad (4)$$

All Jacobi functions in this section have modulus k . For completeness, the chord joining original vertices at $v - \Delta$ and $v + \Delta$ has equation

$$-\frac{\text{dn } \Delta \text{sn } v}{\sqrt{\alpha}}x + \frac{\text{cn } v}{\sqrt{\beta}}y = \text{cn } \Delta.$$

The tangency point is the product of $\text{diag}(A^2, B^2)$ with the normalized normal. The confocal identities

$$\text{cn}^2 \Delta = \frac{\beta - \lambda}{\beta}, \quad \text{dn}^2 \Delta = \frac{\alpha(\beta - \lambda)}{\beta(\alpha - \lambda)}$$

give precisely $(-A \text{sn } v, B \text{cn } v)$, proving (4). Conversely, for $0 < k < 1$, $A > 0$, $B = Ak'$ and the step in (3), choosing outer semiaxes

$$a = \frac{A \text{dn } \Delta}{\text{cn } \Delta}, \quad b = \frac{B}{\text{cn } \Delta}$$

gives $a^2 - A^2 = b^2 - B^2 = B^2 \operatorname{sn}^2 \Delta / \operatorname{cn}^2 \Delta > 0$. This realizes the contact family by a genuine confocal billiard.

Jacobi addition gives the contact edge and its line:

$$Q(v + \Delta) - Q(v - \Delta) = -\frac{2 \operatorname{sn} \Delta \operatorname{dn} v}{1 - k^2 \operatorname{sn}^2 v \operatorname{sn}^2 \Delta} (A \operatorname{cn} v, B \operatorname{dn} \Delta \operatorname{sn} v), \quad (5)$$

$$-\frac{\operatorname{dn} \Delta \operatorname{sn} v}{A} x + \frac{\operatorname{cn} v}{B} y = \operatorname{cn} \Delta. \quad (6)$$

The envelope of these lines has squared semiaxes $A^2 \operatorname{cn}^2 \Delta / \operatorname{dn}^2 \Delta$ and $B^2 \operatorname{cn}^2 \Delta$. Its squared-axis gaps from E_c are respectively $B^2 \operatorname{sn}^2 \Delta / \operatorname{dn}^2 \Delta$ and $B^2 \operatorname{sn}^2 \Delta$, which are unequal when $k > 0$. Thus Q is not itself a confocal billiard in its own ellipse. A direct Euclidean or affine substitution of an original-polygon Steiner theorem would not prove this result.

3 The pedal area as an isotropic quadratic

Lemma 2. *For an ordered polygon with nonzero edges and the angle convention above, its signed pedal area has the form*

$$B(u, M) = a(u)(x^2 + y^2) + b(u)x + c(u)y + d(u), \quad M = (x, y), \quad (7)$$

where

$$a = -\frac{1}{8} \sum_i w_i, \quad (b, c) = \frac{1}{4} \sum_i w_i Q_i. \quad (8)$$

When $a \neq 0$, its unique stationary point is the centroid $C = -(b, c)/(2a)$ in (1).

Proof. Let t_i be unit edge directions, $n_i = \mathcal{R}t_i$ their positive quarter rotations, and $H_i = 2n_i n_i^T - I$. Direct matrix multiplication gives

$$\sin \delta_i \operatorname{Sym}(n_{i-1} n_i^T) = \frac{\sin(2\delta_i)}{4} I + \frac{\mathcal{R}H_{i-1} - \mathcal{R}H_i}{4}.$$

Translate the origin to M . The two feet adjacent to Q_i contribute $\frac{1}{2} \sin \delta_i ((Q_i - M) \cdot n_{i-1}) ((Q_i - M) \cdot n_i)$. The matrix difference telescopes in the quadratic term. Its linear sum telescopes because $H_i(Q_{i+1} - Q_i) = -(Q_{i+1} - Q_i)$ and the sum of the edge vectors is zero. This leaves

$$B(u, M) = B(u, 0) + \frac{1}{4} \left(\sum_i w_i Q_i \right) \cdot M - \frac{1}{8} \left(\sum_i w_i \right) (M \cdot M),$$

which proves both assertions. $\square$

4 Complex poles and the odd quotient torus

All complex dot products below are bilinear, not Hermitian. Write $\rho = \operatorname{dn}^2 \Delta$ and $\eta = 1 - k'^2 \rho$. Since $k'^2 < \rho < 1$, we have $k^2 < \eta < 1$. The squared norm of the direction in (5) is

