

Stationary Focal Antipedal Centroids for Even Elliptic Billiard Periods

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Abstract

Fix a noncircular ellipse and a nondegenerate confocal elliptic caustic supporting a family of billiard orbits of effective even period. Form the outer polygon by intersecting consecutive boundary tangents, then take its antipedal with respect to either focus. We prove that its vertex centroid is fixed throughout the family and lies on the major axis. An exact opposite-edge calculation reduces this centroid to a bilinear trace of a Poncelet polygon inscribed in a circle and circumscribed about a concentric ellipse. We supply a compact-curve proof of trace constancy, adapting the pole cancellation method of Akopyan, Schwartz and Tabachnikov. The calculation proves the constancy assertion k_{407} of Reznik, Garcia and Koiller, with the effective-period convention stated explicitly. The constant need not be the center of the ellipse. We make no signed-area-centroid or hyperbolic-caustic claim.

1 Statement and conventions

Let E be the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a > b > 0,$$

centered at $O = 0$. Its foci are $F = (\pm c, 0)$, where $c^2 = a^2 - b^2$. Fix the confocal elliptic caustic

$$E_\lambda : \frac{x^2}{a^2 - \lambda} + \frac{y^2}{b^2 - \lambda} = 1, \quad 0 < \lambda < b^2.$$

Suppose it supports a prescribed Poncelet family of periodic billiard orbits $P = (P_1, \dots, P_N)$ of effective even period $N \geq 4$. All indices are cyclic. Let X_i be the intersection of the tangents to E at P_i and P_{i+1} . These are the vertices of the outer tangent polygon P' .

For a chosen focus F , the antipedal side through X_i is

$$(X_i - F) \cdot (Y - X_i) = 0. \tag{1}$$

Let Q_i be the intersection of the sides at X_i and X_{i+1} . The vertex centroid is $C_0 = N^{-1} \sum_i Q_i$. This is the primed antipedal construction in [1, Section 3.5 and Table 5], not a polygon of pedal feet or focal inverses. The entry k_{407} reports constancy for even N and either focus but leaves the value unspecified. It also appears in the published version [2, Table 5].

Here “effective period” means the least orbit period, not a list length enlarged by repetition. Equivalently, the standard Jacobi parameterization uses a winding number τ coprime to N . For even N , τ is odd and

$$P_{i+N/2} = -P_i, \quad X_{i+N/2} = -X_i. \tag{2}$$

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Indeed the parameter step is 2Δ , with $\Delta = 2\tau K/N$; after $N/2$ steps the shift is $2\tau K$, and the Jacobi half-period identities change both coordinate signs. See [5, Theorem 2 and its proof]. This includes genuine even star periods with elliptic caustics. Repeating an odd primitive orbit does not create the pairing in (2).

Theorem 1. *Under these hypotheses, every consecutive antipedal intersection is finite, and C_0 is independent of the initial point in the fixed Poncelet family. It lies on the major axis. The conclusion holds for either focus and for either traversal direction.*

No simplicity or nonzero signed area of the antipedal is required for its vertex mean. The theorem does not concern its area centroid, the antipedal of the original orbit, hyperbolic caustics or degenerate diameter trajectories. The distinct four-periodic row $k_{406,b}$ concerns both vertex and area centroids; its elementary rectangle calculation is given in [6] and is not counted anew here.

2 The outer conic and focal pair formula

We use $F = (c, 0)$; replacing F by $-F$ gives the other case. Put $D = \text{diag}(a^2, b^2) = b^2 I + FF^\top$. The chord through P_i, P_{i+1} is the polar of X_i with respect to E , namely $X_i^\top D^{-1} p = 1$. Tangency of that chord to E_λ gives

$$X_i^\top (D^{-1} - \lambda D^{-2}) X_i = 1. \quad (3)$$

The outer vertices therefore lie on a fixed ellipse whose inverse matrix is $\alpha I + \beta FF^\top$, where

$$\begin{aligned} \alpha &= \frac{b^2 - \lambda}{b^4}, & \beta &= \frac{\lambda(a^2 + b^2) - a^2 b^2}{a^4 b^4}, \\ \gamma &= \alpha + \beta c^2 = \frac{a^2 - \lambda}{a^4}, & \kappa &= \alpha + \frac{\lambda}{a^2 b^2}. \end{aligned} \quad (4)$$

