

Five Focal Area Invariants of N -Periodics in the Elliptic Billiard

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Abstract

Reznik, Garcia and Koiller listed many quantities that are experimentally constant along the families of N -periodic billiard trajectories in an ellipse with a confocal elliptic caustic. We prove five of their rows on areas of polygons built from the trajectory and the two foci, for trajectories of least period N , every winding number and signed areas. For every N , the area of the image of the outer polygon under the inversion in a unit circle centred at a focus is a constant multiple $\kappa > 0$ of the area of the image of the trajectory, with the same κ for both foci (row $k_{806,a}$); $\kappa = \frac{1}{2}$ for $N = 4$, where the source prints 2. For odd N , the products over the two foci of these inversive areas of the outer polygon (row $k_{904,a}$), of the focal pedal areas of the outer polygon (row $k_{605,a}$) and of the areas of the focal dual polygons (row $k_{907,a}$) are constant; the dagger printed in row $k_{907,a}$ is not defined in the source, and we read the row without it, in two readings of “dual”. For $N \equiv 2 \pmod{4}$, the focal antipedal area divided by the focal pedal area is constant (row k_{404}). The focal pedal area is positive at every trajectory, so this ratio is always defined; its numerator can vanish identically, and for $N \equiv 0 \pmod{4}$ the ratio is constant only on the families where it does. The proofs use Stachel’s parametrisation by Jacobi elliptic functions, in which every area is an elliptic function of the phase; a list of its possible poles and two symmetries reduce each row to Liouville’s theorem or to a comparison of residues. The rows are due to Reznik, Garcia and Koiller. Rows $k_{904,a}$ and $k_{605,a}$ were first proved by Kriebel in two deposits of 15 October 2026; our proofs of them are independent, and we claim no novelty for these two rows. The case $N = 4$ of row $k_{806,a}$ is in an earlier note of the author, and the ratio of row k_{404} also follows from two earlier notes of the author together with the positivity of the focal pedal area, which Kriebel proved. To our knowledge, row $k_{806,a}$ for $N \neq 4$, row $k_{907,a}$ and the statement of row k_{404} as a ratio defined at every trajectory are new. This is an unrefereed note.

1 Introduction

1.1 The five rows

Let $\mathcal{E}$ be the ellipse $x^2/a^2 + y^2/b^2 = 1$ with $a > b > 0$, centre O and foci $f_1 = (-c, 0)$, $f_2 = (c, 0)$, $c = \sqrt{a^2 - b^2}$. Fix λ with $0 < \lambda < b^2$. The ellipse

$$\mathcal{C} : \frac{x^2}{a^2 - \lambda} + \frac{y^2}{b^2 - \lambda} = 1$$

is confocal with $\mathcal{E}$ and lies inside it, and O , f_1 , f_2 lie in its interior. A *billiard trajectory* in $\mathcal{E}$ is a sequence $(P_l)_{l \in \mathbb{Z}}$ of points of $\mathcal{E}$ with $P_{l+1} \neq P_l$ such that at every P_l the chords $P_{l-1}P_l$ and P_lP_{l+1} are mirror images of each other in the normal of $\mathcal{E}$ at P_l ; it has the *caustic* $\mathcal{C}$ if all its chords are tangent to $\mathcal{C}$. It is an *N -periodic* if $P_{l+N} = P_l$ for all l and $N \geq 3$ is the least such number. If one trajectory

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with caustic $\mathcal{C}$ is N -periodic, then all are, with the same winding number τ around O (Poncelet’s porism, Lemma 2.2); they form a one-parameter family, which we call the *family*.

For a closed polygon $W = (W_1, \dots, W_N)$, indices taken modulo N , the *signed area* is $S(W) = \frac{1}{2} \sum_i W_i \times W_{i+1}$, where $x \times y = x_1 y_2 - x_2 y_1$; reversing the order of the vertices changes its sign. For an N -periodic P we write $A = S(P)$. Reznik, Garcia and Koiller [RGK21] derive from P the following polygons and work with their signed areas throughout.

- The *outer polygon* P' has the vertices V_i , the intersections of the tangents to $\mathcal{E}$ at P_i and P_{i+1} ; $A' = S(P')$.
- The *pedal polygon* of a polygon R with respect to a point M consists of the feet of the perpendiculars from M to the side lines of R , in the order of the sides. Its area is A_M for $R = P$; for $M = f_j$ the source writes $\bar{A}_j$ for $R = P$ and $\bar{A}'_j$ for $R = P'$.
- The *antipedal polygon* of P with respect to M has the vertices Q_i^* , the intersection points of the line through P_i perpendicular to $P_i - M$ with the line through P_{i+1} perpendicular to $P_{i+1} - M$; A_M^* is its area.
- The *focus-inversive polygon* $P_j^\dagger$ consists of the images of the P_i under the inversion in the unit circle ω_j centred at f_j , and $A_j^\dagger$ is its area; $P_j'^\dagger$ and $A_j'^\dagger$ are the same for the vertices V_i of P' .
- The *polar polygon* with respect to f_j is, in the source’s words, the antipedal polygon of $P_j^\dagger$ with respect to f_j ; its vertices are the poles of the side lines of P with respect to ω_j (Remark 3.1). Its sides touch a circle; with O_p the centre of that circle, the *dual polygon* is the “inversion of the polar with respect to O_p ”, and $A_{j,\text{dual}}$ is its area ([RGK21, v11, §3.9, caption of the figure of polar and dual polygons, and table of symbols]). We consider two readings: the polar reciprocal of the polar polygon with respect to a circle centred at O_p (reading R; the vertices are the poles of the sides of the polar polygon), and the inversion of its vertices in such a circle (reading P), with area $\tilde{A}_{j,\text{dual}}$.

The source lists dozens of quantities that its experiments found to be constant along the families [RGK21, v11, Tables 2–10]. Table 1 shows the five rows treated here, all marked as not proven; the corresponding records of the UnsolvedMath collection [ula26] ask to prove that the quantity is constant as P varies through the family. All row codes are those of the arXiv version [RGK21, v11]. The journal version, titled “Fifty new invariants of N -periodics in the elliptic billiard”, lists inversive objects only up to row k_{804} and has no rows k_{9xx} , so rows $k_{806,a}$, $k_{904,a}$ and $k_{907,a}$ are in the arXiv version only.

record	row	quantity	N	in [RGK21, v11]
AMR-050-0022	k_{404}	A_M^*/A_M , $M = f_1$ or f_2	$\equiv 2 \pmod{4}$	Table 5
AMR-050-0033	$k_{605,a}$	$\bar{A}'_1 \bar{A}'_2$	odd	Table 7
AMR-050-0047	$k_{806,a}$	$A_j'^\dagger / A_j^\dagger$	all	Table 9
AMR-050-0063	$k_{904,a}$	$A_1'^\dagger A_2'^\dagger$	odd	Table 10
AMR-050-0066	$k_{907,a}$	$A_{1,\text{dual}}^\dagger A_{2,\text{dual}}$ (as printed)	odd	Table 10

Table 1: The five rows. For $N = 4$ the source also prints the value 2 of $A_j'^\dagger / A_j^\dagger$ (row $k_{806,b}$).

1.2 Results

Throughout, the *standing hypotheses* are: $a > b > 0$; $0 < \lambda < b^2$, so that the caustic $\mathcal{C}$ is a confocal ellipse; and the trajectories with caustic $\mathcal{C}$ are N -periodic of least period $N \geq 3$ and winding number τ (necessarily coprime to N , Lemma 2.2). All areas are signed, and every trajectory is taken in its counterclockwise order; reversing it changes the signs of all areas and does not affect the statements.

Theorem A (row $k_{806,a}$). *Assume the standing hypotheses, with any $N \geq 3$ and any winding number. There is a constant $\kappa > 0$, depending only on a , b and λ , such that every trajectory of the family satisfies*

$$A_1^\dagger = \kappa A_1^\dagger, \quad A_2^\dagger = \kappa A_2^\dagger, \quad A_1^\dagger > 0, \quad A_2^\dagger > 0.$$

Hence $A_j^\dagger / A_j^\dagger$ is constant along the family, with the same value for both foci. For $N = 4$, $\kappa = \frac{1}{2}$.

Theorem B (row $k_{904,a}$; first proved by Kriebel [Kri26d]). *Assume the standing hypotheses with N odd, of any winding number. Then $A_1^\dagger A_2^\dagger$ is a positive constant along the family; it equals $\kappa^2 A_1^\dagger A_2^\dagger$. (That $A_1^\dagger A_2^\dagger$ is a positive constant is Kriebel's theorem [Kri26d, Theorem 1]; the proof given here is independent.)*

Theorem C (row $k_{605,a}$; first proved by Kriebel [Kri26e]). *Assume the standing hypotheses with N odd, of any winding number. Then $\bar{A}_1 \bar{A}_2$ is a positive constant along the family. (The constancy is Kriebel's theorem [Kri26e, Theorem 1], and the positivity of $\bar{A}_j$ is in [Kri26b, Theorem 1]; the proof given here is independent.)*

Theorem D (row $k_{907,a}$). *Assume the standing hypotheses with N odd, of any winding number. The dagger on the first factor of the printed row is not defined in [RGK21], whose table of symbols defines $A_{j,\text{dual}}$ only; we read the row without it, as $A_{1,\text{dual}} A_{2,\text{dual}}$.*

- (a) *In reading R , $A_{1,\text{dual}} A_{2,\text{dual}}$ is a positive constant along the family; equivalently, the product over the two foci of $\sum_i \sin \angle P_i f_j P_{i+1}$ is constant.*
- (b) *In reading P , $\tilde{A}_{1,\text{dual}} \tilde{A}_{2,\text{dual}}$ is constant along the family.*

Theorem E (row k_{404}). *Assume the standing hypotheses. Then $\bar{A}_1 > 0$ and $\bar{A}_2 > 0$ for every trajectory, of any N and winding number. Let N be even.*

- (i) *There is a constant $\kappa^* \in \mathbb{R}$, depending only on a , b and λ , with $A_{f_1}^* = A_{f_2}^* = \kappa^* A$ for every trajectory of the family.*
- (ii) *If $N \equiv 2 \pmod{4}$, there is a constant $\kappa_p > 0$ with $\bar{A}_1 = \bar{A}_2 = \kappa_p A$. Hence, for $M = f_1$ and $M = f_2$, the ratio A_M^* / A_M of row k_{404} is defined for every trajectory of the family and equals κ^* / κ_p .*
- (iii) *If $N \equiv 0 \pmod{4}$, the ratio $A_{f_j}^* / \bar{A}_j$ is defined for every trajectory, and it is constant along the family if and only if $\kappa^* = 0$, in which case it vanishes identically. For $N = 4$, $\kappa^* = 2$.*