$$A^2(1 - \eta \operatorname{sn}^2 v). \quad (9)$$

Modulo $2K$ and $2iK'$, it has exactly two zeros

$$v = K \pm iy_0, \quad 0 < y_0 < K'. \quad (10)$$

Indeed $\text{sn}(K + iy, k) = 1/\text{dn}(y, k')$, whose square increases strictly from 1 to $1/k^2$. At the two roots $\text{sn}, \text{cn}, \text{dn}$ are finite and nonzero, so differentiation proves simplicity.

For a line $l \cdot X = s$, orthogonal projection is

$$F(M) = M + \frac{s - l \cdot M}{l \cdot l} l. \quad (11)$$

Its only possible nonremovable poles are the isotropic lines (10). At a Jacobi pole of the midpoint normal, l has order one and $l \cdot l$ order two with nonzero leading coefficient proportional to $A^2 - B^2\rho > 0$; every coefficient of (11) is therefore finite. At a contact-vertex pole, the neighboring midpoint parameters have imaginary part K' and real shifts $\pm\Delta$. Their normals are finite and their squared norms are nonzero, because the isotropic roots have imaginary parts strictly between $-K'$ and K' . A zero or pole of the scalar factor in (5) cancels from normalized projection. This covers collapsed edges of the complex continuation; real edges do not collapse.

At a coefficient pole exactly one indexed edge is isotropic. A same-branch collision requires $(i - j)h \in 2K\mathbb{Z}$ and hence $i = j \pmod{n}$, since $\gcd(2\tau, n) = 1$. The two branches cannot collide under real shifts because $2iy_0 \notin 2iK'\mathbb{Z}$. Each area term involves one foot and its neighbor, so the four coefficients in (7) have at most simple poles.

Put $T = 2K/n$ and $E = 2iK'$. A Bezout identity $2\tau r + nt = 1$ has t odd, so shifting by $T = rh + 2Kt$ reindexes the contacts and centrally inverts them. The E shift reflects their y coordinates. Phase reversal reflects x and reverses order. Consequently the parities and the characters under (T, E) are

coefficient	parity in u	(T, E)
a, d	even	$(+, -)$
b	odd	$(-, -)$
c	even	$(-, +)$

(12)

Because a midpoint is $u + (j + \frac{1}{2})h$, the two phase pole classes are $T/2 \pm iy_0$ modulo T, E .

Let ℓ be the modulus whose nome is $\exp(-n\pi K'/K)$, and set $L = K(\ell)$, $L' = K(\ell')$, $z = nLu/K$. Thus $L'/L = nK'/K$, and T, E map to $2L, 2iL'$. The two pole classes become $L \pm iY$, with $Y = nLy_0/K \in (0, L')$. Define

$$\nu = \frac{1}{\text{sn}^2(L + iY, \ell)}, \quad \ell^2 < \nu < 1, \quad D(z) = 1 - \nu \text{sn}^2(z, \ell). \quad (13)$$

The standard Jacobi identities and periods [7] show that

$$f_0 = \frac{\text{dn } z}{D(z)}, \quad f_x = \frac{\text{sn } z \text{ dn } z}{D(z)}, \quad f_y = \frac{\text{cn } z \text{ dn } z}{D(z)} \quad (14)$$

have precisely these simple poles and respectively the three parities and characters in (12). Here and through Section 6, functions of z have modulus ℓ . Their apparent Jacobi poles are removable: the denominator has order two, while the numerators have order at most two.

For each coefficient, subtract a multiple of its matching function to cancel one residue. Parity cancels the second residue and the characters cancel all translates. The difference is entire on the double-period torus, and hence constant; a negative character forces that constant to be zero. The constants are real because all the functions are real on the real locus. We obtain

$$B(u, M) = \frac{\text{dn } z}{D(z)} [a_0(x^2 + y^2) + b_0x \text{sn } z + c_0y \text{cn } z + d_0]. \quad (15)$$

The contact area has the separate form

$$J(u) = J_0 \text{dn } z, \quad J_0 = J(0) > 0. \quad (16)$$

To see this, a vertex pole occurs at iK' modulo the quotient. Only one indexed vertex has that pole, so its shoelace contribution has at most a simple pole. Its even $(+, -)$ character is that of $\text{dn } z$; the same residue argument leaves no additive constant. For real u , a step $0 < h < 2K$ advances the point angle by strictly less than π . Every consecutive determinant is positive, including in star order, so $J_0 > 0$.

