

Original Antipedal Centroids in Elliptic Billiards

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Abstract

For a periodic billiard between two confocal ellipses, take the antipedal of the original billiard polygon with respect to its center or either focus. We give an explicit formula for its unweighted vertex centroid when the billiard has even least period. The centroid at the center is the center; each focal centroid is a fixed scalar multiple of the corresponding focus, with scalar determined by the semiaxes, caustic parameter and mean side length. All antipedal intersections are finite. A paired-chord identity and perimeter stationarity prove the formula for every admissible coprime star winding and for repetitions of even primitive cycles. The circle is treated separately. An exact triangle and its negative show that an even number of listed vertices alone does not suffice. This gives a precisely scoped proof of the even-period interpretation of invariant k_{405} and distinguishes it from an unrestricted repeated-list assertion. No absolute-priority claim is made.

1 The original antipedal and the statement

Let the center be the origin and write

$$E : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad E_c : \frac{x^2}{a^2 - \lambda} + \frac{y^2}{b^2 - \lambda} = 1, \quad a > b > 0, \quad 0 < \lambda < b^2. \quad (1)$$

The foci are $F_{\pm} = (\pm c, 0)$, where $c^2 = a^2 - b^2$. Let $(P_i)_{i \in \mathbb{Z}/N\mathbb{Z}}$ be an ordered, closed reflected billiard on E tangent to E_c . Its *least period* is denoted by n ; thus $n \mid N$, and the listed polygon may traverse the same primitive cycle several times. Its positive perimeter, counted with multiplicity, is

$$L = \sum_{i=0}^{N-1} \ell_i, \quad \ell_i = |P_{i+1} - P_i|.$$

For a pole M , define the antipedal sidelines and vertices by

$$A_i(M) = \{X : (P_i - M) \cdot (X - P_i) = 0\}, \quad W_i(M) = A_i(M) \cap A_{i+1}(M). \quad (2)$$

We study the unweighted vertex centroid

$$C_0(M) = \frac{1}{N} \sum_{i=0}^{N-1} W_i(M).$$

The reference polygon is the *original* billiard polygon, not the outer polygon of boundary-tangent intersections. This is neither the pedal-foot construction nor a focal polar, and C_0 is not an area, perimeter or Steiner centroid.

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The antipedal is defined in [1, 2, Section 3.5]. Table 5, row k_{405} , asks for the stationarity of its vertex centroid at the center and foci for even periods. The row is retained in the final journal article [2, p. 348]. We explicitly require even *least* period. Section 5 explains why this distinction matters when repetitions are permitted.

Theorem 1. *Suppose the billiard in (1) has even least period n . All intersections in (2) are finite for $M = 0, F_+, F_-$, and*

$$C_0(0) = 0, \quad C_0(F_{\pm}) = \gamma F_{\pm}, \quad (3)$$

where

$$\gamma = -1 + \frac{a^2 b^2 - \lambda c^2}{(b^2 - \lambda) c^2} \left(1 - \frac{b^2 L}{2Nab\sqrt{\lambda}} \right). \quad (4)$$

These centroids are constant in each Poncelet family of fixed caustic and admissible coprime winding. The conclusion includes star orbits and any repeated traversal of an even primitive cycle.

For a center O in arbitrary coordinates, the focal formula becomes $C_0(F) = O + \gamma(F - O)$. Repetition leaves L/N unchanged. The theorem does not concern hyperbolic or degenerate caustics and does not compare centroids for different caustics. We prove it below without any area denominator.

2 Regularity and antipodal pairing

The center and foci lie strictly inside E_c : at a focus this follows from $a^2 - \lambda - c^2 = b^2 - \lambda > 0$. Every billiard sideline supports E_c , so it cannot contain one of these three poles. Consequently $P_i - M$ and $P_{i+1} - M$ are linearly independent. The two antipedal equations therefore have a unique finite solution.

We use classical central symmetry in its primitive-period form. Stachel's Jacobi parametrization [3, Theorem 2] is

$$P_i = \left(-a \operatorname{sn} \left(u + \frac{4i\tau K}{n} \right), b \operatorname{cn} \left(u + \frac{4i\tau K}{n} \right) \right), \quad \gcd(\tau, n) = 1. \quad (5)$$

Here K is the complete elliptic integral for the caustic modulus $m = c/\sqrt{a^2 - \lambda}$, with m denoting the modulus rather than its square. For even n , coprimality forces τ to be odd. A shift by $n/2$ adds $2\tau K$ and negates both Jacobi functions. Thus

$$P_{i+n/2} = -P_i. \quad (6)$$

This also follows from [4, Corollary 4.2(i)], with its odd-winding hypothesis. It applies to each admissible coprime star winding. We pair vertices in the primitive n -cycle first, then repeat the resulting sum; no parity condition on N/n is needed. At the center, negation sends each antipedal intersection to its opposite partner. Hence $C_0(0) = 0$.