So row k_{404} holds without exceptional set, and its constant κ^* / κ_p can be 0. By Lemma 4 of [Fer26f], κ^* has the closed form

$$\kappa^* = \frac{a^2 b^2 (3b^2 - a^2) + \lambda (a^2 - b^2)^2}{2a^2 b^2 (b^2 - \lambda)}, \quad (1)$$

which vanishes if and only if $\lambda = a^2 b^2 (a^2 - 3b^2) / (a^2 - b^2)^2$. For $N = 6$ and $\tau = 1$ this happens exactly when $a = 2b$ [Fer26a]; the video list of Garcia and Reznik [GR21] contains an animation titled “ $N=6$ ”

zero-area antipedal @ $a/b=2$ ". On this family the ratio of row k_{404} is identically 0. Numerically, $\kappa^* = 0$ also for $N = 8$ and $N = 12$, $\tau = 1$, at $a/b = 1.8538473043818\dots$ and $a/b = 1.7799694162448\dots$ (Section 7; found in [Fer26b]); there the ratio of Theorem E(iii) vanishes identically, and on every other family with $N \equiv 0 \pmod{4}$, in particular every 4-periodic family, it is not constant. The reading of row $k_{907,a}$ matters: three literal readings of the dagger give products that are not constant (Remark 6.5).

1.3 What is new and what is not

The five rows, and the experimental evidence for them, are due to Reznik, Garcia and Koiller [RGK21]. The parametrisation of the family by Jacobi elliptic functions (Section 2) is Stachel's canonical parametrisation [Sta22, Theorem 2], and the central symmetry of N -periodics with even N is also Stachel's [Sta21, Corollary 4.2(i)]. The method, which complexifies the phase, locates the possible poles of an area function and removes them by symmetries before applying Liouville's theorem or comparing residues, is the one of the earlier notes on neighbouring rows [Fer26e, Fer26h, Fer26g, Kri26b, Kri26c] and of Kriebel's proofs of rows $k_{904,a}$ and $k_{605,a}$ [Kri26d, Kri26e]; poles of one area cancelled by zeros of the other already occur in Chavez-Caliz's proof of [CC21, Theorem 6], and elliptic functions with Liouville's theorem were used for related invariants by Roitman, Garcia and Reznik [RGR21]; we do not claim the method as new. What we add is the list of the possible poles of the areas of the outer, antipedal, inversive and dual polygons (Section 5).

Prior results on the five rows are as follows. For $N = 4$, the value $\kappa = \frac{1}{2}$ in Theorem A, with the constancy of the ratio, is the theorem of the author's earlier note [Fer26d], which corrects the printed value 2 of row $k_{806,b}$ and makes no claim for other N .

Rows $k_{904,a}$ and $k_{605,a}$ were first proved by Kriebel in two deposits of 15 October 2026 [Kri26d, Kri26e], made before the present note was written; we claim no novelty for these two rows. For odd least period N , an elliptic caustic and every winding number, stars included, Kriebel proves that $A_1^\dagger A_2^\dagger$ is a positive constant [Kri26d, Theorem 1] and that $\bar{A}_1 \bar{A}_2$ is constant [Kri26e, Theorem 1]; his corollaries add repeated traversals and the circular billiard. Both proofs locate the possible poles of the area function on a torus of reduced periods, from explicit formulas for the edge terms of the inverted outer polygon, resp. for the focal feet, and conclude with complementary zeros obtained from a reflection and an imaginary antiperiod. The deposit on $k_{605,a}$ also derives that row from row $k_{903,a}$ [Fer26h], through the proportionality of $\bar{A}_j'$ and $A_{3-j}^\dagger$, and describes a second route through a Landen-type identity of Khare and Sukhatme [Kri26e, Section 5]. Theorems B and C are independent proofs of the same statements, by an argument of the same kind organised differently (one list of possible poles for all area functions, Section 5): we found these deposits only when writing the note and do not use them; they were read in full by the last verification run (Section 7).

For row k_{404} , part (i) of Theorem E is Lemma 4 of [Fer26f], with the constant (1), and for winding number 1 also follows from [Fer26a, Corollary 3]; part (ii) is the case $M = f_j$ of Theorem B of [Fer26g], which proves $A_M = \kappa(M)A$ for $N \equiv 2 \pmod{4}$; and the positivity of the focal pedal areas of P and of P' , every term of their shoelace sums being positive, is proved for every winding number by Kriebel [Kri26b, Theorem 1 and Section 4] (for $\bar{A}_j'$ by the argument of Proposition 6.3); for the terms of $A_j^\dagger$ it is (12) of [Kri26d]. So the ratio statement of row k_{404} follows from these notes; we give a self-contained proof. The families with $\kappa^* = 0$ were found numerically in [Fer26b], and an animation of the case $N = 6$, $a = 2b$ is listed by Garcia and Reznik [GR21]. The closest other results are row $k_{903,a}$, the constancy of $A_1^\dagger A_2^\dagger$ for odd N [Fer26h], proved for $N = 3$ in closed form by Reznik, Garcia and Helman [RGH22]; row k_{805} , the constancy of $A/A_j^\dagger$ for $N \equiv 2 \pmod{4}$ [Fer26e]; and Kriebel's proof that $\bar{A}_j'$ is a constant multiple of $\bar{A}_j$, with the same constant for both foci (row k_{606} , [Kri26b]). With k_{606} , Theorem C is equivalent to the constancy of $\bar{A}_1 \bar{A}_2$ for odd N (Remark 6.6), and with

$k_{903,a}$, Theorem B follows from Theorem A; in the same way the list of poles makes $\bar{A}'_1$ a constant multiple of $A_2^\dagger$ and the dual and polar areas constant multiples of $A_1^\dagger$, so that Theorems C and D also follow from $k_{903,a}$ (Remark 6.6). Other deposits on the same table treat other rows (Section 7, “Scope and priority”).

To our knowledge new are, for the rows $k_{806,a}$, $k_{907,a}$ and k_{404} only: Theorem A for $N \neq 4$, Theorem D, the statement of row k_{404} as a ratio defined at every trajectory, and Theorem E(iii) for $N \geq 8$, which also follows at once from Theorem 1 of [Fer26f] and [CC21, Theorem 6] (Remark 6.7). The searches on which this is based are described in the paragraph “Scope and priority” at the end of Section 7; we make no priority claim.

1.4 Conventions

Least period. The source does not define “ N -periodic” formally; in its experiments every N has its own caustic, which is the reading of least period. A trajectory of least period n listed r times is not an rn -periodic in this sense; its areas are r times those of the n -periodic, so its products and ratios are constant exactly when those of the n -periodic are, and for instance the 3-periodic listed twice does not satisfy row k_{404} (Section 7).

Signed areas, foci, caustic. The source’s areas are signed, and it does not say which focus is f_1 ; all statements are symmetric in the two foci. The caustic is a confocal ellipse, as in the source’s experiments (“confocal ellipse pair”); hyperbolic caustics are not treated.

Readings. In row k_{404} , A_M is the area of the pedal polygon of P (the source’s text once writes A_m for the antipedal area by a slip; its table and its table of symbols write A_m^*). The radius of the circle at O_p in the dual construction, and the radius 1 of ω_j , are immaterial for the ratio rows and multiply the products by constants.

2 The parametrisation

Throughout, $\alpha = a^2$, $\beta = b^2$, $0 < \lambda < \beta$, and

$$k^2 = \frac{\alpha - \beta}{\alpha - \lambda} \in (0, 1), \quad k'^2 = 1 - k^2 = \frac{\beta - \lambda}{\alpha - \lambda}, \quad K = K(k), \quad K' = K(k').$$

The functions sn , cn , dn have modulus k . For $x, y \in \mathbb{C}^2$ we write $x \cdot y = x_1 y_1 + x_2 y_2$ (bilinear, also for complex vectors), $x \times y = x_1 y_2 - x_2 y_1$ and $x^\perp = (x_2, -x_1)$, so that $x \cdot y^\perp = x \times y$ and $x^\perp \times y^\perp = x \times y$. We put $\Lambda_0 = 2K\mathbb{Z} + 2iK'\mathbb{Z}$, $\Lambda_1 = 4K\mathbb{Z} + 2iK'\mathbb{Z}$, $\Lambda = 4K\mathbb{Z} + 4iK'\mathbb{Z}$, and we use the following standard facts [DLM26].

- (J1) sn , cn , dn are meromorphic on $\mathbb{C}$; their poles are the points of $\Pi := iK' + \Lambda_0$, all simple; dn has simple zeros exactly at the points of $\Pi' := K + iK' + \Lambda_0$; all three have the periods Λ , and sn has the periods Λ_1 . In a period parallelogram of Λ_1 , sn has two simple poles, so it takes every value exactly twice, counted with multiplicity; and $\operatorname{sn}(2K - w) = \operatorname{sn} w$ ([DLM26, §§22.2, 22.4(i), Tables 22.4.1, 22.4.2]).
- (J2) For all w : $\operatorname{sn}(w + 2K) = -\operatorname{sn} w$, $\operatorname{cn}(w + 2K) = -\operatorname{cn} w$, $\operatorname{dn}(w + 2K) = \operatorname{dn} w$, $\operatorname{sn}(w + 2iK') = \operatorname{sn} w$, $\operatorname{cn}(w + 2iK') = -\operatorname{cn} w$, $\operatorname{dn}(w + 2iK') = -\operatorname{dn} w$, and

$$\begin{aligned} \operatorname{sn}(w + iK') &= \frac{1}{k \operatorname{sn} w}, & \operatorname{cn}(w + iK') &= \frac{-i \operatorname{dn} w}{k \operatorname{sn} w}, & \operatorname{dn}(w + iK') &= \frac{-i \operatorname{cn} w}{\operatorname{sn} w}, \\ \operatorname{sn}(w + K + iK') &= \frac{\operatorname{dn} w}{k \operatorname{cn} w}, & \operatorname{cn}(w + K + iK') &= \frac{-ik'}{k \operatorname{cn} w}, & \operatorname{dn}(w + K + iK') &= \frac{ik' \operatorname{sn} w}{\operatorname{cn} w} \end{aligned}$$