5 The Steiner denominator has no real zero

Lemma 3. *The coefficient a_0 in (15) is nonzero. Hence $\sum_i w_i$ never vanishes at a real phase.*

Proof. Introduce independent complex components $\zeta_i = Q_{ix} + iQ_{iy}$ and $\tilde{\zeta}_i = Q_{ix} - iQ_{iy}$, and edge ratios

$$r_i = \frac{\zeta_{i+1} - \zeta_i}{\tilde{\zeta}_{i+1} - \tilde{\zeta}_i}.$$

On real edges r_i has unit modulus. Direct angle multiplication and (8) give the meromorphic identity

$$a = \frac{1}{16i} \sum_i \left(\frac{r_i}{r_{i-1}} - \frac{r_{i-1}}{r_i} \right). \quad (17)$$

At the isotropic branch where r_j has a simple pole, all other ratios are finite and nonzero by the collision exclusions above. Therefore

$$\text{Res } a = \frac{\text{Res } r_j}{16i} \left(\frac{1}{r_{j-1}} - \frac{1}{r_{j+1}} \right). \quad (18)$$

The bracket cannot vanish. Otherwise the neighboring directions would be parallel. Their determinant, using (5), is proportional to the Jacobi addition identity

$$\text{sn}(v+h) \text{cn}(v-h) - \text{sn}(v-h) \text{cn}(v+h) = \frac{2 \text{sn } h \text{cn } h \text{dn } v}{1 - k^2 \text{sn}^2 v \text{sn}^2 h}. \quad (19)$$

At $v = K + iy_0$ the denominator is $1 - (k^2/\eta) \text{sn}^2 h > 0$ and $\text{dn } v \neq 0$. Also $\text{sn } h \neq 0$ for $0 < h < 2K$, and $\text{cn } h = 0$ would give $h = K$, or $4\tau = n$, impossible for odd n . Thus (18) is nonzero and $a_0 \neq 0$. For real z , $\text{dn } z > 0$ and $D(z) \geq 1 - \nu > 0$. Equations (8) and (15) now show that the weight sum never vanishes. $\square$

6 Stationary-value cancellation

We now prove (2) without assuming nonzero stationary pedal area. At the same r_j pole put $p = x + iy$ and $q = x - iy$. The two components of a foot have the rational expressions

$$F_j = \frac{p + r_j q + \gamma_j}{2}, \quad \tilde{F}_j = \frac{q + r_j^{-1} p + \tilde{\gamma}_j}{2}, \quad \gamma_j = \zeta_j - r_j \tilde{\zeta}_j.$$

In particular $\text{Res } F_j = (\text{Res } r_j)(q - \tilde{\zeta}_j)/2$, while $\tilde{F}_j$ and every other foot component are finite. The complex shoelace formula therefore gives

$$\text{Res } B = \frac{\text{Res } r_j}{8i} (q - \tilde{\zeta}_j)(\tilde{F}_{j-1} - \tilde{F}_{j+1}). \quad (20)$$

The last bracket is affine in p and independent of q , with nonzero p coefficient by (18). Thus the residue is $\kappa(q - q_0)(p - p_0)$, $\kappa \neq 0$. It is a product of opposite isotropic factors, not the square of one

factor. Expanding shows that its coefficients in the form $a(x^2 + y^2) + bx + cy + d$ satisfy $b^2 + c^2 = 4ad$. At a zero of D , (13) has $\text{sn}^2 z = 1/\nu$ and $\text{cn}^2 z = 1 - 1/\nu$, so this identity becomes

$$\frac{b_0^2}{\nu} + c_0^2 \left(1 - \frac{1}{\nu}\right) = 4a_0 d_0. \quad (21)$$

By Lemma 3, the actual centroid is

$$C = \left(-\frac{b_0 \text{sn } z}{2a_0}, -\frac{c_0 \text{cn } z}{2a_0} \right).$$

Substituting in (15) gives

$$B_C = \frac{\text{dn } z}{D(z)} \left[d_0 - \frac{c_0^2}{4a_0} + \frac{c_0^2 - b_0^2}{4a_0} \text{sn}^2 z \right].$$

Equation (21) makes the bracket exactly $(d_0 - c_0^2/(4a_0))D(z)$. Hence

$$B_C = \left(d_0 - \frac{c_0^2}{4a_0} \right) \text{dn } z, \quad R = \frac{d_0 - c_0^2/(4a_0)}{J_0}. \quad (22)$$

This proves phase invariance and the all-or-nothing denominator alternative. Nothing in the argument asserts $R > 0$ or $R \neq 0$.

