

Steiner-Centroid Pedal Area Ratios in Elliptic Billiards

Alper Ferudun*

1 October 2026

Abstract

For an odd-periodic billiard family between fixed confocal ellipses, we prove that the signed area of the original orbit divided by the signed area of its pedal polygon at its own Steiner curvature centroid is constant. Both the centroid and this ratio are defined at every real phase. Coprime star trajectories and odd repeated traversals are included. Reflection identities express the pedal area as an isotropic quadratic in the pedal point. Complementary complex pole sets and an odd-degree quotient of the Jacobi parameter identify the moving centroid ellipse and reduce the ratio to a constant coefficient. A strict theta-function grid inequality proves the necessary nonvanishing in the positive-quadratic regime. This establishes invariant k_{501} of Reznik, Garcia and Koiller. The circular case has the elementary value $\sec^2(\pi\tau/n)$. No absolute priority claim is made.

1 The original-side Steiner pedal

Fix $a > b > 0$ and $0 < \lambda < b^2$, and consider the confocal pair

$$E : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad E_\lambda : \frac{x^2}{a^2 - \lambda} + \frac{y^2}{b^2 - \lambda} = 1.$$

Let $P = (P_i)_{i \in \mathbb{Z}/N\mathbb{Z}}$ be a billiard polygon on E whose sides are tangent to E_λ . For a cyclic vector sequence V , write $\mathcal{A}(V) = \frac{1}{2} \sum_i \det(V_i, V_{i+1})$. These are signed shoelace areas, including for self-intersecting stars, not sums of absolute face areas.

Orient the orbit with its caustic to the left of each edge. Its local turn δ_i belongs to $(0, \pi)$; the internal ray angle is $\theta_i = \pi - \delta_i$. Define the Steiner curvature centroid by

$$C = \frac{\sum_i w_i P_i}{\sum_i w_i}, \quad w_i = \sin(2\theta_i) = -\sin(2\delta_i). \quad (1)$$

This specifies the angle convention at crossed stars explicitly. Let $Q_i(M)$ be the perpendicular foot of M on the original supporting sideline $P_i P_{i+1}$, and put

$$A = \mathcal{A}(P), \quad B(M) = \mathcal{A}(Q(M)), \quad B_C = B(C).$$

The sidelines here are the original billiard chords, not boundary tangents or the sides of a caustic-contact polygon.

Theorem 1. *For every odd $N \geq 3$, the denominator in (1) is nonzero, $B_C > 0$ in the chosen orientation, and A/B_C is independent of the initial vertex in the fixed-caustic billiard family. All admitted star windings and repeated odd traversals are included.*

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: [AlperTheKing](https://github.com/AlperTheKing).

This is the invariant k_{501} in Table 6 of Reznik, Garcia and Koiller [1, 2]. Their signed-area and Steiner-centroid definitions are retained. Their table says odd N without explicitly requiring least period; Section 6 explains why repetitions do not require that extra restriction here. Hyperbolic and degenerate caustics are excluded. Reversing the orbit leaves C unchanged and reverses both areas, so the ratio is unaffected.

The fixed-polygon Steiner theorem gives circular level sets for pedal area as the pedal point moves; see [3, §2]. It does not directly give the present ratio: both the original polygon and its own centroid vary with the billiard phase. The source also observes that the odd-period centroid loci are ellipses. We prove the precise parametrization of that locus needed here. The Jacobi billiard parametrization is due to the established theory; we use Stachel's explicit form [4]. Complex pole cancellation in Poncelet area problems is also prior methodology [5]. Our argument supplies the original-side pedal trace, the quotient-torus axis relation, and the strict inequality required to invert the area ratio.

2 Reflection identities and the pedal quadratic

It is enough initially to use least period n , odd, and winding τ with $\gcd(\tau, n) = 1$ and $0 < \tau < n/2$. Set

$$\alpha = a^2, \quad \beta = b^2, \quad d = \alpha - \beta, \quad D = \alpha\beta, \quad s = \alpha + \beta.$$

The canonical parametrization [4, Theorem 2] is

$$\begin{aligned} k^2 &= \frac{d}{\alpha - \lambda}, \quad k' = \sqrt{1 - k^2}, \quad K = K(k), \quad K' = K(k'), \\ P(u) &= (-a \operatorname{sn}(u, k), b \operatorname{cn}(u, k)), \quad P_i = P(u + ih), \quad h = 2\Delta = \frac{4\tau K}{n}. \end{aligned} \tag{2}$$