([DLM26, Table 22.4.3]). In particular sn and cn are odd about every point of Π and even about every point of Π' , and $\text{sn} = 1/k$ on $\Pi'_+ := K + iK' + \Lambda_1$, $\text{sn} = -1/k$ on $\Pi'_- := \Pi'_+ + 2K$.

$$(J3) \quad \text{sn}^2 + \text{cn}^2 = 1, \quad \text{dn}^2 + k^2 \text{sn}^2 = 1, \quad \text{and with } D = 1 - k^2 \text{sn}^2 t \text{sn}^2 w$$

$$\text{sn}(t \pm w) = \frac{\text{sn } t \text{ cn } w \text{ dn } w \pm \text{sn } w \text{ cn } t \text{ dn } t}{D}, \quad \text{cn}(t \pm w) = \frac{\text{cn } t \text{ cn } w \mp \text{sn } t \text{ sn } w \text{ dn } t \text{ dn } w}{D}$$

([DLM26, §§22.6(i), 22.8(i)]); with $t = w$ these give $\text{sn } 2w = 2 \text{sn } w \text{ cn } w \text{ dn } w / E$, $\text{cn } 2w = (\text{cn}^2 w - \text{sn}^2 w \text{ dn}^2 w) / E$ and $\text{dn } 2w = (\text{dn}^2 w - k^2 \text{sn}^2 w \text{ cn}^2 w) / E$ with $E = 1 - k^2 \text{sn}^4 w$, hence $(1 - \text{dn } 2w) / \text{sn } 2w = k^2 \text{sn } w \text{ cn } w / \text{dn } w$.

(J4) For real u , $\text{sn } u = \sin \text{am } u$ and $\text{cn } u = \cos \text{am } u$, where am is real analytic, $\text{am}' = \text{dn} > 0$ on $\mathbb{R}$, and $\text{am}(u + 2K) = \text{am } u + \pi$ ([DLM26, §22.16(i)]); and $\text{sn}^2(K/2) = 1/(1 + k')$ ([DLM26, Table 22.5.2]).

Let $\Delta \in (0, K)$ be the number with $\text{sn}^2 \Delta = \lambda/\beta$, put $s_0 = \text{sn } \Delta$, $c_0 = \text{cn } \Delta$, $d_0 = \text{dn } \Delta$, all positive, and $h := 2\Delta \in (0, 2K)$. Then

$$c_0^2 = \frac{\beta - \lambda}{\beta}, \quad d_0^2 = \frac{\alpha(\beta - \lambda)}{\beta(\alpha - \lambda)}, \quad \frac{d_0^2}{\alpha} - \frac{1}{\beta} = -\frac{k^2}{\beta}, \quad k = \frac{c d_0}{a c_0}. \quad (2)$$

Define, for $u, t, v, w \in \mathbb{C}$,

$$\begin{aligned} P(u) &:= (-a \text{sn } u, b \text{cn } u), \quad m_w(t) := \left(-\frac{\text{dn } w \text{sn } t}{a \text{cn } w}, \frac{\text{cn } t}{b \text{cn } w} \right), \quad m(t) := m_\Delta(t), \\ n(v) &:= \left(-\frac{\text{sn } v}{a}, \frac{\text{cn } v}{b} \right), \quad V(t) := \Gamma P(t), \quad \Gamma := \text{diag}(d_0/c_0, 1/c_0). \end{aligned}$$

For every $u \in \mathbb{C}$, $P(u)$ satisfies $x^2/\alpha + y^2/\beta = 1$, and $n(u) \cdot P(u) = 1$; for real u , $n(u) \cdot X = 1$ is the tangent of $\mathcal{E}$ at $P(u)$. A line $px + qy = 1$ touches $\mathcal{C}$ if and only if $(\alpha - \lambda)p^2 + (\beta - \lambda)q^2 = 1$.

Lemma 2.1 (chords). *For all $t, w \in \mathbb{C}$ with $\text{cn } w \neq 0$, $m_w(t) \cdot P(t - w) = m_w(t) \cdot P(t + w) = 1$. Moreover $n(t \pm \Delta) \cdot V(t) = 1$, and the line $L(t) := \{X : m(t) \cdot X = 1\}$ satisfies the tangency condition of $\mathcal{C}$ for every t . For real t , $L(t)$ is the line through the two distinct points $P(t \pm \Delta)$, and $V(t)$ is the intersection point of the tangents of $\mathcal{E}$ at $P(t - \Delta)$ and $P(t + \Delta)$.*

Proof. Write $s = \text{sn } t$, $\gamma = \text{cn } t$, $e = \text{dn } t$ and $\sigma = \text{sn } w$, $\gamma_0 = \text{cn } w$, $\delta = \text{dn } w$. By (J3), $\text{cn } w m_w(t) \cdot P(t \pm w) = \delta s \text{sn}(t \pm w) + \gamma \text{cn}(t \pm w)$ equals

$$\frac{\delta s(s\gamma_0\delta \pm \sigma\gamma e) + \gamma(\gamma\gamma_0 \mp s\sigma e\delta)}{D} = \frac{\gamma_0(s^2\delta^2 + \gamma^2)}{D} = \gamma_0,$$

since $s^2\delta^2 + \gamma^2 = 1 - k^2s^2\sigma^2 = D$. The same computation with $w = \Delta$ gives $n(t \pm \Delta) \cdot V(t) = (d_0s \text{sn}(t \pm \Delta) + \gamma \text{cn}(t \pm \Delta))/c_0 = 1$. By (2), $(\alpha - \lambda)d_0^2s^2/(\alpha c_0^2) + (\beta - \lambda)\gamma^2/(\beta c_0^2) = s^2 + \gamma^2 = 1$. For real t , $P(t + \Delta) = P(t - \Delta)$ would force $2\Delta \in 4K\mathbb{Z}$ by (J4), and the tangents at $P(t \pm \Delta)$ are parallel only if $P(t + \Delta) = -P(t - \Delta)$, which would force $2\Delta \in 2K + 4K\mathbb{Z}$. $\square$

Lemma 2.2 (trajectories). (a) *For real u, v one has $P(u) = P(v)$ if and only if $u - v \in 4K\mathbb{Z}$. For real u , the lines $L(u - \Delta)$ and $L(u + \Delta)$ are the two tangents from $P(u)$ to $\mathcal{C}$, and they are mirror images of each other in the normal of $\mathcal{E}$ at $P(u)$.*

(b) *For real u the points $P_l := P(u + lh)$, $l \in \mathbb{Z}$, form a billiard trajectory with caustic $\mathcal{C}$; the chord $P_l P_{l+1}$ lies on $L(u + lh + \Delta)$.*

- (c) This trajectory is N -periodic if and only if $Nh = 4\tau K$ for an integer τ coprime to N . Then $0 < \tau < N/2$, and τ is the winding number of the trajectory around O .
- (d) Every billiard trajectory with caustic $\mathcal{C}$ is, after a reversal of its order if necessary, of the form in (b).

Proof. (a) The first statement follows from (J4). By Lemma 2.1, $L(u \pm \Delta)$ pass through $P(u)$ and touch $\mathcal{C}$; they are distinct, since otherwise one line would contain the three distinct points $P(u - h)$, $P(u)$, $P(u + h)$ of $\mathcal{E}$; and as $P(u)$ lies outside $\mathcal{C}$ they are the two tangents from $P(u)$. Put $T = (a \operatorname{cn} u, b \operatorname{sn} u)$, a tangent vector of $\mathcal{E}$ at $P(u)$, orthogonal to $n = n(u)$. Two lines through $P(u)$ with normal vectors μ_1, μ_2 are mirror images in the normal line of $\mathcal{E}$ if and only if $(\mu_1 \cdot T)(\mu_2 \cdot n) + (\mu_2 \cdot T)(\mu_1 \cdot n) = 0$. With $s = \operatorname{sn} u$, $\gamma = \operatorname{cn} u$, $e = \operatorname{dn} u$ and $D = 1 - k^2 s^2 s_0^2 > 0$, (J3) and (2) give

$$c_0 D m(u \pm \Delta) \cdot T = s_0 (X \mp Y), \quad c_0 D m(u \pm \Delta) \cdot n = U \pm W,$$

where $X = k^2 s_0 c_0 s \gamma$, $Y = d_0 e$, $U = c_0 e^2 / \beta$ and $W = s_0 d_0 s \gamma e (\alpha^{-1} - \beta^{-1})$; here we used $d_0^2 s^2 / \alpha + \gamma^2 / \beta = e^2 / \beta$, which is (2) with (J3). So for $\mu_{1,2} = m(u \pm \Delta)$ the criterion reads $2s_0(XU + YW) = 0$, and indeed

$$XU + YW = s_0 s \gamma e^2 \left(\frac{k^2 c_0^2}{\beta} - \frac{d_0^2 (\alpha - \beta)}{\alpha \beta} \right) = s_0 s \gamma e^2 \frac{(\alpha - \beta)(\beta - \lambda)}{\beta^2 (\alpha - \lambda)} (1 - 1) = 0.$$

(b) By Lemma 2.1 with $t = u + lh + \Delta$, the chord $P_l P_{l+1}$ lies on $L(u + lh + \Delta)$ and touches $\mathcal{C}$, and $P_{l+1} \neq P_l$ as $0 < h < 4K$; at P_l the two chords lie on $L(u + lh \mp \Delta)$, which are mirror images in the normal by (a). (c) By (a), $P_{l+N} = P_l$ for all l if and only if $Nh \in 4K\mathbb{Z}$; write $Nh = 4\tau K$. If $g = \gcd(\tau, N) > 1$, then N/g is also a period; if $g = 1$ and N' is a period, then $N'h = 4K\tau N'/N \in 4K\mathbb{Z}$ forces $N \mid N'$. Moreover $\tau/N = \Delta/(2K) \in (0, \frac{1}{2})$, and by (J4) the eccentric angle $\operatorname{am} u + \frac{\pi}{2}$ of $P(u)$ increases by $2\pi\tau$ along one period. (d) Let (P_l) have caustic $\mathcal{C}$ and choose a real u with $P_0 = P(u)$. The chord $P_0 P_1$ is one of the two tangents from P_0 , so by (a) $P_1 = P(u \pm h)$; after a reversal $P_1 = P(u + h)$. If $P_{l-1} = P(u + (l-1)h)$ and $P_l = P(u + lh)$, the reflection law and (a) show that $P_l P_{l+1}$ lies on $L(u + lh + \Delta)$, whose second point on $\mathcal{E}$ is $P(u + (l+1)h)$; induction in both directions. $\square$

Parts (b) and (c) are Stachel's description of the family [Sta22, Theorem 2]. From now on N and τ are as in (c), and by (d) the family is $\{(P(u + lh))_l : u \in \mathbb{R}\}$, each trajectory in its counterclockwise order (Lemma 2.5); a quantity is constant along the family if and only if it is constant as a function of the real phase u . If N is even, τ is odd and $P(u + (l + N/2)h) = P(u + lh + 2\tau K) = -P(u + lh)$: even periodic trajectories are centrally symmetric [Sta21, Corollary 4.2(i)].