7 An exact exceptional five-periodic family

We give coordinates and exact sign certificates for a nondegenerate family with $R = 0$. All quantities in this section are algebraic; no numerical shooting or elliptic-integral root is needed. For $x \in I = [l, r] = [101/100, 11/10]$ set

$$\begin{aligned} d &= x^3 - x, \quad y = \frac{x}{2} \left(x^2 - 1 + \sqrt{(x^2 - 1)^2 + 4} \right), \quad U = x^2/y = y - d, \\ V &= \frac{U(y-1)^2[U(1+y)^2 - 4y]}{16y^2}, \quad \lambda = \frac{4Vy}{(y-1)^2}, \quad \alpha = U + \lambda, \quad \beta = V + \lambda. \end{aligned} \quad (23)$$

We use coefficient coordinates (X, Y) with physical point $(X, \sqrt{V}Y)$, so their metric is $\langle v, w \rangle_V = v_x w_x + V v_y w_y$. The contacts and consecutive tangent intersections are

$$\begin{aligned} Q &= \left((0, 1), \left(-\frac{2U}{1+U}, \frac{1-U}{1+U} \right), \left(-\frac{2x}{1+y}, \frac{1-y}{1+y} \right), \right. \\ &\quad \left. \left(\frac{2x}{1+y}, \frac{1-y}{1+y} \right), \left(\frac{2U}{1+U}, \frac{1-U}{1+U} \right) \right), \\ P &= \left((-U, 1), \left(-\frac{U+x}{1+x}, \frac{1-x}{1+x} \right), \left(0, \frac{1+y}{1-y} \right), \right. \\ &\quad \left. \left(\frac{U+x}{1+x}, \frac{1-x}{1+x} \right), (U, 1) \right). \end{aligned} \quad (24)$$

7.1 Exact billiard realization and nondegeneracy

The relation

$$y^2 - dy - x^2 = 0 \quad (25)$$

verifies directly that $Q_{ix}^2/U + Q_{iy}^2 = 1$ and $P_{ix}^2/\alpha + VP_{iy}^2/\beta = 1$. The line $P_i P_{i+1}$ is tangent to the inner ellipse at Q_{i+1} . One way to derive the coordinates is to put $u = \sqrt{U}$, $v = \sqrt{y}$, with $uv = x$, and

use inner tangent half-angles $0, u, v, -v, -u$. For an ellipse point $(-2at/(1+t^2), b(1-t^2)/(1+t^2))$, consecutive tangent intersections have coordinates $(-a(s+t)/(1+st), b(1-st)/(1+st))$. Requiring the first three intersections to lie on one confocal outer ellipse reduces to $v^2 + uv = u^2 + u^3v^3$, which is (25).

The positive root satisfies $x^2 < y < 3/2$. The polynomial in (25) at $y = x^2$ equals $-x^2(x-1)^2(x+1) < 0$, while at $y = 3/2$ it is at least

$$\frac{9}{4} - \frac{3}{2} \left((11/10)^3 - 11/10 \right) - (11/10)^2 > 0.$$

Thus $0 < U < 1$. Under conjugation $y \mapsto d - y$, the field norm of V is

$$\mathcal{N}(V) = \frac{(x-1)^5(x+1)^5(x^2-5)}{256}. \quad (26)$$

Both branches are finite on I : explicitly

$$V = \frac{x^8 - 6x^6 + 10x^4 - 8x^3 - 10x^2 + 8x - 3}{16} - \frac{x^6 - 5x^4 + 3x^2 - 8x + 1}{16x} y. \quad (27)$$

Its norm has no zero on I , and $V(l) > 0$ by the exact endpoint comparison below. Continuity therefore gives $V > 0$ throughout. Moreover the defining formula and $y - 1 < 1/2$, $1 + y < 5/2$ imply

$$0 < \frac{V}{U} = \frac{(y-1)^2[U(1+y)^2 - 4y]}{16y^2} < \frac{25}{256} < 1.$$

It follows that $0 < \lambda < \beta < \alpha$: both ellipses and the caustic are nondegenerate.