The caustic condition gives

$$\operatorname{sn}^2 \Delta = \frac{\lambda}{\beta}, \quad \operatorname{cn}^2 \Delta = \frac{\beta - \lambda}{\beta}, \quad \operatorname{dn}^2 \Delta = \frac{\alpha(\beta - \lambda)}{\beta(\alpha - \lambda)}. \tag{3}$$

We suppress k when no confusion is possible. After complexification, dot products are bilinear, not Hermitian. All sums below are over $i = 0, \dots, n-1$ at the arguments $u + ih$. Introduce

$$\begin{aligned} W(u) &= \alpha - d \operatorname{sn}^2 u, \quad H = \sqrt{\lambda} \sqrt{\alpha - \lambda}, \quad S = \sum \frac{\operatorname{dn} u_i}{W_i}, \\ X &= \sum \frac{\operatorname{dn} u_i \operatorname{sn} u_i}{W_i^2}, \quad Y = \sum \frac{\operatorname{dn} u_i \operatorname{cn} u_i}{W_i^2}, \\ q &= \frac{D - \lambda s}{2D}, \quad c = \frac{sD - \lambda(\alpha^2 + \beta^2)}{2D}. \end{aligned} \tag{4}$$

For real u , $S > 0$. Also $c > 0$, since its numerator is greater than $\beta^2 d > 0$.

Lemma 2. *The following identities hold:*

$$B(x, y) = H[cS + 2a(\beta - \lambda)xX - 2b(\alpha - \lambda)yY + qS(x^2 + y^2)], \tag{5}$$

$$C_x = -\frac{a(\beta - \lambda)X}{qS}, \quad C_y = \frac{b(\alpha - \lambda)Y}{qS}, \tag{6}$$

$$B_C = HS(c - q(C \cdot C)). \tag{7}$$

Here $q \neq 0$ for odd n , and $q > 0$ exactly when $\Delta < K/2$.

Proof. First consider any ordered polygon with nonzero edges. Let t_i be its unit edge direction, $n_i = Jt_i$, with J positive quarter rotation. Put $\mathcal{H}_i = 2n_i n_i^T - I$. Direct multiplication gives

$$\sin \delta_i \operatorname{Sym}(n_{i-1} n_i^T) = \frac{\sin 2\delta_i}{4} I + \frac{J\mathcal{H}_{i-1} - J\mathcal{H}_i}{4}.$$

Translate the origin to the pedal point M . The two feet adjacent to vertex P_i contribute $\frac{1}{2} \sin \delta_i ((P_i - M) \cdot n_{i-1})((P_i - M) \cdot n_i)$. The matrix difference telescopes in the quadratic term. Its linear sum is $\sum J\mathcal{H}_i(P_{i+1} - P_i) = -J \sum (P_{i+1} - P_i) = 0$. Consequently, writing $w = \sum_i w_i$ and $T = \sum_i w_i P_i$, we have

$$B(M) = B(0) + \frac{T \cdot M}{4} - \frac{w(M \cdot M)}{8}. \quad (8)$$

This proves the classical stationary-point identification $C = T/w$ whenever $w \neq 0$.

For the ellipse, at $P = (a \cos \theta, b \sin \theta)$ use the outward normal $n = (b \cos \theta, a \sin \theta)/\sqrt{W}$ and tangent $t = (-a \sin \theta, b \cos \theta)/\sqrt{W}$, where $W = \beta \cos^2 \theta + \alpha \sin^2 \theta$. The incoming and outgoing unit velocities are

$$v_- = \eta t + j n, \quad v_+ = \eta t - j n, \quad j = \sqrt{\lambda/W}, \quad \eta = \sqrt{(W - \lambda)/W} > 0.$$

These signs hold for the elliptic caustic, including stars. For an explicit check, a chord with endpoint parameters $m \pm \epsilon$ has $\sin^2 \epsilon = \lambda U/D$, where $U = \alpha \sin^2 m + \beta \cos^2 m$. The endpoint tangent-component numerators $U \cos \epsilon \pm d \sin m \cos m \sin \epsilon$ have positive sum and product $U(DU - \lambda V)/D > 0$, where $V = \alpha^2 \sin^2 m + \beta^2 \cos^2 m$ and $DU - \beta V = \beta^2 d \cos^2 m \geq 0$.