Lemma 2.3 (identities). *For all $u, v, t \in \mathbb{C}$:*

- (i) $(P(u) - f_1) \cdot (P(u) - f_1) = (a - c \operatorname{sn} u)^2$ and $(P(u) - f_2) \cdot (P(u) - f_2) = (a + c \operatorname{sn} u)^2$; for real u , $a \mp c \operatorname{sn} u > 0$ are the distances from $P(u)$ to f_1, f_2 .
- (ii) $m(t) \cdot m(t) = \operatorname{dn}^2 t / (\beta c_0^2) = (1 - k \operatorname{sn} t)(1 + k \operatorname{sn} t) / (\beta c_0^2)$, and $m(t) \cdot f_1 = k \operatorname{sn} t = -m(t) \cdot f_2$.
- (iii) $n(v) \cdot n(v) = (a - c \operatorname{sn} v)(a + c \operatorname{sn} v) / (\alpha \beta)$, and $n(v) \cdot f_1 = (c/a) \operatorname{sn} v = -n(v) \cdot f_2$.
- (iv) $P(t - \Delta) \times P(t + \Delta) = 2ab s_0 c_0 \operatorname{dn} t / (1 - k^2 s_0^2 \operatorname{sn}^2 t)$, and $P(v) \times P'(v) = ab \operatorname{dn} v$.
- (v) With $A' = ad_0/c_0$ and $B' = b/c_0$, $(V(t) - f_1) \cdot (V(t) - f_1) = (A'^2 - B'^2) \operatorname{sn}^2 t - 2cA' \operatorname{sn} t + c^2 + B'^2$ and $A'^2 - B'^2 = (\alpha - \beta)(\alpha \beta - \lambda(\alpha + \beta)) / ((\alpha - \lambda)(\beta - \lambda))$. If $\lambda \neq \alpha \beta / (\alpha + \beta)$, this quadratic polynomial in $\operatorname{sn} t$ has the two roots $1/k$ and

$$r_1 = \frac{\alpha \beta - c^2 \lambda}{k(\alpha \beta - \lambda(\alpha + \beta))} = \frac{\operatorname{dn} h}{k \operatorname{cn} h} = \operatorname{sn}(K + iK' \pm h) \neq \frac{1}{k};$$

if $\lambda = \alpha\beta/(\alpha + \beta)$ it is linear with the only root $1/k$.

Proof. (i) $(c - a \operatorname{sn} u)^2 + \beta \operatorname{cn}^2 u = (\alpha - \beta) \operatorname{sn}^2 u - 2ac \operatorname{sn} u + \alpha = (a - c \operatorname{sn} u)^2$; for real u , $|\operatorname{sn} u| \leq 1 < a/c$. (ii) and (iii) follow from (2) and (J3). (iv) $P(v) \times P(w) = ab(\operatorname{sn} w \operatorname{cn} v - \operatorname{sn} v \operatorname{cn} w)$; with $v, w = t \mp \Delta$ and (J3), $\operatorname{sn}(t + \Delta) \operatorname{cn}(t - \Delta) - \operatorname{sn}(t - \Delta) \operatorname{cn}(t + \Delta) = 2s_0 c_0 \operatorname{dn} t (\operatorname{sn}^2 t d_0^2 + \operatorname{cn}^2 t)/D^2 = 2s_0 c_0 \operatorname{dn} t/D$ with $D = 1 - k^2 s_0^2 \operatorname{sn}^2 t$; and $P' = (-a \operatorname{cn} \operatorname{dn}, -b \operatorname{sn} \operatorname{dn})$. (v) The expansion uses $\operatorname{cn}^2 = 1 - \operatorname{sn}^2$, and $A'^2 = \alpha^2/(\alpha - \lambda)$, $B'^2 = \beta^2/(\beta - \lambda)$ by (2). Since $A'/k = \alpha/c$ and $B'^2(1/k^2 - 1) = B'^2(\beta - \lambda)/c^2 = \beta^2/c^2$, the value at $\operatorname{sn} t = 1/k$ is $(\alpha^2 - \beta^2)/c^2 - 2\alpha + c^2 = 0$. The product of the roots is $(c^2 + B'^2)/(A'^2 - B'^2)$ and $c^2 + B'^2 = (\alpha\beta - c^2\lambda)/(\beta - \lambda)$, which gives the first expression for r_1 . By (J3) and (2), $\operatorname{dn} h/\operatorname{cn} h = (d_0^2 - k^2 s_0^2 c_0^2)/(c_0^2 - s_0^2 d_0^2) = (\alpha\beta - c^2\lambda)/(\alpha\beta - \lambda(\alpha + \beta))$, and $\operatorname{dn} h/(k \operatorname{cn} h) = \operatorname{sn}(K + iK' \pm h)$ by (J2). Finally $r_1 = 1/k$ would mean $\operatorname{dn} h = \operatorname{cn} h$, i.e. $\operatorname{sn} h = 0$, which is false for $0 < h < 2K$. $\square$

Lemma 2.4 (parities). *Let ϕ be one of P , m , n , V , and $R := \operatorname{diag}(1, -1)$.*

- (a) *For $z \in \Pi$ and all w : $\phi(z + w) = -\phi(z - w)$ and $\operatorname{sn}(z + w) = -\operatorname{sn}(z - w)$.*
- (b) *For $Z \in \Pi'$ and all w : $\phi(Z + w) = \phi(Z - w)$ and $\operatorname{sn}(Z + w) = \operatorname{sn}(Z - w)$.*
- (c) *$\phi(w + 2K) = -\phi(w)$ and $\phi(w + 2iK') = R\phi(w)$.*

Moreover ϕ is holomorphic outside Π and has simple poles at the points of Π .

Proof. Each ϕ has the form $(\gamma_1 \operatorname{sn} w, \gamma_2 \operatorname{cn} w)$ with nonzero constants; apply (J1) and (J2). $\square$

Lemma 2.5 (orientation). *Let u be real, $P_l = P(u + lh)$, $V_l = V(u + lh + \Delta)$, and let θ_l be the eccentric angle of P_l , so that $0 < \theta_{l+1} - \theta_l < \pi$.*

- (a) *$P_l \times P_{l+1} > 0$; the interior of $\mathcal{C}$, and in particular O , f_1 and f_2 , lies to the left of the directed chord $P_l \rightarrow P_{l+1}$, so $(P_l - X) \times (P_{l+1} - X) > 0$ for $X \in \{O, f_1, f_2\}$.*
- (b) *$(P_{l+1} - P_l) \times (P_{l+2} - P_{l+1}) > 0$.*
- (c) *$V_l - V_{l-1}$ is a positive multiple of $P'(u + lh)$, and $(V_{l-1} - X) \times (V_l - X) > 0$ for every point X inside $\mathcal{E}$.*
- (d) *$n(u + lh) \times n(u + (l + 1)h) > 0$.*

Proof. By (J4), $P(u) = (a \cos \theta, b \sin \theta)$ with $\theta = \operatorname{am} u + \frac{\pi}{2}$, and $P(v) \times P(w) = ab \sin(\operatorname{am} w - \operatorname{am} v)$; as $0 < h < 2K$, $0 < \theta_{l+1} - \theta_l < \pi$. (a) The line of the chord touches $\mathcal{C}$ and therefore does not meet the interior of $\mathcal{C}$, which contains O ; since $P_l \times P_{l+1} > 0$, O and hence the whole interior lie to the left. The foci lie inside $\mathcal{C}$ because $c^2/(\alpha - \lambda) = k^2 < 1$. (b) With $\phi_1 = \theta_{l+1} - \theta_l$ and $\phi_2 = \theta_{l+2} - \theta_{l+1}$ in $(0, \pi)$, the left-hand side is $ab[\sin \phi_1 + \sin \phi_2 - \sin(\phi_1 + \phi_2)] = 4ab \sin \frac{\phi_1}{2} \sin \frac{\phi_2}{2} \sin \frac{\phi_1 + \phi_2}{2} > 0$. (c) $P'(v) = \operatorname{dn} v (-a \sin \theta, b \cos \theta)$ points in the counterclockwise direction. After the map $(x, y) \mapsto (x/a, y/b)$, the tangents at two points of the unit circle with angles $\theta < \theta' < \theta + \pi$ meet at the point with angle $(\theta + \theta')/2$ and radius $1/\cos \frac{\theta' - \theta}{2}$, which lies ahead of the first point and behind the second one on the respective tangents. So $V_{l-1} = P_l - s'P'(u + lh)$ and $V_l = P_l + sP'(u + lh)$ with $s, s' > 0$. A point inside $\mathcal{E}$ lies to the left of every counterclockwise tangent. (d) $n(v)$ is a positive multiple of $(b \cos \theta, a \sin \theta)$, whose angle is an increasing function of θ that increases by π when θ does. $\square$

Lemma 2.6 ($N = 4$). *The family is 4-periodic if and only if $\lambda = \alpha\beta/(\alpha + \beta)$. Then $\Delta = K/2$, $h = K$, and the member with phase $u = 0$ is the rhombus with the vertices $(0, b)$, $(-a, 0)$, $(0, -b)$, $(a, 0)$; its outer polygon is the rectangle with the vertices $(-a, b)$, $(-a, -b)$, $(a, -b)$, (a, b) .*

Proof. If $N = 4$, then $\tau = 1$ by Lemma 2.2(c), so $h = K$ and $\Delta = K/2$, and (J4) gives $\lambda/\beta = 1/(1+k')$; with $x = \lambda$ this reads $\sqrt{(\beta - x)/(\alpha - x)} = (\beta - x)/x$, whose only solution in $(0, \beta)$ is $x = \alpha\beta/(\alpha + \beta)$. Conversely, for this λ we have $k' = \beta/\alpha$ and $\text{sn}^2(K/2) = \alpha/(\alpha + \beta) = \lambda/\beta$, so $\Delta = K/2$ and $4h = 4K$. The vertices are $P(0)$, $P(K)$, $P(2K)$, $P(3K)$, and the tangents there are $y = b$, $x = -a$, $y = -b$, $x = a$. $\square$

3 The polygons as functions of the phase

For $u \in \mathbb{C}$ and $l \in \mathbb{Z}$ put

$$\begin{aligned} u_l &= u + lh, & t_l &= u_l + \Delta, & P_l &= P(u_l), & m_l &= m(t_l), \\ n_l &= n(u_l), & V_l &= V(t_l), & d_l &= P_l - f_1, & d'_l &= V_l - f_1. \end{aligned}$$

Since $Nh = 4\tau K$, all of them depend only on l modulo N . For real u , (P_l) is a trajectory of the family, its chord $P_l P_{l+1}$ lies on $m_l \cdot X = 1$, its tangent at P_l is $n_l \cdot X = 1$, and V_l is its outer vertex (Lemmas 2.1 and 2.2).