Its semiaxes are

$$L_x = \gamma^{-1/2} = \frac{a^2}{\sqrt{a^2 - \lambda}}, \quad L_y = \alpha^{-1/2} = \frac{b^2}{\sqrt{b^2 - \lambda}}. \quad (5)$$

Both exceed the corresponding semiaxes of E . Boundary support lines consequently meet this outer ellipse in two distinct points.

Take the shared boundary tangent containing consecutive outer vertices X, Y . Choose its unit normal n , directed away from the origin, and an orthogonal unit tangent t . Write

$$X = hn + st, \quad Y = hn + rt, \quad F = fn + gt, \quad h > 0.$$

The boundary support value satisfies

$$h^2 = b^2 + f^2, \quad h > |f|, \quad f^2 + g^2 = c^2. \quad (6)$$

Subtracting the two antipedal equations (1) and then substituting into either one gives their intersection

$$Q = \left[h - \frac{(s-g)(r-g)}{h-f} \right] n + (s+r-g)t. \quad (7)$$

The determinant of the two line normals is $(h-f)(r-s)$, up to orientation. It is nonzero by (6) and distinctness of the two outer vertices. Thus every Q is finite.

Substituting $hn + ut$ into (3), with $u = s, r$, yields

$$A = \alpha + \beta g^2 > 0, \quad s+r = -\frac{2h\beta fg}{A}, \quad sr = \frac{h^2(\alpha + \beta f^2) - 1}{A}. \quad (8)$$

The positivity of A follows from the positive eigenvalues α, γ ; it holds even when β is negative.

Let Q_{opp} be the antipedal vertex from the opposite edge $-X, -Y$, with respect to the same fixed focus. Adding its version of (7) gives

$$Q + Q_{\text{opp}} = \frac{2hg(s+r) - 2f(sr+g^2)}{b^2} n - 2gt.$$

For an explicit simplification, the normal numerator after (8) is $-2fB/A$, where

$$B = h^2\alpha + h^2\beta(f^2 + 2g^2) + \alpha g^2 + \beta g^4 - 1 = \alpha a^2 + \beta a^4 - \beta h^2 b^2 - 1.$$

Using (4),

$$A - \frac{B}{b^2} = \alpha \left(1 - \frac{a^2}{b^2}\right) + \beta a^2 - \frac{\beta a^4}{b^2} + \frac{1}{b^2} = \kappa.$$

Hence the exact opposite-pair formula is

$$Q + Q_{\text{opp}} = -2F + 2\kappa \frac{(F \cdot n)n}{\alpha + \beta(F \cdot t)^2}. \quad (9)$$

Sum over the actual pairs supplied by (2). Equivalently, take half of the full cyclic sum of (9). The centroid is

$$C_0 = -F + \frac{\kappa}{N} \sum_{i=1}^N \frac{(F \cdot n_i)n_i}{\alpha + \beta(F \cdot t_i)^2}. \quad (10)$$

3 Reduction to a circle trace

Apply the affine map $(x, y) \mapsto (x/L_x, y/L_y)$. The fixed outer ellipse becomes the unit circle, while E becomes the concentric ellipse with squared semiaxes

$$U = 1 - \frac{\lambda}{a^2}, \quad V = 1 - \frac{\lambda}{b^2}, \quad 0 < V < U < 1. \quad (11)$$

Write its circle vertices as $z_i = e^{i\theta_i}$. If $m_i = (\cos \psi_i, \sin \psi_i)$ is a unit normal to the transformed chord, it is proportional to $(L_x n_{i,x}, L_y n_{i,y})$. From $\alpha = L_y^{-2}$ and $\gamma = L_x^{-2}$ one obtains

$$\frac{(F \cdot n_i)n_i}{\alpha + \beta(F \cdot t_i)^2} = c(L_y^2 \cos^2 \psi_i, L_x L_y \cos \psi_i \sin \psi_i). \quad (12)$$