The coordinate-product jump is

$$(v_-)_x (v_-)_y - (v_+)_x (v_+)_y = \frac{2\sqrt{\lambda}\sqrt{W - \lambda}(2D - sW)}{dW^2}.$$

It telescopes on a closed orbit. Since $\sqrt{W - \lambda} = \sqrt{\alpha - \lambda} \operatorname{dn} u$ on the real locus,

$$\sum \frac{\operatorname{dn} u_i}{W_i^2} = \frac{s}{2D} S. \quad (9)$$

The two centered pedal feet adjacent to P contribute $\eta j(\eta^2 P_n^2 - j^2 P_t^2)$, with $P_n = ab/\sqrt{W}$ and $P_t = -d \sin \theta \cos \theta/\sqrt{W}$. Use $d^2 \sin^2 \theta \cos^2 \theta = (W - \beta)(\alpha - W)$ and sum:

$$B(0) = \sqrt{\lambda}[\lambda R_1 + (D - s\lambda)R_2] = HcS, \quad R_j = \sum \frac{\sqrt{W_i - \lambda}}{W_i^j}. \quad (10)$$

The vector velocity jump in Jacobi coordinates is $2\sqrt{\lambda}(-b \operatorname{sn} u, a \operatorname{cn} u)/W$, so closure also gives $\sum \operatorname{sn} u_i/W_i = \sum \operatorname{cn} u_i/W_i = 0$. Differentiate these identities with respect to the common phase, using

$$(\operatorname{cn} u/W)' = \operatorname{dn} u \operatorname{sn} u/W - 2\beta \operatorname{dn} u \operatorname{sn} u/W^2, \quad (\operatorname{sn} u/W)' = 2\alpha \operatorname{dn} u \operatorname{cn} u/W^2 - \operatorname{dn} u \operatorname{cn} u/W.$$

Thus

$$X = \frac{1}{2\beta} \sum \frac{\operatorname{dn} u_i \operatorname{sn} u_i}{W_i}, \quad Y = \frac{1}{2\alpha} \sum \frac{\operatorname{dn} u_i \operatorname{cn} u_i}{W_i}. \quad (11)$$

The turn has $\sin \delta = 2\sqrt{\lambda}\sqrt{W - \lambda}/W$ and $\cos \delta = 1 - 2\lambda/W$. Therefore $w_i = -4H \operatorname{dn} u_i (W_i - 2\lambda)/W_i^2$. Equations (9) and (11) give

$$w = -8HqS, \quad T_x = 8aH(\beta - \lambda)X, \quad T_y = -8bH(\alpha - \lambda)Y.$$

Substitute in (8) and (10) to obtain (5)–(7).

Finally, (3) shows that $q = 0$ is equivalent to $\operatorname{dn}^2 \Delta = k'$. Since dn is strictly decreasing on $(0, K)$ and $\operatorname{dn}^2(K/2) = k'$, this is equivalent to $\Delta = K/2$, or $4\tau = n$, impossible for odd n . The same comparison gives $q > 0$ precisely when $\Delta < K/2$. $\square$

3 The complementary pole identity

Lemma 3. *For odd least period, $A(u)S(u) = \Lambda$ is a positive constant. The zeros of S are precisely simple zeros at the shifted Jacobi vertex poles; its poles at the shifted zeros of W are simple.*

Proof. The common Jacobi pole positions are $z_p = 2mK + (2j+1)iK'$. The standard shift formulas [6] give, near iK' ,

$$\operatorname{sn}(iK' + w) = \frac{1}{k \operatorname{sn} w}, \quad \operatorname{cn}(iK' + w) = -\frac{i \operatorname{dn} w}{k \operatorname{sn} w}, \quad \operatorname{dn}(iK' + w) = -\frac{i \operatorname{cn} w}{\operatorname{sn} w}.$$

Thus $f = \operatorname{dn}/W$ is holomorphic, odd, and zero at each z_p . Put $Z = z_p + K$. The point function satisfies $P(Z+w) = P(Z-w)$; at $Z = K + iK'$ this follows from

$$\operatorname{sn}(Z+w) = \frac{\operatorname{dn} w}{k \operatorname{cn} w}, \quad \operatorname{cn}(Z+w) = -\frac{ik'}{k \operatorname{cn} w}.$$

The zeros of W are exactly $Z \pm \Delta$ modulo $2K\mathbb{Z} + 2iK'\mathbb{Z}$. They exhaust its order-two divisor and are simple: at them

$$\operatorname{sn}^2 u = \alpha/d, \quad \operatorname{cn}^2 u = -\beta/d, \quad \operatorname{dn}^2 u = -\lambda/(\alpha - \lambda), \quad (12)$$

so $W' = -2d \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u \neq 0$.