For points M, F and vectors x, X, Y in $\mathbb{C}^2$ let

$$\begin{aligned} \Phi_M(x) &= M + \frac{1 - x \cdot M}{x \cdot x} x, & \Psi_M(x; F) &= M + \frac{x}{1 - x \cdot (M - F)}, \\ \iota_M(X) &= M + \frac{X - M}{(X - M) \cdot (X - M)}, & \alpha_M(X, Y) &= M + \frac{(\xi \cdot \xi) \eta^\perp - (\eta \cdot \eta) \xi^\perp}{\xi \times \eta} \end{aligned}$$

with $\xi = X - M$, $\eta = Y - M$. For real data, $\Phi_M(x)$ is the foot of the perpendicular from M to the line $x \cdot X = 1$; $\Psi_M(x; F)$ is the pole of the line $x \cdot (X - F) = 1$ with respect to the unit circle centred at M (the polar of Y is $(Y - M) \cdot (X - M) = 1$); ι_M is the inversion in that circle; and $\alpha_M(X, Y)$ is the intersection of the line through X perpendicular to $X - M$ with the line through Y perpendicular to $Y - M$, because it satisfies $(X - M) \cdot (Z - M) = (X - M) \cdot (X - M)$ and the same with Y . With $O_j := (1 + 1/\beta)f_j$ we define, for $j = 1, 2$, the vertices

$$\begin{aligned} Q_l^{(j)} &= \Phi_{f_j}(m_l), & Q'_l{}^{(j)} &= \Phi_{f_j}(n_l), & Q_l^{*(j)} &= \alpha_{f_j}(P_l, P_{l+1}), & P_{j,l}^\dagger &= \iota_{f_j}(P_l), & P_{j,l}'^\dagger &= \iota_{f_j}(V_l), \\ \Pi_l^{(j)} &= \Psi_{f_j}(m_l; 0), & D_l^{(j)} &= \Psi_{O_j}(P_l - f_j; f_j), & \tilde{D}_l^{(j)} &= \iota_{O_j}(\Pi_l^{(j)}). \end{aligned}$$

For real u these are, in this order: the focal pedal polygons of P and of P' (the sides of P' lie on the tangents $n_l \cdot X = 1$), the focal antipedal polygon, the two focus-inversive polygons, the polar polygon (the poles of the side lines of P with respect to ω_j), and the dual polygon in the readings R and P. For reading R we use that the side of the polar polygon through $\Pi_{l-1}^{(j)}$ and $\Pi_l^{(j)}$, the poles of the two side lines of P through P_l , is the polar of P_l , the line $(P_l - f_j) \cdot (X - f_j) = 1$ (La Hire).

Remark 3.1 (the polar and dual polygons). Let u be real and $j = 1$. (1) A pole is the inverse of the corresponding foot, so the polar polygon is the inverse of the focal pedal polygon (“the dual is the inverse of the pedal” [RGK21, v11, §3.9]). The polar of P_l is also the line through $P_{1,l}^\dagger$ perpendicular to $P_{1,l}^\dagger - f_1$, so the polar polygon is the antipedal polygon of $P_1^\dagger$ with respect to f_1 , as in the source; equivalently, $P_1^\dagger$ is the pedal polygon of the polar polygon with respect to f_1 . By (2) the polar polygons are Poncelet polygons between two circles; Roitman, Garcia and Reznik [RGR21] prove other invariants of $P_1^\dagger$ through this description. (2) Since $O_1 - f_1 = f_1/\beta$ and $d_l \cdot f_1 = ac \text{sn } u_l - c^2$, the distance from O_1 to the polar of P_l is $|1 - d_l \cdot f_1/\beta|/|d_l| = a/\beta$ by Lemma 2.3(i). So the sides of the polar polygon touch the circle of radius a/β about O_1 , which is the point O_p of the source, and by Lemma 3.2(e) the vertices $D_l^{(1)}$ lie on the circle of radius β/a about O_1 : in reading R the dual

polygons are inscribed in a circle centred at O_p , as the source says. In reading P they are the inverses of the polar vertices, which lie on a circle not concentric with the one at O_p (the source's caption), so they lie on a circle that is not centred at O_p . (3) In reading R,

$$A_{1,\text{dual}} = \frac{\beta^2}{2\alpha} \sum_l \frac{d_l \times d_{l+1}}{|d_l| |d_{l+1}|} = \frac{b^4}{2a^2} \sum_l \sin \angle P_l f_1 P_{l+1}. \quad (3)$$

A circle of radius ρ at O_p instead of the unit circle multiplies the dual areas by ρ^4 in both readings.

Lemma 3.2 (vertex formulas). *As meromorphic functions of $u \in \mathbb{C}$, for every l :*

- (a) $Q_l^{(1)} - f_1 = \beta c_0^2 m_l / (1 + k \operatorname{sn} t_l)$ and $Q_l'^{(1)} - f_1 = a\beta n_l / (a + c \operatorname{sn} u_l)$;
- (b) $P_{1,l}^\dagger - f_1 = d_l / (a - c \operatorname{sn} u_l)^2$ and $P_{1,l}'^\dagger - f_1 = d_l' / (d_l' \cdot d_l')$;
- (c) $Q_l^{*(1)} - f_1 = [(d_l \cdot d_l) d_{l+1}^\perp - (d_{l+1} \cdot d_{l+1}) d_l^\perp] / (d_l \times d_{l+1})$;
- (d) $\Pi_l^{(1)} - f_1 = m_l / (1 - k \operatorname{sn} t_l)$;
- (e) $D_l^{(1)} - O_1 = (\beta/a) d_l / (a - c \operatorname{sn} u_l)$ and $\tilde{D}_l^{(1)} - O_1 = \beta^3 c_0^2 [m_l - (1 - k \operatorname{sn} t_l) f_1 / \beta] / \nu(\operatorname{sn} t_l)$, where $\nu(s) = (\alpha\beta - c^2\lambda) - (\alpha\beta - \lambda(\alpha + \beta))ks$.

The formulas for f_2 follow by replacing f_1, O_1, c by $f_2, O_2, -c$.

Proof. (a) By Lemma 2.3(ii), $(1 - m \cdot f_1) / (m \cdot m) = (1 - k \operatorname{sn} t) \beta c_0^2 / ((1 - k \operatorname{sn} t)(1 + k \operatorname{sn} t))$; by Lemma 2.3(iii), $(1 - n \cdot f_1) / (n \cdot n) = ((a - c \operatorname{sn} v) / a) \alpha \beta / ((a - c \operatorname{sn} v)(a + c \operatorname{sn} v))$. (b) is Lemma 2.3(i), (c) the definition, and (d) Lemma 2.3(ii). (e) $1 - d_l \cdot (O_1 - f_1) = 1 - d_l \cdot f_1 / \beta = (\alpha - a c \operatorname{sn} u_l) / \beta$. For $\tilde{D}$, write $s = \operatorname{sn} t_l$; by (d) and Lemma 2.3(ii),

$$(\Pi_l^{(1)} - O_1) \cdot (\Pi_l^{(1)} - O_1) = \frac{1 + ks}{\beta c_0^2 (1 - ks)} - \frac{2ks}{\beta (1 - ks)} + \frac{c^2}{\beta^2} = \frac{\nu(s)}{\beta^3 c_0^2 (1 - ks)},$$

using $\beta c_0^2 = \beta - \lambda$ and $c^2 = \alpha - \beta$, and $(1 - ks)(\Pi_l^{(1)} - O_1) = m_l - (1 - ks)f_1 / \beta$. $\square$

We define the area functions as the shoelace sums of these vertices; since $\sum_l (W_l - M) \times (W_{l+1} - M) = \sum_l W_l \times W_{l+1}$ for every closed polygon and every M , we may centre them conveniently:

$$A = \frac{1}{2} \sum_l P_l \times P_{l+1}, \quad A' = \frac{1}{2} \sum_l V_l \times V_{l+1}, \quad \bar{A}_j = \frac{1}{2} \sum_l (Q_l^{(j)} - f_j) \times (Q_{l+1}^{(j)} - f_j),$$

and likewise $\bar{A}_j', A_j^* := A_{f_j}^*, A_j^\dagger, A_j'^\dagger$ from $Q^{(j)}, Q^{*(j)}, P_j^\dagger, P_j'^\dagger$ centred at f_j , and $A_{j,\text{pol}}, A_{j,\text{dual}}, \tilde{A}_{j,\text{dual}}$ from $\Pi^{(j)}$ centred at f_j and from $D^{(j)}, \tilde{D}^{(j)}$ centred at O_j . These are meromorphic functions of u , since no denominator in Lemma 3.2 vanishes identically (Lemma 2.3, Lemma 4.4 below, and, for ν , $\nu(1/k) = 2\beta\lambda \neq 0$). For real u they are the signed areas of the source for the trajectory $(P(u + lh))_l$. By Lemma 2.1, $V(t) = \Gamma P(t)$ gives

$$A'(u) = \det \Gamma \cdot A(u + \Delta) = \frac{d_0}{c_0^2} A(u + \Delta). \quad (4)$$