3 The paired-chord identity

Let ν_i be the unit normal to the i th chord, with the sign chosen so that its supporting equation is $\nu_i \cdot X = h_i > 0$. Put

$$H_i = a^2 \nu_{ix}^2 + b^2 \nu_{iy}^2, \quad h_i^2 = H_i - \lambda, \quad \ell_i = \frac{2ab\sqrt{\lambda}}{H_i}. \quad (7)$$

The support equation follows from tangency to E_c . For the length formula, restrict the quadratic form of E to the supporting line. Its two roots along a unit tangent have separation $2ab\sqrt{H_i - h_i^2}/H_i$, which gives (7). This is the positive chord length, even for star orbits.

Fix either focus F and a chord. In an orthonormal frame (ν, t) , write $F = (f, g)$ and the two endpoints as $(h, s), (h, r)$. Set $B = b^2$. Then

$$a^2 = B + f^2 + g^2, \quad H = B + f^2, \quad h^2 = H - \lambda.$$

The ellipse equation in this frame is

$$(B + g^2)x^2 - 2fgxy + (B + f^2)y^2 = a^2B.$$

Consequently its endpoint roots satisfy

$$s + r = \frac{2hfg}{H}, \quad sr = \frac{(B + g^2)h^2 - a^2B}{H}. \quad (8)$$

Subtracting the two antipedal equations at these endpoints gives

$$\begin{aligned} W_\nu &= h - \frac{(s - g)(r - g)}{h - f}, & W_t &= s + r - g, \\ W'_\nu &= -h + \frac{(s + g)(r + g)}{h + f}, & W'_t &= -s - r - g, \end{aligned}$$

where W' is the intersection for the opposite chord, with endpoints $(-h, -s), (-h, -r)$. The denominators are nonzero since $h^2 - f^2 = B - \lambda > 0$. Summing and using (8),

$$\begin{aligned} W_\nu + W'_\nu &= \frac{2hg(s + r) - 2f(sr + g^2)}{B - \lambda} \\ &= \frac{2f((B - \lambda)g^2 + B\lambda)}{H(B - \lambda)}, & W_t + W'_t &= -2g. \end{aligned}$$

Define

$$\kappa = \frac{a^2b^2 - \lambda c^2}{b^2 - \lambda} = a^2 + \frac{B\lambda}{B - \lambda}.$$

The preceding calculation is precisely

$$W_i(F) + W_{i+n/2}(F) = -2F + 2\kappa \frac{(F \cdot \nu_i)\nu_i}{H_i}. \quad (9)$$

The tensor $\nu_i\nu_i^\top/H_i$ is unchanged by negating ν_i . Sum over the $n/2$ opposite pairs, then repeat if necessary:

$$C_0(F) = -F + \frac{\kappa}{N} \sum_{i=0}^{N-1} \frac{(F \cdot \nu_i)\nu_i}{H_i}. \quad (10)$$

4 Perimeter stationarity and the weighted trace

Write $u_i = (P_{i+1} - P_i)/\ell_i$. Under independent infinitesimal motions of the vertices tangent to E , the first variation is

$$\delta L = \sum_i (u_{i-1} - u_i) \cdot \delta P_i = 0. \quad (11)$$

Indeed, the reflection law makes $u_{i-1} - u_i$ normal to the boundary at P_i . This applies to self-crossing and repeated polygons because the positive length functional is a sum over labels and every side has nonzero length.

Take the particular variation $\delta P_i = TP_i$, where

$$T = \begin{pmatrix} 0 & -a/b \\ b/a & 0 \end{pmatrix}.$$

It is tangent to E since $x(TP)_x/a^2 + y(TP)_y/b^2 = 0$. Differentiating edge by edge also gives

$$0 = \sum_i \ell_i u_i \cdot T u_i = \left(\frac{b}{a} - \frac{a}{b} \right) \sum_i \ell_i u_{ix} u_{iy}.$$

The prefactor is nonzero. Since $u_i \perp \nu_i$, $u_{ix} u_{iy} = -\nu_{ix} \nu_{iy}$, independently of orientation. Equation (7) yields

$$\sum_i \frac{\nu_{ix} \nu_{iy}}{H_i} = 0. \quad (12)$$

Also $H_i = b^2 + c^2 \nu_{ix}^2$, so

$$\sum_i \frac{\nu_{ix}^2}{H_i} = \frac{N - b^2 \sum_i 1/H_i}{c^2} = \frac{N - b^2 L / (2ab\sqrt{\lambda})}{c^2}. \quad (13)$$

Insert $F = (\pm c, 0)$ and (12)–(13) into (10). This proves (4).