A shifted W zero can coincide with a shifted vertex pole only if $jh = K \pm \Delta \pmod{2K}$, or $2\tau(2j \pm 1) = n(2l + 1)$. Parity excludes this for odd n . Two different indexed vertex poles do not coincide either, because $\gcd(2\tau, n) = 1$. Therefore A has at most simple vertex poles. At each of them S is analytic and vanishes by pairing its terms under $i \mapsto n - i$, using oddness of f and $nh = 4\tau K$. The product AS has no pole there.

At $u_0 = Z - \Delta$, the vertices are finite and $P_i(u_0) = P_{1-i}(u_0)$. Reversing the ordered edges thus gives $A(u_0) = 0$. Since S has at most a simple pole there, AS is regular. The other branch is an h shift; lattice translates preserve the cancellations. These exhaust the possible poles. The entire elliptic product is constant. On the real locus, each edge has the origin strictly on its left, so $\det(P_i, P_{i+1}) > 0$; hence $A > 0$ and $\Lambda = AS > 0$.

At u_0 exactly two summands have W poles, at $Z - \Delta$ and $Z + \Delta$. Their residues are equal and nonzero because f is odd about Z . Thus the pole of S is nonremovable and simple. At every vertex pole, the zero of S , the at-most-simple pole of A , and $AS = \Lambda \neq 0$ force both orders to be exactly one. The same identity excludes any further zeros of S . $\square$

4 The odd quotient torus

Let $Q = \exp(-\pi K'/K)$. Choose the unique modulus $\ell \in (0, 1)$ whose nome is Q^n , and put

$$L = K(\ell), \quad L' = K(\ell'), \quad \ell' = \sqrt{1 - \ell^2}, \quad v = \frac{nL}{K}u, \quad \frac{L'}{L} = \frac{nK'}{K}.$$

Lemma 4. *There are constants $A_0, \sigma > 0$ and nonzero real c_x, c_y such that*

$$A = A_0 \operatorname{dn}(v, \ell), \quad S = \frac{\sigma}{\operatorname{dn}(v, \ell)}, \quad C_x = c_x \operatorname{sn}(v, \ell), \quad C_y = c_y \operatorname{cn}(v, \ell), \quad (13)$$

with $A_0\sigma = \Lambda$. Moreover,

$$c - qc_y^2 = \frac{q(c_x^2 - c_y^2)}{\ell^2}, \quad (14)$$

$$\frac{c_x^2}{c_y^2} = \left(\frac{\alpha}{\beta}\right)^4 k'^4 \ell'^2 = \frac{\operatorname{dn}^8(\Delta, k) \ell'^2}{k'^4}. \quad (15)$$

Proof. Equation (11) and the simple, nonzero residues of S show that C_x, C_y are regular at the shifted W poles. At Jacobi vertex poles, $\text{dn sn}/W$ and $\text{dn cn}/W$ are individually holomorphic, while S has a simple zero. Thus C_x, C_y have at most simple poles at the vertex-pole lattice and no others.

Put $T = 2K/n$ and $E = 2iK'$. The functions are h -periodic. The characters of (A, C_x, C_y) under $2K$ are $(+, -, -)$, and under E they are $(-, +, -)$. The latter follow from the Jacobi characters $(\text{sn}, \text{cn}, \text{dn}) \mapsto (\text{sn}, -\text{cn}, -\text{dn})$. Choose integers r, t with $2\tau r + nt = 1$. Since n is odd, t is odd, and $T = rh + t(2K)$ also has characters $(+, -, -)$.