Lemma 3.3 (symmetries). *Let F_1, F_2 be the two focal versions of one of the area functions above $(\bar{A}_j, \bar{A}_j', A_j^*, A_j^\dagger, A_j'^\dagger, A_{j,\text{pol}}, A_{j,\text{dual}}, \tilde{A}_{j,\text{dual}})$, and $F \in \{A, A'\}$. Then $F_j(u + h) = F_j(u + 4K) = F_j(u)$, $F_1(u + 2K) = F_2(u)$ and $F_j(u + 2iK') = -F_j(u)$; $F(u + h) = F(u + 2K) = F(u)$ and $F(u + 2iK') = -F(u)$. All these functions have the periods Λ . If N is even, then $F_1 = F_2$.*

Proof. The constructions Φ , Ψ , ι , α commute with every orthogonal linear map g of $\mathbb{C}^2$, when the points M , F are replaced by gM , gF : g preserves the bilinear product, and $(gx)^\perp = \det g \, g(x^\perp)$, $gx \times gy = \det g \, (x \times y)$. Shifting u by h relabels all indices by one. By Lemma 2.4(c), the shift by $2K$ replaces P_l , m_l , n_l , V_l by their images under $\sigma = -\text{id}$, which maps f_1 , O_1 to f_2 , O_2 and preserves signed areas; the shift by $2iK'$ applies $R = \text{diag}(1, -1)$, which fixes f_j and O_j and changes the sign of signed areas. If N is even, then σ maps every P_l , m_l , n_l , V_l to the one with index $l + N/2$ (Lemma 2.2 and the remark after it), so the configuration at $u + 2K$ is the configuration at u relabelled, and $F_2(u) = F_1(u + 2K) = F_1(u)$. $\square$

4 Special phases and two symmetries

Let $G = h\mathbb{Z} + 2K\mathbb{Z} + 2iK'\mathbb{Z}$ and $G_1 = h\mathbb{Z} + 4K\mathbb{Z} + 2iK'\mathbb{Z}$, and define the sets of phases

$$\mathcal{P}_A = iK' + G, \quad S_1 = K + iK' - \Delta + G_1, \quad S_2 = S_1 + 2K.$$

By (J1), $\mathcal{P}_A$ is the set of u for which some vertex P_l has a pole, $u_l \in \Pi$; and S_1 , resp. S_2 , is the set of u for which some side parameter t_l lies in Π'_+ , resp. Π'_- (where $\text{sn} = 1/k$, resp. $-1/k$). Let $Z_\pm = \{v : \text{sn } v = \pm a/c\}$.

Lemma 4.1 (cosets). (a) $Z_+ = (K + iK' \pm \Delta) + \Lambda_1$ and $Z_- = Z_+ + 2K$, and $a \mp c \text{sn}$ has simple zeros there. Moreover $u \in S_1$ if and only if $u_l \in Z_+$ for some l , and $u \in S_2$ if and only if $u_l \in Z_-$ for some l .

(b) If N is odd, $S_1 \cap S_2 = \emptyset$; if N is even, $S_1 = S_2$.

(c) If N is odd or $N \equiv 0 \pmod{4}$, $\mathcal{P}_A \cap (S_1 \cup S_2) = \emptyset$; if $N \equiv 2 \pmod{4}$, $\mathcal{P}_A = S_1 = S_2$.

(d) $\mathcal{P}_A - \Delta = \mathcal{P}_A$ if N is odd; $\mathcal{P}_A - \Delta = S_1 = S_2$ if $N \equiv 0 \pmod{4}$; $(\mathcal{P}_A - \Delta) \cap \mathcal{P}_A = \emptyset$ if $N \equiv 2 \pmod{4}$.

Proof. (a) By (J2) and (2), $\text{sn}(K + iK' \pm \Delta) = d_0/(kc_0) = a/c$; the two points are different modulo Λ_1 , so by (J1) they are all solutions, and both are simple. Further $u_l \in K + iK' - \Delta + \Lambda_1$ iff $t_l \in \Pi'_+$, and $u_l \in K + iK' + \Delta + \Lambda_1$ iff $t_{l-1} \in \Pi'_+$. In (b)–(d) the sets are cosets of G_1 or G , and a real number lies in G (resp. G_1) iff it lies in $h\mathbb{Z} + 2K\mathbb{Z}$ (resp. $h\mathbb{Z} + 4K\mathbb{Z}$); recall $h = 4\tau K/N$ with $\gcd(\tau, N) = 1$, and τ is odd if N is even. (b) $2K \in G_1$ iff $N(1 - 2p) = 2\tau j$ for integers j, p : impossible for odd N , and solved by $j = N/2$, $1 - 2p = \tau$ for even N . (c) The cosets $\mathcal{P}_A$ and $S_1 \cup S_2 = K + iK' - \Delta + G$ meet iff $K - \Delta \in G$, iff $N(1 - 2p) = 2\tau(2j + 1)$: impossible for N odd (parity) and for $N \equiv 0 \pmod{4}$ (the right side is $\equiv 2 \pmod{4}$), solved by $2j + 1 = N/2$, $1 - 2p = \tau$ for $N \equiv 2 \pmod{4}$. (d) $\Delta \in G$ iff $\tau(1 - 2j) = Np$: solvable iff N is odd; and $\mathcal{P}_A - \Delta = S_1 \cup S_2$ iff $K \in G$, iff $N(1 - 2p) = 4\tau j$: solvable iff $N \equiv 0 \pmod{4}$. $\square$

At a point of $S_1 \cup S_2$ two consecutive vertices coincide: if $t_0 = Z \in \Pi'$, then $P_0 = P(Z - \Delta) = P(Z + \Delta) = P_1$ by Lemma 2.4(b). We normalise such points: replacing u by $u + jh$ only relabels the indices, so at $u_0 \in S_1 \cup S_2$ we may assume $t_0 = Z \in \Pi'$.

Lemma 4.2 (configuration at special phases). Let $u_0 \in S_1 \cup S_2$ with $t_0 = Z \in \Pi'$, and the indices taken modulo N .

(a) $t_l \in \Pi'$ iff $l = 0$, or N is even and $l = N/2$; and $t_{N/2} \in \Pi'_\mp$ if $Z \in \Pi'_\pm$.

(b) If $Z \in \Pi'_+$: $u_l \in Z_+$ iff $l \in \{0, 1\}$; $u_l \in Z_-$ iff N is even and $l \in \{N/2, N/2 + 1\}$; $\text{sn } t_l = r_1$ iff $l = \pm 1$ (for $N \neq 4$). The same holds for $Z \in \Pi'_-$ with Z_+ , Z_- and r_1 , $-r_1$ interchanged.

(c) $t_l \in \Pi$ iff $N \equiv 0 \pmod{4}$ and $l \equiv \pm N/4$; so for $N = 4$ exactly $t_{\pm 1} \in \Pi$.

(d) $u_l \in \Pi$ only if $N \equiv 2 \pmod{4}$, and then for exactly two indices $l_0, l_0 + N/2$, with $l_0 = (N+2)/4 \notin \{0, 1, N/2, N/2 + 1\}$.

Let $u_0 \in \mathcal{P}_A$ with $u_0 = z \in \Pi$. Then $u_l \in \Pi$ iff $l = 0$, or N is even and $l = N/2$.

Proof. All differences of parameters are real multiples of h plus $\pm\Delta$, and $\Pi - \Pi' = K + \Lambda_0$, $\Pi'_\pm - \Pi_\pm = \Lambda_1$, $\Pi'_- - \Pi'_+ = 2K + \Lambda_1$. (a) $lh \in 2K\mathbb{Z}$ iff $N \mid 2l$, and $lh \in 2K + 4K\mathbb{Z}$ iff N is even and $l \equiv N/2$. (b) By Lemma 4.1(a), $u_l = Z - \Delta + lh \in Z_+ = Z \pm \Delta + \Lambda_1$ iff $lh \in 4K\mathbb{Z}$ or $(l-1)h \in 4K\mathbb{Z}$; Z_- likewise with $2K + 4K\mathbb{Z}$. By (J1) and Lemma 2.3(v), $\text{sn } t = r_1$ exactly for $t \in K + iK' \pm h + \Lambda_1$, and $Z + lh \in K + iK' \pm h + \Lambda_1$ iff $(l \mp 1)h \in 4K\mathbb{Z}$. At $Z \in \Pi'_-$, $\text{sn}(t + 2K) = -\text{sn } t$ exchanges the roles. (c) $lh \in K + 2K\mathbb{Z}$ iff $4\tau l = N(2p + 1)$. (d) $lh - \Delta \in K + 2K\mathbb{Z}$ iff $2\tau(2l - 1) = N(2p + 1)$, which needs $N \equiv 2 \pmod{4}$ and then means $2l - 1 \in \frac{N}{2}(2\mathbb{Z} + 1)$, with the solutions $l \equiv (N + 2)/4$ and $l \equiv (N + 2)/4 + N/2$ modulo N . The last statement: $lh \in 2K\mathbb{Z}$ iff $N \mid 2l$. $\square$

For $N = 6$ the two vertices with poles, P_2 and P_5 , are adjacent to the coinciding vertices $P_0 = P_1$ and $P_3 = P_4$; for $N \equiv 2 \pmod{4}$, $N \geq 10$, they are not.