The chord normal has angle $(\theta_i + \theta_{i+1})/2$ modulo π , so $e^{2i\psi_i} = z_i z_{i+1}$, without a square-root choice. Put $S_N = \sum_i z_i z_{i+1}$. Equation (10) becomes

$$\begin{aligned} (C_0)_x &= c \left[-1 + \frac{\kappa L_y^2}{2} + \frac{\kappa L_y^2}{2N} \text{Re } S_N \right], \\ (C_0)_y &= \frac{\kappa c L_x L_y}{2N} \text{Im } S_N. \end{aligned} \quad (13)$$

We prove that this trace is constant and real. The use of an affine map here is only an explicit calculation; no projective invariance of Euclidean centroids is being assumed.

4 The bilinear trace lemma

The compactification and pole cancellation approach is established in [4] and, for circle/concentric-ellipse polygons, in [3, Section 4, proof of Theorem 5]. The explicit invariant in the latter theorem is an adjacent dot-product sum. We adapt its cancellation method to the symmetric complex-bilinear product needed here, and give the exact global calculation. This lemma is not presented as a new analytic technique.

Lemma 2. *Let $0 < V < U < 1$. In an infinite nonbacktracking family of closed N -vertex Poncelet polygons on the unit circle, tangent to the concentric ellipse with squared semiaxes U, V , the sum $S_N = \sum_i z_i z_{i+1}$ is constant and real.*

Proof. Set $d = U - V$ and $\eta = U + V - 1$. Then $0 < d < 1$ and $|\eta| < 1 - d$. The condition that the chord between $z = e^{i\theta}$ and $w = e^{i\phi}$ be tangent is

$$\cos^2 \frac{\theta - \phi}{2} = U \cos^2 \frac{\theta + \phi}{2} + V \sin^2 \frac{\theta + \phi}{2}.$$

In exponential coordinates this is

$$dz^2 w^2 - z^2 - w^2 + 2\eta zw + d = 0. \quad (14)$$

Let $\mathcal{C}$ be its closure in $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$.

We first check the compact curve, rather than discarding exceptional points of an affine formula. As a quadratic in w , the discriminant divided by four is

$$Q(z) = dz^4 + (\eta^2 - d^2 - 1)z^2 + d = d(z^4 - 2\chi z^2 + 1), \quad \chi = \frac{1 + d^2 - \eta^2}{2d} > 1.$$

It has four distinct nonzero roots and is not a square in $\mathbb{C}(z)$. The three quadratic coefficients have no common factor: at $dz^2 = 1$ the constant coefficient is $d - 1/d \neq 0$. Thus the affine equation is irreducible. Homogenization adds no vertical or horizontal component. On $dz^2 \neq 1$, the change $y = (dz^2 - 1)w + \eta z$ gives $y^2 = Q(z)$, which is smooth because Q is square-free. At $dz^2 = 1$, finite points are smooth when $\eta \neq 0$, since the w derivative is $2\eta z$; when $\eta = 0$ there are no finite points at that coordinate value.

At $z = \infty$, use $x = 1/z$. The equation is

$$(d - x^2)w^2 + 2\eta xw + dx^2 - 1 = 0. \quad (15)$$

There are two points $x = 0$, $w = \pm 1/\sqrt{d}$; the w derivative $2dw$ is nonzero. The chart at $w = \infty$ is symmetric. The point (∞, ∞) is absent because the leading bihomogeneous coefficient is $d \neq 0$. These checks show that $\mathcal{C}$ is smooth, irreducible, connected and compact.

The projection to the first coordinate has degree two. Its deck involution exchanges the two possible neighbors and extends across ramification points. Composing it with coordinate exchange gives a global automorphism T whose affine expression is

$$T(z, w) = \left(w, -z - \frac{2\eta w}{dw^2 - 1} \right). \quad (16)$$

This can also be seen by extending the corresponding function-field automorphism to the nonsingular projective curve. Apparent coefficient zeros in (16) represent the appropriate infinity points. If $\eta = 0$, $T(z, w) = (w, -z)$, so T^4 is the identity. On the physical unit-circle locus the two tangent choices are distinct; the nonbacktracking step selects the other root.