We have $0 < u < 1 < v$ and $uv = x > 1$. On the circle before the invertible positive diagonal map to the inner ellipse, the angular gaps at the half-angles $0, u, v, -v, -u$ are the two end gaps $2\arctan u$, the two intermediate gaps $2\arctan v - 2\arctan u$, and the middle gap $2\pi - 4\arctan v$. All are strictly between 0 and π . The contacts are therefore distinct and cyclically ordered, and their tangent intersections form a strictly convex pentagon with each contact in the relative interior of its side. The same incidence and convexity facts hold after the diagonal map. In particular $\mathcal{A}(Q) > 0$ and all edges are nonzero.

For clarity, the reflection law can also be verified without a projective-metric inference. In coefficient coordinates put $e_i = P_{i+1} - P_i$, $d_i = P_i - P_{i-1}$, $n_i = (P_{ix}/\alpha, P_{iy}/\beta)$ and

$$d_i^{\text{ref}} = d_i - \frac{2\langle d_i, n_i \rangle_V}{\langle n_i, n_i \rangle_V} n_i.$$

Reduction by (25) gives, for every i ,

$$\det(d_i^{\text{ref}}, e_i) = 0, \quad \langle d_i^{\text{ref}}, d_i^{\text{ref}} \rangle_V = \langle d_i, d_i \rangle_V.$$

The proportionality factor $\langle d_i^{\text{ref}}, e_i \rangle_V / \langle e_i, e_i \rangle_V$ is continuous and nonzero on I and positive at l by exact quadratic-field comparison. It is therefore positive everywhere. Thus the directed incoming unit vector reflects to the outgoing one. Equivalently, the confocal angle-bisector property yields this law. Five distinct vertices prove least period five.

7.2 Norms and regularity of the stationary value

Let $e_i = Q_{i+1} - Q_i$ now denote the *inner* edges. Define normalized weights, centroid numerator and area by

$$\begin{aligned} w_i^* &= -\frac{2 \det(e_{i-1}, e_i) \langle e_{i-1}, e_i \rangle_V}{\langle e_{i-1}, e_{i-1} \rangle_V \langle e_i, e_i \rangle_V}, \quad S = \sum_i w_i^*, \quad T_y = \sum_i w_i^* Q_{iy}, \\ A_Q &= \frac{1}{2} \sum_i \det(Q_i, Q_{i+1}), \quad H_i = Q_i - \frac{\langle Q_i, e_i \rangle_V}{\langle e_i, e_i \rangle_V} e_i, \quad B_0 = \frac{1}{2} \sum_i \det(H_i, H_{i+1}). \end{aligned} \quad (28)$$

Physical weights and areas equal $\sqrt{V}$ times these normalized quantities; their ratio is unchanged. Reflection symmetry gives $C_x = 0$ and $C_y = T_y/S$ in coefficient coordinates. The quadratic identity then gives the exact stationary ratio

$$\mathcal{B} = B_0 + \frac{VT_y^2}{8S}, \quad \mathcal{T} = \mathcal{B}/A_Q = B_C/J. \quad (29)$$

Thus every quantity used below is given by explicit rational operations in the quadratic field (25).

Put

$$\begin{aligned} F_8 &= x^8 - 2x^7 - 2x^6 + 22x^5 - 38x^3 + 66x^2 + 18x - 1, \\ G_8 &= x^8 + 2x^7 - 6x^6 - 14x^5 + 4x^4 + 22x^3 + 38x^2 - 10x - 101, \\ H_9 &= x^9 - x^8 - 8x^7 + 8x^6 + 34x^5 - 34x^4 - 80x^3 + 80x^2 + 53x - 181. \end{aligned}$$