All possible vertex poles form the one class iK' modulo $T\mathbb{Z} + E\mathbb{Z}$. Under $u \mapsto v$, T maps to $2L$ and iK' to iL' . The functions $\text{dn}(v, \ell), \text{sn}(v, \ell), \text{cn}(v, \ell)$ have respectively the same characters and simple-pole class. Subtract from each of A, C_x, C_y a multiple of the appropriate function to cancel the residue at iK' . The characters cancel every translated residue. The difference is entire with periods $2T, 2E$, hence constant, and a negative character forces that constant to vanish. This proves the asserted representations, initially allowing a zero axis scale. Evaluation at zero gives $A_0 = A(0) > 0$, and Lemma 3 gives $\sigma = S(0) > 0$. Evaluation at real phases where the Jacobi functions are nonzero shows c_x, c_y are real.

At $u_0 = K + iK' - \Delta$, the two branch values of sn and cn are equal, by evenness about $K + iK'$. Taking the ratios of residues in (11) and (6) gives

$$C_x(u_0) = -\frac{a(\beta - \lambda) \text{sn } u_0}{2\beta q}, \quad C_y(u_0) = \frac{b(\alpha - \lambda) \text{cn } u_0}{2\alpha q}. \quad (16)$$

Both coordinates are nonzero. Equation (12) and the factorization

$$\begin{aligned} \alpha^4(\beta - \lambda)^2 - \beta^4(\alpha - \lambda)^2 \\ = d(D - \lambda s)[sD - \lambda(\alpha^2 + \beta^2)] \end{aligned}$$

show $C(u_0) \cdot C(u_0) = c/q$. Now $v(u_0) = (n - 2\tau)L + iL'$, with $n - 2\tau$ odd. Its squared Jacobi values are $\text{sn}^2(v, \ell) = 1/\ell^2$ and $\text{cn}^2(v, \ell) = -\ell'^2/\ell^2$. Substitution proves (14). Taking the ratio of squared coordinates in (16) proves (15), using $k'^2 = (\beta - \lambda)/(\alpha - \lambda)$ and (3). It also proves both axis scales nonzero. $\square$

From (13), (14), and $\text{sn}^2 v + \text{cn}^2 v = 1$ we obtain

$$c - q(C \cdot C) = \frac{q(c_x^2 - c_y^2)}{\ell^2} \text{dn}^2(v, \ell), \quad \frac{B_C}{A} = \frac{H\sigma^2 q(c_x^2 - c_y^2)}{\ell^2 \Lambda}. \quad (17)$$

This already proves constancy of the reciprocal ratio. Nonvanishing is a separate obligation: it is not legitimate to invert a constant that might be zero.

5 A strict odd-grid inequality and nonvanishing

Lemma 5. *Let $0 < k < 1$, and let ℓ have nome Q^n , where $Q = e^{-\pi K'/K}$ and $n \geq 5$ is odd. If τ is a positive integer with $\Delta = 2\tau K/n < K/2$, then*

$$\text{dn}^4(\Delta, k) > \frac{k'^2}{\ell'}. \quad (18)$$

Proof. Define $F(x) = \log \text{dn}(Kx/\pi, k) - \frac{1}{2} \log k'$. The half-period identity $\text{dn}(u + K) = k'/\text{dn } u$ gives $F(x + \pi) = -F(x)$, and F is even. On $[0, \pi/2]$ it is decreasing, positive before the right endpoint, and $F(\pi/2) = 0$. It is strictly concave in the interior: with $z = \text{sn}^2 u$,

$$(\log \text{dn } u)'' = -\frac{k^2(1 - 2z + k^2 z^2)}{\text{dn}^2 u} < 0 \quad (0 < u < K/2).$$

The smaller zero of the numerator polynomial is $1/(1+k')$, attained at $u = K/2$.

The theta representation of dn and the products for θ_3, θ_4 [7, 6] yield the uniformly absolutely convergent real Fourier series

$$F(x) = 4 \sum_{\substack{m \geq 1 \\ m \text{ odd}}} \frac{Q^m \cos(mx)}{m(1-Q^{2m})}. \quad (19)$$

Indeed $\text{dn } u = \sqrt{k'} \theta_3(\pi u/(2K), Q)/\theta_4(\pi u/(2K), Q)$, and expanding $\log((1+z)/(1-z))$ in the paired products gives (19). Summation over the odd n -grid gives

$$G := \sum_{j=0}^{n-1} F(2\pi j/n) = 4 \sum_{\substack{m \geq 1 \\ m \text{ odd}}} \frac{Q^{nm}}{m(1-Q^{2nm})} = -\frac{1}{2} \log \ell' > 0. \quad (20)$$

The grid factor n cancels the denominator of the surviving frequency nm .

