

Focal Antipedal Centroids of Rectangles Circumscribed about an Ellipse

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30 September 2026

Abstract

Let a rectangle be circumscribed about an ellipse with center O and a focus F . We show that its antipedal quadrilateral with respect to F has vertex centroid O and area centroid $O + (F - O)/3$, independently of the rectangle's orientation. The quadrilateral is always finite and strictly convex. Applied to the known outer tangent rectangles of simple four-periodic elliptic billiards, this calculation proves both constancy assertions in the experimental invariant $k_{406,b}$ of Reznik, Garcia and Koiller and identifies their values. The proof uses an orthonormal support frame and exact polygon first moments; it does not assert the corresponding general-even-period invariant.

1 The centroid assertion

For an ordered polygon $X = (X_1, \dots, X_n)$ and a point F , its antipedal polygon is formed by the consecutive intersections of the lines through X_i perpendicular to $X_i - F$:

$$(X_i - F) \cdot (Y - X_i) = 0. \quad (1)$$

This is the construction in [1, Section 3.5]. It differs from the polar of X or an antipedal of an inverted polygon. For vertices Q_i in cyclic order, define

$$\begin{aligned} S &= \frac{1}{2} \sum_{i=1}^n \det(Q_i, Q_{i+1}), & C_0 &= \frac{1}{n} \sum_{i=1}^n Q_i, \\ C_2 &= \frac{1}{6S} \sum_{i=1}^n (Q_i + Q_{i+1}) \det(Q_i, Q_{i+1}), & Q_{n+1} &= Q_1. \end{aligned} \quad (2)$$

Here S is signed shoelace area, C_0 is the vertex centroid and C_2 is the signed-area centroid, defined when $S \neq 0$. These are the area conventions in [1, Section 2].

The row $k_{406,b}$ of [1, Table 5], also present in the published version [2, Table 5], reports that both centroids of the focal antipedal of the outer tangent polygon are constant for $N = 4$. The row gives neither their values nor a proof. We prove the assertion through a slightly more general elementary result.

Theorem 1. *Let E be an ellipse with center O , semiaxes $a > b > 0$, and either focus F . Let R be any rectangle whose four sides are supporting tangents of E . The antipedal of R with respect to F is a finite strictly convex quadrilateral, and its centroids are*

$$C_0 = O, \quad C_2 = O + \frac{F - O}{3}.$$

In particular, both points are independent of the rectangle's orientation.

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2 Support coordinates and four line intersections

Translate O to the origin and choose positively oriented orthonormal axes along the side normals of R . Write $F = (f, g)$. The ellipse has the matrix equation

$$X^T D^{-1} X = 1, \quad D = b^2 I + F F^T, \quad a^2 = b^2 + f^2 + g^2.$$

Indeed D has eigenvalues a^2, b^2 , with its major axis along F . The support value in a unit direction e is $\sqrt{e^T D e}$. Since the ellipse is centrally symmetric, its two support lines in each coordinate direction are opposite. Thus R is centered on O and has ordered vertices

$$(h, k), \quad (-h, k), \quad (-h, -k), \quad (h, -k), \quad h^2 = b^2 + f^2, \quad k^2 = b^2 + g^2, \quad (3)$$

where $h, k > 0$. In particular $h > |f|$ and $k > |g|$.

The antipedal line at (sh, tk) , with $s, t \in \{-1, 1\}$, is

$$(sh - f)x + (tk - g)y = h^2 + k^2 - shf - tkg. \quad (4)$$

Subtracting the two equations with $t = 1$ gives $x = -f$. Substitution and $b^2 = k^2 - g^2$ then give

$$(k - g)y = h^2 + k^2 - f^2 - kg = b^2 + k^2 - kg = (k - g)(g + 2k).$$

The top intersection is therefore $(-f, g + 2k)$. The four consecutive intersections, in the order induced by (3), are

$$\begin{aligned} Q_1 &= (-f, g + 2k), & Q_2 &= (f - 2h, -g), \\ Q_3 &= (-f, g - 2k), & Q_4 &= (f + 2h, -g). \end{aligned} \quad (5)$$

All intersections are unique: the four consecutive normal determinants in (4) are

$$2h(k - g), \quad 2k(h + f), \quad 2h(k + g), \quad 2k(h - f),$$

which are strictly positive.

Relative to the point $-F$, the vertices in (5) are

$$(0, U), \quad (-B, 0), \quad (0, -V), \quad (A, 0), \quad (6)$$

where

$$A = 2(h + f), \quad B = 2(h - f), \quad U = 2(k + g), \quad V = 2(k - g).$$

All four quantities are positive. The quadrilateral joining these four axis intercepts in the displayed order is strictly convex. Its area is

$$S = \frac{(A + B)(U + V)}{2} = 8hk > 0. \quad (7)$$

This establishes finiteness, convexity and existence of the area centroid throughout the family, including orientations with $f = 0$ or $g = 0$.