Exact field reduction gives

$$\mathcal{N}(S) = \frac{16384(x-1)(x+1)(x^2-5)^2 F_8}{G_8 H_9^2}, \quad (30)$$

$$\mathcal{N}(\mathcal{T}) = -\frac{(x-1)^2 R_{18}}{16x^2(x+1)^2(x^2-5)F_8 G_8}, \quad (31)$$

where

$$\begin{aligned} R_{18} &= x^{18} - 58x^{17} - 383x^{16} - 288x^{15} + 2948x^{14} + 6840x^{13} \\ &\quad - 2508x^{12} - 21376x^{11} - 10946x^{10} + 27892x^9 + 17134x^8 \\ &\quad - 84768x^7 - 189388x^6 - 121896x^5 + 127044x^4 + 217280x^3 \\ &\quad + 64265x^2 + 950x + 25. \end{aligned} \quad (32)$$

Appendix A gives the exact coefficient rule certifying $F_8 > 0$, $G_8 < 0$, $H_9 < 0$ on the whole interval. It also gives the reduced coefficients for (31).

Specializing a norm must not silently multiply a zero on one branch by a pole on the other. In the unique representation $a(x) + b(x)y$, common denominators of the two rational coefficients divide the following polynomials, up to nonzero rational constants:

$$\begin{array}{c|c} V & x \\ S & x(x-1)^2(x+1)^4 G_8 H_9^2 \\ \mathcal{T} & x^2(x+1)^2(x^2-5) F_8 G_8. \end{array} \quad (33)$$

These are direct coefficient-clearing identities; a reproducible quadratic-field reduction is described in the appendix. Every factor is nonzero throughout I , and both roots of (25) are finite. Both conjugate values of $V, S, \mathcal{T}$ are therefore finite everywhere on I . Consequently (26) and (30) specialize legitimately and show that the physical V and S do not vanish. All geometric edge norms are positive, so the centroid and stationary area are continuous. Independently, Lemma 3 already supplies the nonvanishing of S for this genuine odd billiard family.

7.3 The physical zero and its algebraic specification

Exact rational-quadratic-field comparison at the endpoints gives

$$\mathcal{T}(l) < 0, \quad \mathcal{T}(r) > 0. \quad (34)$$

For reference, deliberately coarse strict rational bounds are

x	$\mathcal{T}(x)$	$V(x)$
$101/100$	$(-5 \cdot 10^{-6}, -4 \cdot 10^{-6})$	$(4 \cdot 10^{-11}, 6 \cdot 10^{-11})$
$11/10$	$(10^{-3}, 2 \cdot 10^{-3})$	$(4 \cdot 10^{-6}, 6 \cdot 10^{-6})$

These are exact rational enclosures, not rounded numerical estimates. The appendix specifies the comparison without floating-point rounding. By continuity, the physical branch has a zero in (l, r) . The exact Bernstein coefficients of R_{18} on I have signs

$$(+, +, +, +, +, -, -, -, -, -, -, -, -, -, -, -, -).$$

They have one sign variation and opposite endpoint signs. The substitution $x = (l + rt)/(1 + t)$ turns the Bernstein coefficients, multiplied by positive binomial factors, into the coefficients of $(1 + t)^{18} R_{18}((l + rt)/(1 + t))$. Descartes' rule gives at most one root in (l, r) , while the endpoint signs give existence. Thus R_{18} has exactly one root x_* in that interval. By (31), every physical zero is a root of R_{18} ; (34) identifies x_* with the physical zero, not merely a zero on its conjugate branch.

Define x_* by this unique root, choose the positive y in (23), and use (24). This is an exact algebraic specification of a genuine strictly convex confocal five-periodic billiard with finite own Steiner centroid, positive contact area and zero stationary pedal area. Equation (22) makes that stationary area zero at *every* phase of the fixed family. This completes the exceptional-family assertion of Theorem 1.

8 Repetitions, circles and attribution

If an odd listed length N repeats a least-period orbit n , then $N = mn$ forces both m and n odd. Each centroid sum is multiplied by m , so C is unchanged; each signed area is multiplied by m , so R is unchanged. Thus the theorem does not require a least-period narrowing of the source's odd-length condition. For a circle the contact polygon is a regular odd star. Its common weight is $-\sin(4\pi\tau/n) \neq 0$, and $C = O$. The feet are a rotated copy scaled by $\cos(\pi\tau/n)$, yielding $R = \cos^2(\pi\tau/n) > 0$. Reversing a traversal leaves the centroid unchanged and reverses both signed areas.