The prescribed family has infinitely many distinct oriented pairs (z_0, z_1) satisfying $T^N(z_0, z_1) = (z_0, z_1)$. The two meromorphic coordinate differences $z \circ T^N - z$ and $w \circ T^N - w$ consequently have infinitely many zeros on connected compact $\mathcal{C}$ and vanish identically. The coordinate maps separate points, so T^N is the identity on all of $\mathcal{C}$. This uses the full periodic family, not one isolated sample orbit.

Write $z_j = z \circ T^j$. Vieta's identity gives, meromorphically,

$$z_{j-1} + z_{j+1} = -\frac{2\eta z_j}{dz_j^2 - 1}. \quad (17)$$

If z_j has a pole at a point p of $\mathcal{C}$, both neighbors are finite, because (∞, ∞) is absent. Group its two incident edge products:

$$z_{j-1}z_j + z_jz_{j+1} = -\frac{2\eta z_j^2}{dz_j^2 - 1} = -\frac{2\eta}{d - z_j^{-2}}. \quad (18)$$

The last expression is holomorphic at p , since z_j^{-1} vanishes there and $d > 0$. Multiple pole indices at the same point are cyclically nonadjacent; their incident edge-label sets are disjoint. Thus every potentially singular summand occurs once in a regular group, and every ungrouped summand has two finite coordinates. This includes simultaneous poles and the parameter $\eta = 0$. Therefore S_N has no poles on $\mathcal{C}$ and is constant.

Complex conjugation preserves the real-coefficient curve and commutes with T . Hence the constant equals its complex conjugate and is real. $\square$

The affine image of the original prescribed billiard family meets all the lemma's hypotheses: its vertices lie on the circle, its consecutive edges are tangent to the concentric image of E , and the nonbacktracking family remains infinite and closed. Combining Lemma 2 with (13) proves Theorem 1. Reversing traversal reverses the same cyclic edge list and leaves the sum unchanged. Reflection in the minor axis handles the other focus. Translation and rotation restore a general center and major axis.

5 Values, exact checks and scope

Equation (13) specifies the fixed centroid in terms of the fixed caustic and the real trace. It need not equal O . For $a = 5$, $b = 3$ and $\lambda = 225/64$, a genuine six-periodic family has focal antipedal centroid $-F/3$ when $O = 0$. For $a^2 = 40$, $b^2 = 15$ and $\lambda = 24/7$, a genuine eight-periodic family has centroid $-7F/18$. These values can be computed from one member once the theorem proves family constancy. The eight-periodic affine circle vertices may be taken as

$$(1, 0), (4/5, 3/5), (0, 1), (-4/5, 3/5), (-1, 0), (-4/5, -3/5), (0, -1), (4/5, -3/5),$$

whose adjacent-product trace is $4/5$.

The accompanying standard-library checkers verify the actual defining antipedal intersections, reflection law, confocal tangency, positive flights and primitive closure using exact fractions and quadratic fields. They include five six-periodic starts in both directions, the eight-periodic example, and nonperiodic controls that reject stronger unsupported assertions. Finite tests supplement the general argument; they do not prove its constancy by sampling.

Arbitrary centrally symmetric polygons circumscribed about E do not suffice for the proof: their vertices need not lie on the fixed outer conic (3). Nor is the four-period value O a general answer. The mean is well-defined even if antipedal vertices coincide or the antipedal is self-intersected. We

make no assertion about its signed-area centroid. The circular and degenerate boundary cases of the compact-curve parameter range are not required here.

This resolves the constancy assertion in the literal row k_{407} with the effective-period convention stated above, not the entire multi-invariant source paper. A bounded primary-literature search did not locate an explicit proof of this focal-antipedal application; it does not certify absolute priority. The compact-curve method and periodic symmetry are credited prior work. This self-audited, unrefereed manuscript was prepared with AI assistance; no independent human review or proof-assistant formalization is claimed.

References

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