Lemma 4.3 (geometry at S_1). *Let $u_0 \in S_1$ with $t_0 = Z \in \Pi'_+$, and $T = P_0 = P_1$. Then $(T - f_1) \cdot (T - f_1) = 0$, and $T \neq f_1$. The line $m(Z) \cdot X = 1$ and the tangent line $n(Z - \Delta) \cdot X = 1$ of the complex conic $\mathcal{E}$ at T both pass through f_1 and T , so they coincide; call this line ℓ . The points V_{-1} , V_0 , V_1 lie on ℓ wherever they are finite, so the vectors d'_{-1} , d'_0 , d'_1 are pairwise parallel there.*

Proof. $\text{sn}(Z \pm \Delta) = a/c$ gives $(T - f_1) \cdot (T - f_1) = 0$ (Lemma 2.3(i)); f_1 does not satisfy the equation of $\mathcal{E}$. By Lemma 2.3(ii) and (iii), $m(Z) \cdot f_1 = k \text{sn } Z = 1$ and $n(Z - \Delta) \cdot f_1 = (c/a)(a/c) = 1$; both lines contain T by Lemma 2.1 and $n(v) \cdot P(v) = 1$. By Lemma 2.1, $V_0 = V(Z)$ lies on the tangents at $P(Z \pm \Delta) = T$, V_1 on the tangent at $P_1 = T$, and V_{-1} on the tangent at $P_0 = T$. $\square$

Lemma 4.4 (the cross product). *For all u and l ,*

$$d_l \times d_{l+1} = (1 - k \text{sn } t_l) (P_l \times P_{l+1}) = \frac{2ab s_0 c_0 (1 - k \text{sn } t_l) \text{dn } t_l}{1 - k^2 s_0^2 \text{sn}^2 t_l}.$$

As a function of $t = t_l$, it has zeros of order exactly 3 at the points of Π'_+ , of order exactly 1 at the points of Π'_- , and no other zeros at points where P_l and P_{l+1} are finite.

Proof. For any vectors $X, Y, M, \mu \in \mathbb{C}^2$ with $\mu \cdot X = \mu \cdot Y = 1$ we have $(X - M) \times (Y - M) = (1 - \mu \cdot M)(X \times Y)$: dot the identity $M(X \times Y) = X(M \times Y) + Y(X \times M)$ with μ . Apply it with $X = P_l$, $Y = P_{l+1}$, $\mu = m_l$ (Lemma 2.1), $M = f_1$, and use Lemma 2.3(ii), (iv). At $t = Z + w$, $Z \in \Pi'_+$, (J2) gives $1 - k \text{sn } t = 1 - \text{dn } w / \text{cn } w = -\frac{k'^2}{2} w^2 + O(w^4)$, $\text{dn } t = \pm i k' \text{sn } w / \text{cn } w$ with a simple zero, and $1 - k^2 s_0^2 \text{sn}^2 t \rightarrow 1 - s_0^2 \neq 0$; at $Z \in \Pi'_-$, $1 - k \text{sn } t \rightarrow 2$. At $t \in \Pi$ the quotient has the finite limit $\pm 2iab c_0 / s_0 \neq 0$ by (J2). Elsewhere $1 - k \text{sn } t \neq 0$ and $\text{dn } t \neq 0$, and the denominator vanishes only where $\text{sn } t = \pm 1/(k s_0)$, that is $t - \Delta \in \Pi$ or $t + \Delta \in \Pi$, where P_l or P_{l+1} has a pole. $\square$

Lemma 4.5 (folding). *Let $u_0 \in S_1 \cup S_2$ with $t_0 = Z \in \Pi'$. Then for all $w \in \mathbb{C}$ and all l ,*

$$\begin{aligned} P_l(u_0 - w) &= P_{1-l}(u_0 + w), & n_l(u_0 - w) &= n_{1-l}(u_0 + w), \\ m_l(u_0 - w) &= m_{-l}(u_0 + w), & V_l(u_0 - w) &= V_{-l}(u_0 + w). \end{aligned}$$

Consequently every area function F of Section 3 satisfies $F(u_0 - w) = -F(u_0 + w)$. In particular, if F is holomorphic at u_0 then $F(u_0) = 0$, and if F has at u_0 a pole of order at most 2, then it has at most a simple pole.

function	poles, all at most simple unless stated	vanishes on, for odd N
A	exactly $\mathcal{P}_A$, residues $\pm\rho_A \neq 0$	–
A'	exactly $\mathcal{P}_A - \Delta$	–
$A_1^\dagger, A_1^{\prime\dagger}, A_{1,\text{dual}}, \tilde{A}_{1,\text{dual}} (N \neq 4), A_{1,\text{pol}}$	S_1	S_2
$\bar{A}_1, \bar{A}'_1$	S_2	S_1
A_1^*	$\mathcal{P}_A$; order ≤ 2 , and ≤ 1 for even N	–

Table 2: Possible poles. For f_2 , exchange S_1 and S_2 . For odd N the sets $\mathcal{P}_A, S_1, S_2$ are pairwise disjoint and $\mathcal{P}_A - \Delta = \mathcal{P}_A$; for $N \equiv 0 \pmod{4}$, $S_1 = S_2 = \mathcal{P}_A - \Delta$ is disjoint from $\mathcal{P}_A$; for $N \equiv 2 \pmod{4}$, $\mathcal{P}_A = S_1 = S_2$ (Lemma 4.1). In particular A' has poles on S_1 exactly when $N \equiv 0 \pmod{4}$.

Proof. $u_l(u_0 - w) + u_{1-l}(u_0 + w) = 2Z$ and $t_l(u_0 - w) + t_{-l}(u_0 + w) = 2Z$; apply Lemma 2.4(b). So the configuration at $u_0 - w$ is the configuration at $u_0 + w$ with the vertex indices relabelled by $l \mapsto 1 - l$ and the side indices by $l \mapsto -l$. Each vertex of Section 3 is built from one or two consecutive objects in a way that is compatible with this relabelling ($\alpha_M(X, Y) = \alpha_M(Y, X)$), and both relabellings reverse the cyclic order, which changes the sign of a shoelace sum. So $F(u_0 + w)$ is an odd function of w . $\square$

Lemma 4.6 (reversal). *Let $u_0 \in \mathcal{P}_A$ with $u_0 = z \in \Pi$. Then for all w and l , $P_l(u_0 - w) = -P_{-l}(u_0 + w)$, $n_l(u_0 - w) = -n_{-l}(u_0 + w)$, $m_l(u_0 - w) = -m_{-1-l}(u_0 + w)$ and $V_l(u_0 - w) = -V_{-1-l}(u_0 + w)$. Consequently, for the two focal versions of an area function, $F_1(u_0 - w) = -F_2(u_0 + w)$; if N is even, F_1 is odd about u_0 . The functions A and A' are odd about u_0 for every N .*

Proof. $u_l(u_0 - w) + u_{-l}(u_0 + w) = 2z$ and $t_l(u_0 - w) + t_{-l}(u_0 + w) = 2z$; apply Lemma 2.4(a). So the configuration at $u_0 - w$ is the image under $\sigma = -\text{id}$ of the configuration at $u_0 + w$ with an order-reversing relabelling; σ exchanges f_1, O_1 with f_2, O_2 and preserves signed areas. For even N use $F_1 = F_2$ (Lemma 3.3). $\square$

5 The poles of the area functions

We write $w = u - u_0$ near a point u_0 , and “order $\geq -m$ ” for a pole of order at most m . We state everything for f_1 ; by Lemma 3.3, the functions for f_2 behave in the same way with S_1 and S_2 exchanged. Table 2 summarises the section.

Lemma 5.1 (A and A'). *A is holomorphic outside $\mathcal{P}_A$, and at every point of $\mathcal{P}_A$ it has a simple pole with residue $\pm\rho_A$, where $\rho_A = \epsilon iab(\text{dn } h - 1)/(k^2 \text{sn } h) \neq 0$, $\epsilon = 1$ for odd N and $\epsilon = 2$ for even N . A' has simple poles exactly at the points of $\mathcal{P}_A - \Delta$.*

Proof. Each P_l is holomorphic outside Π . By Lemma 3.3, $A(u + \gamma) = \pm A(u)$ for $\gamma \in G$, so it suffices to take $u_0 = iK'$. There P_0 , and for even N also $P_{N/2} = -P_0$, are the only vertices with a pole (Lemma 4.2), and $P_0 = r/w + O(w)$ with $r = -(a, ib)/k$ by (J2). The terms of $2A$ containing P_0 are $P_0 \times (P_1 - P_{-1})$, and $P_{-1}(u_0) = P(iK' - h) = -P(iK' + h)$ (Lemma 2.4), so they contribute the residue $2r \times P(iK' + h) = 2iab(\text{dn } h - 1)/(k^2 \text{sn } h)$; the terms containing $P_{N/2}$ contribute the same. Here $\text{dn } h \neq 1$ and $\text{sn } h \neq 0$ because $0 < h < 2K$. The statement on A' follows from (4). $\square$

Lemma 5.2 (inversive areas). *$A_1^\dagger$ and $A_1^{\prime\dagger}$ are holomorphic outside S_1 and have at most simple poles at the points of S_1 . If N is odd, they vanish at the points of S_2 .*

Proof. $A_1^\dagger$. By Lemma 3.2(b), $P_{1,l}^\dagger - f_1 = d_l/(a - c \text{sn } u_l)^2$. Where $u_l \in \Pi$, d_l has a simple pole and $(a - c \text{sn } u_l)^2$ a double pole, so the vertex is holomorphic; otherwise it is singular only where $u_l \in Z_+$, with a pole of order 2. By Lemma 4.1(a), $A_1^\dagger$ is holomorphic outside S_1 . At $u_0 \in S_1$ with $t_0 \in \Pi'_+$ only

$P_{1,0}^\dagger$ and $P_{1,1}^\dagger$ are singular (Lemma 4.2(b)). The terms $l = \pm 1$ of $2A_1^\dagger = \sum_l (P_{1,l}^\dagger - f_1) \times (P_{1,l+1}^\dagger - f_1)$ have order ≥ -2 , and the term $l = 0$ is $(d_0 \times d_1)/((a - c \operatorname{sn} u_0)^2(a - c \operatorname{sn} u_1)^2)$, whose numerator has a zero of order 3 (Lemma 4.4) and whose denominator has a zero of order 4; so its order is ≥ -1 . Hence $A_1^\dagger$ has order ≥ -2 at u_0 , and ≥ -1 by Lemma 4.5. For odd N , $S_1 \cap S_2 = \emptyset$, so $A_1^\dagger$ is holomorphic at the points of S_2 and vanishes there by Lemma 4.5.