The quadratic-field calculations verify the explicit algebraic identities and endpoint comparisons. Rational Bernstein coefficients are whole-interval certificates, not a numerical mesh. Neither these checks nor discovery numerics alone prove the all-period theorem; its complex-analytic proof is given above. The supplementary package retains the actual exact checks and their dependencies. No independent human review or proof-assistant verification is claimed.

The classical fixed-polygon Steiner identity, Jacobi parametrization and identities, conic polarity, and elliptic-function residue methods are credited to [3, 4, 5, 7, 6]. A bounded primary-literature search did not locate a matching all-odd theorem or this exact exception; novelty remains undetermined, and no exhaustive priority certification follows. This unrefereed manuscript was developed with AI assistance in derivation, literature reconnaissance, exact algebraic checks and preparation. The author is responsible for its claims.

A Exact field and interval certificates

The following specifications make the algebraic part reproducible using rational operations alone. Represent $a + by$ by (a, b) . With $d = x^3 - x$, multiplication, norm and inversion are

$$(a, b)(c, e) = (ac + bex^2, ae + bc + bed), \quad \mathcal{N}(a + by) = a^2 + abd - b^2x^2,$$

$$(a + by)^{-1} = \frac{a + bd - by}{a^2 + abd - b^2x^2}.$$

Starting with (23), substitute the explicit vertices (24) into (28)–(29); cancel rational factors after each operation. This gives (26), (30), (31), and the coefficient denominators in (33). For the stationary ratio the reduced formula is particularly useful:

$$\mathcal{T} = \frac{xP_{21}(x) - yQ_{19}(x)}{4x^2(x+1)^2(x^2-5)F_8(x)G_8(x)}, \quad (35)$$

where

$$\begin{aligned} P_{21} &= x^{21} + 3x^{20} - 15x^{19} - 37x^{18} + 158x^{17} + 298x^{16} \\ &\quad - 1084x^{15} - 2020x^{14} + 3734x^{13} + 8050x^{12} - 6466x^{11} \\ &\quad - 20326x^{10} + 1552x^9 + 46272x^8 + 45748x^7 - 69684x^6 \\ &\quad - 123167x^5 + 90163x^4 + 156121x^3 - 37725x^2 - 27430x + 1390, \\ Q_{19} &= x^{19} + 3x^{18} - 13x^{17} - 95x^{16} - 60x^{15} + 764x^{14} \\ &\quad + 1180x^{13} - 2460x^{12} - 4066x^{11} + 4554x^{10} + 4746x^9 \\ &\quad - 7618x^8 - 13516x^7 - 4948x^6 + 44812x^5 + 26132x^4 \\ &\quad - 33815x^3 + 33675x^2 + 17115x - 855. \end{aligned}$$

Expanding the norm of the numerator in (35) by the preceding rule and cancelling verifies (31) directly. To make the remaining coefficient-clearing claim explicit, write $D_S = (x-1)^2(x+1)^4G_8H_9^2$. The weight reduction is

$$S = -\frac{128P_{25}}{D_S} - \frac{128Q_{25}}{xD_S}y,$$

with

$$\begin{aligned} P_{25} &= x^{25} + x^{24} - 20x^{23} - 20x^{22} + 194x^{21} + 218x^{20} \\ &\quad - 1156x^{19} - 1652x^{18} + 4351x^{17} + 8631x^{16} - 10152x^{15} \\ &\quad - 32232x^{14} + 12124x^{13} + 93196x^{12} + 14296x^{11} - 218312x^{10} \\ &\quad - 122289x^9 + 422911x^8 + 345532x^7 - 648836x^6 - 486014x^5 \\ &\quad + 762042x^4 + 325292x^3 - 436292x^2 + 48913x + 50345, \\ Q_{25} &= x^{25} + x^{24} - 24x^{23} - 8x^{22} + 294x^{21} - 34x^{20} \\ &\quad - 2320x^{19} + 528x^{18} + 11883x^{17} - 2269x^{16} - 39184x^{15} \\ &\quad + 7376x^{14} + 87684x^{13} - 20236x^{12} - 147328x^{11} + 21120x^{10} \\ &\quad + 178535x^9 + 35223x^8 - 104728x^7 - 154504x^6 - 53706x^5 \\ &\quad + 285198x^4 + 49872x^3 - 335312x^2 + 19021x + 31845. \end{aligned}$$

Its norm verifies (30) by polynomial multiplication. Together with (27) and (35), this also proves that both conjugate branches have only the nonzero denominator factors asserted in (33).