3 Vertex and area first moments

The arithmetic mean of (6) is

$$\left(\frac{A-B}{4}, \frac{U-V}{4}\right) = (f, g).$$

Translating back by $-F$ gives $C_0 = 0$.

The four consecutive cross products for (6) are BU, BV, AV, AU . Its x first-moment numerator in (2) is

$$-B(BU) - B(BV) + A(AV) + A(AU) = (A^2 - B^2)(U + V).$$

Dividing by $6S = 3(A + B)(U + V)$ gives $(A - B)/3$. The y numerator is

$$U(BU) - V(BV) - V(AV) + U(AU) = (U^2 - V^2)(A + B),$$

which gives $(U - V)/3$. Hence the area centroid relative to $-F$ is

$$\left(\frac{A-B}{3}, \frac{U-V}{3}\right) = \left(\frac{4f}{3}, \frac{4g}{3}\right).$$

Translating back gives $C_2 = F/3$. Undoing the initial rigid change of coordinates proves Theorem 1. The other focus is handled by replacing (f, g) with $(-f, -g)$; the rectangle is unchanged. Reversing the cyclic order changes the signs of S and both signed first-moment numerators, but leaves both centroids unchanged.

4 Four-periodic elliptic billiards and scope

Corollary 2. *Let P range over a primitive four-periodic billiard family inside a fixed ellipse, with a strictly nested confocal elliptic caustic. Let P' be the outer polygon formed by consecutive boundary-tangent intersections. For either focus F , the vertex and signed-area centroids of the antipedal of P' are O and $O + (F - O)/3$.*

The known geometric ingredient is that P' is a rectangle: see [3, proof of Proposition 4.7], corresponding to Proposition 12 of arXiv:2011.06640v3. One can also recover it from the classical central symmetry recalled in [4, proof of Theorem 5.2]. The four orbit vertices are $P, Q, -P, -Q$. If u, v are the unit directions of the first two sides, the next two directions are $-u, -v$. The reflection law makes the outward normals at P, Q parallel to $-(u + v)$ and $u - v$. These are nonzero and perpendicular because $(u + v) \cdot (u - v) = |u|^2 - |v|^2 = 0$. Thus the four boundary tangents form a rectangle. Theorem 1 proves the corollary.

This proves both quantities in the literal $N = 4$ row $k_{406,b}$. The source's confocal pair comprises two ellipses; hyperbolic-caustic self-intersected four-periodics and degenerate diameter repeats are not included. We make no assertion here for the distinct general-even-period row k_{407} . The outer-rectangle and orthoptic geometry are prior art, not new claims of this note.

An accompanying standard-library rational checker constructs the intersections from their defining lines and checks both foci, boundary tangency, reflected directions, rigid motions and reversed cyclic order. It includes 228 noncircular support frames and 5,472 corresponding focus, rigid-motion and orientation cases; circle-limit tests are reported separately. These bounded tests supplement the displayed general proof, not replace it. A bounded review of current primary literature did not locate an explicit proof of both focal antipedal centroids; this is not a certificate of absolute priority or an exhaustive novelty determination. This self-audited, unrefereed note was prepared with AI assistance; no independent human review or proof-assistant formalization is claimed.

References

- [1] D. Reznik, R. Garcia and J. Koiller, *Eighty New Invariants in the Elliptic Billiard*, arXiv:2004.12497v11 (2020), Section 3.5 and Table 5. [arXiv source](#).
- [2] D. Reznik, R. Garcia and J. Koiller, *Fifty New Invariants of N-Periodics in the Elliptic Billiard*, Arnold Mathematical Journal **7** (2021), 341–355, Table 5. [doi:10.1007/s40598-021-00174-y](https://doi.org/10.1007/s40598-021-00174-y).
- [3] R. Garcia and D. Reznik, *Exploring Self-Intersected N-Periodics in the Elliptic Billiard*, Annales Mathematicae et Informaticae (2022), Proposition 4.7. [doi:10.33039/ami.2022.02.001](https://doi.org/10.33039/ami.2022.02.001). Earlier version: arXiv:2011.06640v3, Proposition 12.
- [4] H. Stachel, *On the Motion of Billiards in Ellipses*, European Journal of Mathematics **8** (2022), 1602–1622, proof of Theorem 5.2; preprint arXiv:2105.03624, Theorem 6. [doi:10.1007/s40879-021-00524-2](https://doi.org/10.1007/s40879-021-00524-2).