Along a smooth Poncelet family, (11) gives $dL/dt = 0$. The Jacobi phase in (5) runs over a connected circle for fixed admissible coprime winding. Hence L and (4) are constant throughout that family. Cyclic relabeling and reversal preserve L as well. This completes the proof of Theorem 1.

The circle is handled without dividing by c^2 . For a nondegenerate concentric caustic, a periodic circular billiard is a regular star, possibly repeatedly traversed. Rotational symmetry makes the vertex mean of its central antipedal the center. Both foci coincide there. The diameter orbit is excluded by the nondegenerate-caustic assumption.

5 Why even list length does not suffice

Consider the ellipse $x^2/21 + y^2/16 = 1$ and triangle

$$P = \left((\sqrt{21}, 0), (-3\sqrt{21}/5, 16/5), (-3\sqrt{21}/5, -16/5) \right). \quad (14)$$

Its two slanted sides have length 8, and its vertical side has length $32/5$. At the upper-left vertex the incoming and outgoing unit velocities and a boundary normal are

$$v_{\text{in}} = (-\sqrt{21}/5, 2/5), \quad v_{\text{out}} = (0, -1), \quad q = (-\sqrt{21}/35, 1/5).$$

Here $q \cdot q = 2/35$ and $v_{\text{in}} \cdot q = 1/5$, so $v_{\text{in}} - 2(v_{\text{in}} \cdot q)q/(q \cdot q) = v_{\text{out}}$. Reflection at the right vertex changes only the horizontal velocity; symmetry handles the third vertex. Thus this is a genuine billiard of least period three.

Its caustic parameter is $\lambda = 336/25$, giving semiaxes squared $189/25$ and $64/25$. The vertical edge is tangent at $(-3\sqrt{21}/5, 0)$. The upper slanted sideline is $2x + \sqrt{21}y = 2\sqrt{21}$, and its tangency equation is

$$4\frac{189}{25} + 21\frac{64}{25} = 84.$$

The other slanted side is its reflection. The contact points are inside their chords, at side parameters $2/5, 1/2, 3/5$ in the order of (14).

For the pole 0, directly intersecting the antipedal lines gives

$$W_0 = (\sqrt{21}, 19/2), \quad W_1 = (-89\sqrt{21}/63, 0), \quad W_2 = (\sqrt{21}, -19/2).$$

Therefore

$$C_0(0) = \left(\frac{37}{9\sqrt{21}}, 0 \right) \neq 0. \quad (15)$$

The negated triangle belongs to the same triangular Poncelet family and has the negative centroid. Traversing either triangle twice gives a six-entry list with the same mean, but still least period three. Thus stationarity fails for the unrestricted even-list interpretation. This is a repetition-convention boundary, not a counterexample to the genuine-even theorem proved above.

6 Attribution, scope and reproducibility

The source setting is a confocal pair of ellipses and its even-period discussion invokes antipodal symmetry. We make the least-period convention explicit instead of silently treating padded odd cycles as genuine even orbits. In particular, the qualified theorem here should not be read as a resolution of every interpretation of the abbreviated dataset record AMR-050-0023.

Central symmetry, the Jacobi parametrization and perimeter invariance are classical ingredients, not new claims of this note. Bialy and Tabachnikov [5, Theorem 2.2 and Corollary 3.2] give related variational and boundary-normal identities. Their angle refers to the *boundary* normal at a bounce; ν_i here is a *chord* normal and carries the weight $1/H_i$. Their unweighted boundary-normal identity is therefore not substituted for (12), which was proved directly. Their Theorem 4.1 is about pedals on boundary tangents, not the original-polygon antipedals of k_{405} . The specific calculation in this note is the paired-chord formula (9) and its centroid consequence. A bounded literature search does not certify priority for that formula.

The accompanying verification files contain exact symbolic reductions of the chord, pair and trace identities; exact controls for two four-periodic phases, three six-periodic ellipses, the circle, both foci, the center and Euclidean covariance; and a separate rational certificate for (14)–(15). The symbolic checker requires SymPy; the rational triangle checker uses Python’s standard library. No floating-point tolerance is used to decide these tests. They audit algebra and examples; the written argument, not finite sampling, proves the all-period theorem.

This unrefereed preprint was prepared with AI assistance in mathematical exploration, computation, literature search and drafting. The argument has been self-audited; no independent human review, proof-assistant verification or absolute-priority certification is claimed.

References

- [1] D. Reznik, R. Garcia and J. Koiller, *Eighty New Invariants of N-Periodics in the Elliptic Billiard*, arXiv:2004.12497v11 (2020), Section 3.5 and Table 5. <https://arxiv.org/abs/2004.12497v11>.
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- [3] H. Stachel, *On the Motion of Billiards in Ellipses*, arXiv:2105.03624v2 (2021), Theorem 2 and proof of Theorem 6. <https://arxiv.org/abs/2105.03624v2>.

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