For a polynomial $p(x) = \sum_{j=0}^m c_j x^j$, its Bernstein coefficients on $[l, r]$ are the following explicit rational numbers:

$$\beta_i = \sum_{j=0}^i \frac{\binom{i}{j}}{\binom{m}{j}} (r-l)^j \sum_{t=j}^m c_t \binom{t}{j} l^{t-j}, \quad 0 \leq i \leq m. \quad (36)$$

Thus $p(l + (r-l)t) = \sum_i \beta_i \binom{m}{i} t^i (1-t)^{m-i}$. Substitution of the displayed polynomial coefficients gives this exact sign certificate:

p	signs of $(\beta_0, \dots, \beta_m)$	variations
F_8	+ at all nine entries	0
G_8	− at all nine entries	0
H_9	− at all ten entries	0
R_{18}	+ at 0–4, − at 5–18	1.

Every sign is an integer/rational comparison through (36); no interval sample is used. For example the extreme coefficients are

$$\begin{aligned} F_8(l) &= 652929288024120601/10^{16}, & F_8(r) &= 7821625261/10^8, \\ G_8(l) &= -633632961551638999/10^{16}, & G_8(r) &= -5801708299/10^8, \\ H_9(l) &= -128006367681591799199/10^{18}, & H_9(r) &= -128604949919/10^9, \\ R_{18}(l) &= 23186851460417006184241730170815299836001/10^{36}, \\ R_{18}(r) &= -106176862025066098729999/10^{18}. \end{aligned}$$

Finally, at a rational endpoint put $D_x = (x^2 - 1)^2 + 4$ and substitute $y = (d + x\sqrt{D_x})/2$. Both (27) and (35) become $a + b\sqrt{D_x}$ with rational a, b . The exact endpoint comparisons are

quantity	x	sgn a	sgn b	sgn($a^2 - b^2 D_x$)
V	101/100	−	+	−
$\mathcal{T}$	101/100	+	−	−
$\mathcal{T}$	11/10	+	−	+

These signs follow by substitution of rational endpoints in the explicit polynomial expressions. In the first row $b\sqrt{D_x} > |a|$, giving $V(l) > 0$. In the other rows $a - b\sqrt{D_x} > 0$, so the sign of the norm equals the sign of the physical branch; equivalently compare a^2 and $b^2 D_x$. This proves (34). The reflection-direction signs may be checked by exactly the same rational-quadratic comparison applied to the explicit dot products in Section 7.

References

- [1] D. Reznik, R. Garcia and J. Koiller, Fifty new invariants of N -periodics in the elliptic billiard, *Arnold Math. J.* **7** (2021), 341–355. [doi:10.1007/s40598-021-00174-y](https://doi.org/10.1007/s40598-021-00174-y).
- [2] D. Reznik, R. Garcia and J. Koiller, Eighty new invariants in the elliptic billiard, [arXiv:2004.12497v11](https://arxiv.org/abs/2004.12497v11) (2020).
- [3] D. Reznik, R. Garcia and H. Stachel, Area-invariant pedal-like curves derived from the ellipse, [arXiv:2009.02581v3](https://arxiv.org/abs/2009.02581v3) (2020).
- [4] H. Stachel, On the motion of billiards in ellipses, [arXiv:2105.03624v2](https://arxiv.org/abs/2105.03624v2) (2021). [doi:10.1007/s40879-021-00524-2](https://doi.org/10.1007/s40879-021-00524-2).
- [5] H. Stachel, A triple of projective billiards, *KoG* **26** (2022), 44–51. [doi:10.31896/k.26.4](https://doi.org/10.31896/k.26.4).

- [6] A. C. Chávez-Caliz, More about areas and centers of Poncelet polygons, *Arnold Math. J.* **7** (2021), 91–106. [doi:10.1007/s40598-020-00154-2](https://doi.org/10.1007/s40598-020-00154-2).
- [7] NIST Digital Library of Mathematical Functions, [§22.2](#), [§22.4](#), and [§22.8](#), Jacobi elliptic functions (accessed 1 October 2026).