Put $t = \pi/n$, $r = \lfloor n/4 \rfloor$, and $d_j = F((j-1)t) - F(jt)$. Strict concavity implies $d_{2j} > d_{2j-1}$ for the indices used below. Folding the grid with evenness and $F(\pi - x) = -F(x)$ gives

$$\begin{aligned} n = 4r + 1 : \quad G &= F(0) - 2 \sum_{j=1}^r d_{2j} < F(2rt), \\ n = 4r + 3 : \quad G &= F(0) - 2 \sum_{j=1}^r d_{2j} - 2F((2r+1)t) < F(2rt). \end{aligned}$$

In the second line $(2r+1)t < \pi/2$, so the extra term is strictly negative. The assumption implies $\tau \leq r$. Thus $F(2\tau t) \geq F(2rt) > G > G/2$. Equations (20) and the definition of F now give $4 \log \text{dn } \Delta - 2 \log k' > -\log \ell'$, proving (18). $\square$

Completion of Theorem 1. The centroid is finite by Lemma 2. If $q < 0$, (7) immediately gives $B_C = HS(c - q|C|^2) > 0$ at every real phase. If $q > 0$, then $\Delta < K/2$; in particular $n \geq 5$. Lemma 5 and (15) give $c_x^2 > c_y^2$. Formula (17) is therefore strictly positive in this case too. Its inverse is the desired constant A/B_C . $\square$

6 Boundary cases, verification and scope

If a listed odd orbit of length N repeats an orbit of least period n , then $N = mn$ forces n, m odd. Both weight sums in (1) are multiplied by m , so C is unchanged. Both areas are multiplied by m , and their ratio is unchanged. This proves the repeated-traversal assertion without narrowing the source's odd-length condition.

For $a = b$, the primitive trajectory is a regular star with chord step $2\pi\tau/n$. Its common weight is $-\sin(4\pi\tau/n)$, nonzero for odd n , and $C = O$. The feet form a rotated similar star scaled by $\cos(\pi\tau/n) > 0$, giving $A/B_C = \sec^2(\pi\tau/n)$. For $n = 3$ on a general ellipse, the Steiner centroid is the triangle circumcenter and its feet are the three side midpoints, yielding the familiar ratio 4. This classical case is not claimed as new.

The supporting package retains exact coefficient checks and low-period geometric controls. It also retains numerical tests with their arithmetic precision, geometry residuals, and rejected integrations. These support regression testing, not the all-period theorem, whose proof is analytic above. In particular, positivity of the stationary pedal area of an arbitrary ellipse-inscribed polygon is false; that tempting shortcut is not used here.

Only the original-side own-Steiner ratio k_{501} is claimed. The neighboring outer and inner invariants k_{502}, k_{503} and hyperbolic or degenerate caustics are not addressed. A bounded primary-literature

search did not identify an earlier proof of the same all-odd statement; this does not establish exhaustive novelty or absolute priority. This refereed manuscript was developed with AI assistance in the mathematical derivation, literature reconnaissance, algebraic checks and preparation. The author is responsible for its claims. No independent human review or formal proof-assistant verification is asserted.

References

- [1] D. Reznik, R. Garcia and J. Koiller, Fifty new invariants of N -periodics in the elliptic billiard, *Arnold Math. J.* **7** (2021), 341–355. doi:10.1007/s40598-021-00174-y.
- [2] D. Reznik, R. Garcia and J. Koiller, Eighty new invariants in the elliptic billiard, [arXiv:2004.12497v11](#) (2020).
- [3] D. Reznik, R. Garcia and H. Stachel, Area-invariant pedal-like curves derived from the ellipse, [arXiv:2009.02581v3](#) (2020).
- [4] H. Stachel, On the motion of billiards in ellipses, [arXiv:2105.03624v2](#) (2021).
- [5] A. C. Chávez-Caliz, More about areas and centers of Poncelet polygons, *Arnold Math. J.* **7** (2021), 91–106. doi:10.1007/s40598-020-00154-8.
- [6] NIST Digital Library of Mathematical Functions, §22.2, §22.4, and §22.8, Jacobi elliptic functions (accessed 1 October 2026).
- [7] NIST Digital Library of Mathematical Functions, §20.5, Theta-function infinite products, equations 20.5.3–4 (accessed 1 October 2026).