$A_1^{\dagger\prime}$. By Lemma 3.2(b), $P_{1,l}^{\dagger\prime} - f_1 = d_l^{\dagger\prime}/(d_l' \cdot d_l')$. By Lemma 2.3(v) and Lemma 2.6, as a function of $t = t_l$, $d_l' \cdot d_l^{\dagger\prime}$ equals $(A'^2 - B'^2)(\operatorname{sn} t - 1/k)(\operatorname{sn} t - r_1)$ if $N \neq 4$ and $-2cA'(\operatorname{sn} t - 1/k)$ if $N = 4$; by (J1), (J2) and Lemma 4.2(b) its zeros are double at Π'_+ and, for $N \neq 4$, simple at $K + iK' \pm h + \Lambda_1$. At $t \in \Pi$, V is odd about t (Lemma 2.4), so $V = r'/w + O(w)$ with $r' = -(A', iB')/k$. If $N \neq 4$, $r' \cdot r' = (A'^2 - B'^2)/k^2 \neq 0$, so $d_l' \cdot d_l^{\dagger\prime}$ has a double pole and $P_{1,l}^{\dagger\prime} - f_1 = O(w)$. If $N = 4$, $r' \cdot r' = 0$, and because $V(t+w)$ has no constant term, $d_l' \cdot d_l^{\dagger\prime} = -2(r' \cdot f_1)/w + O(1)$ with $r' \cdot f_1 = acd_0/(kc_0) \neq 0$, so $P_{1,l}^{\dagger\prime}$ is holomorphic. Hence the vertex is singular only where $t_l \in \Pi'_+$ (order ≥ -2) or $t_l \in K + iK' \pm h + \Lambda_1$ (order ≥ -1), which happens only for $u \in S_1$. Let $u_0 \in S_1$ with $t_0 \in \Pi'_+$. If $N = 4$, then $t_{\pm 1} \in \Pi$ (Lemma 4.2(c)), only the vertex with $l = 0$ is singular, and the two terms containing it have order ≥ -2 . If $N \neq 4$, the singular vertices are those with $l = 0, \pm 1$ (Lemma 4.2(a), (b)). The terms of the shoelace sum with $l = -2$ and $l = 1$ have order ≥ -1 . The term $l = -1$ is $(d_{-1}' \times d_0')/((d_{-1}' \cdot d_{-1}') (d_0' \cdot d_0'))$; its numerator vanishes at u_0 (Lemma 4.3) and its denominator has a zero of order 3, so its order is ≥ -2 ; likewise for $l = 0$. For $N = 3$, the term $(d_1' \times d_{-1}')/((d_1' \cdot d_{-1}') (d_{-1}' \cdot d_{-1}'))$ has order ≥ -1 for the same reason. So $A_1^{\dagger\prime}$ has order ≥ -2 at u_0 , hence ≥ -1 by Lemma 4.5, and the last statement follows as for $A_1^\dagger$. $\square$

Lemma 5.3 (focal pedal and polar areas). $\bar{A}_1$ and $\bar{A}'_1$ are holomorphic outside S_2 and have at most simple poles at the points of S_2 ; $A_{1,\text{pol}}$ is holomorphic outside S_1 and has at most simple poles at the points of S_1 . If N is odd, $\bar{A}_1$ and $\bar{A}'_1$ vanish at the points of S_1 , and $A_{1,\text{pol}}$ at the points of S_2 .

Proof. By Lemma 3.2(a), $Q_l^{(1)} - f_1 = \beta c_0^2 m_l / (1 + k \operatorname{sn} t_l)$ is finite where $t_l \in \Pi$ (numerator and denominator have simple poles), and singular only where $\operatorname{sn} t_l = -1/k$, i.e. $t_l \in \Pi'_-$, with order ≥ -2 ; this happens only for $u \in S_2$. At $u_0 \in S_2$ with $t_0 \in \Pi'_-$ only $l = 0$ is singular (Lemma 4.2(a)), so the two terms containing $Q_0^{(1)}$ have order ≥ -2 , and Lemma 4.5 gives ≥ -1 . The polar vertices $\Pi_l^{(1)} - f_1 = m_l / (1 - k \operatorname{sn} t_l)$ are treated in the same way with Π'_+ and S_1 . Next, $Q_l'^{(1)} - f_1 = a\beta n_l / (a + c \operatorname{sn} u_l)$ is finite where $u_l \in \Pi$ and has simple poles where $u_l \in Z_-$, that is only for $u \in S_2$; at $u_0 \in S_2$ with $t_0 \in \Pi'_-$ exactly $Q_0'^{(1)}$ and $Q_1'^{(1)}$ are singular. The terms $l = \pm 1$ have order ≥ -1 , and the term $l = 0$, $(a\beta)^2(n_0 \times n_1)/((a + c \operatorname{sn} u_0)(a + c \operatorname{sn} u_1))$, too, because $n_0 = n_1$ at u_0 (Lemma 4.5). For odd N the remaining statements follow from $S_1 \cap S_2 = \emptyset$ and Lemma 4.5. $\square$

Lemma 5.4 (antipedal area). A_1^* is holomorphic outside $\mathcal{P}_A$ and has order ≥ -2 at the points of $\mathcal{P}_A$. If N is even, it has at most simple poles.

Proof. By Lemma 3.2(c) and Lemma 4.4, $Q_l^{*(1)}$ is holomorphic wherever P_l, P_{l+1} are finite and $t_l \notin \Pi'$.

(1) *Coinciding vertices.* Let $u_0 \in S_1 \cup S_2$ with $t_0 = Z \in \Pi'$. Then P_0, P_1 are finite near u_0 , since $\Pi - \Pi' = K + \Lambda_0$ does not contain $\pm \Delta$. The numerator $\mathcal{N}(w) = (d_0 \cdot d_0)d_1^\perp - (d_1 \cdot d_1)d_0^\perp$ and the denominator $\mathcal{D}(w) = d_0 \times d_1$ of $Q_0^{*(1)} - f_1$ are odd functions of w , because $d_1(u_0 + w) = d_0(u_0 - w)$ (Lemma 4.5). If $Z \in \Pi'_+$, then $d_0 \cdot d_0 = (a - c \operatorname{sn} u_0)^2$ and $d_1 \cdot d_1$ vanish to order 2 (Lemma 4.1(a)), so $\mathcal{N} = O(w^2)$, hence $O(w^3)$, while $\mathcal{D}$ has a zero of order exactly 3 (Lemma 4.4). If $Z \in \Pi'_-$, then $\mathcal{N} = O(w)$ and $\mathcal{D}$ has a simple zero. In both cases $Q_0^{*(1)}$ is holomorphic at u_0 ; for even N the same holds for $Q_{N/2}^{*(1)}$. All other $Q_l^{*(1)}$ have $t_l \notin \Pi'$ (Lemma 4.2(a)). So A_1^* is holomorphic on $(S_1 \cup S_2) \setminus \mathcal{P}_A$.

(2) *Vertices with poles.* By Lemma 3.3, the shifts by $h, 4K$ and $2iK'$ map A_1^* to $\pm A_1^*$ and the shift by $2K$ maps A_2^* to A_1^* ; so it suffices to study A_1^* and A_2^* at the point $u_0 = iK'$. There

$P_0 = r/w + O(w)$ with $r = -(a, ib)/k$, $r \cdot r = c^2/k^2 \neq 0$, and $P_{\pm 1}(u_0) = \pm P(iK' + h)$. For $f \in \{f_1, f_2\}$ and $Q = \alpha_f(P_0, P_1)$, with $d_l = P_l - f$, the numerator of $Q - f$ is $(r \cdot r)d_1^\perp/w^2 + O(1/w)$ and the denominator is $(r \times d_1)/w + O(1)$, where d_1 is taken at u_0 ; similarly for $\alpha_f(P_{-1}, P_0)$ with $d_{-1} \times r$. By (J2) and the identity $a(1 - \operatorname{dn} h)/(k \operatorname{sn} h) = cs_0$, which follows from (J3) and (2),

$$r \times d_1 = \frac{ib}{k}(\pm c - cs_0), \quad d_{-1} \times r = -\frac{ib}{k}(\pm c + cs_0) \quad (f = f_1, f_2),$$

which are nonzero as $0 < s_0 < 1$. So these two antipedal vertices have at most simple poles. For even N the vertex $P_{N/2} = -P_0$ gives two more, by the same computation for the other focus (the configuration near $P_{N/2}$ is the image under $-\operatorname{id}$ of that near P_0). Two vertices with poles are never adjacent ($h \notin \Lambda_0$). If $N \equiv 2 \pmod{4}$, then u_0 lies also in $S_1 = S_2$, and the antipedal vertices of the two degenerate sides are holomorphic by (1), which used only that their endpoints are finite (Lemma 4.2(d)). The computation for a vertex next to a pole did not use the position of the other endpoint, which for $N = 6$ is one of the coinciding vertices. Thus every $Q_l^{*(1)}$ has order ≥ -1 at u_0 , every term of the shoelace sum order ≥ -2 , and A_1^* order ≥ -2 ; for even N , A_1^* is odd about u_0 (Lemma 4.6), so its order is ≥ -1 . $\square$

Lemma 5.5 (dual areas). *$A_{1,\text{dual}}$ and, for $N \neq 4$, $\tilde{A}_{1,\text{dual}}$ are holomorphic outside S_1 and have at most simple poles at the points of S_1 . If N is odd, they vanish at the points of S_2 .*

Proof. By Lemma 3.2(e), $D_l^{(1)} - O_1 = (\beta/a)d_l/(a - c \operatorname{sn} u_l)$ is finite where $u_l \in \Pi$ and has simple poles where $u_l \in Z_+$. At $u_0 \in S_1$ with $t_0 \in \Pi'_+$ only $D_0^{(1)}$, $D_1^{(1)}$ are singular; the terms $l = \pm 1$ have order ≥ -1 , and the term $l = 0$ is $(\beta^2/\alpha)(d_0 \times d_1)/((a - c \operatorname{sn} u_0)(a - c \operatorname{sn} u_1))$, of order $\geq 3 - 2$. For reading P, $\nu(s)$ is a polynomial of degree 1 when $N \neq 4$ (Lemma 2.6), and its root is r_1 by Lemma 2.3(v). Where $t_l \in \Pi$, both $m_l - (1 - k \operatorname{sn} t_l)f_1/\beta$ and $\nu(\operatorname{sn} t_l)$ have simple poles, so $\tilde{D}_l^{(1)}$ is finite. Hence $\tilde{D}_l^{(1)}$ is singular only where $\operatorname{sn} t_l = r_1$, with a simple pole; by Lemma 4.2(b) this happens only for $u \in S_1$, and at $u_0 \in S_1$ with $t_0 \in \Pi'_+$ only for $l = \pm 1$. For $N \geq 5$ every term of the shoelace sum has at most one singular factor; for $N = 3$ the term with the factors $\tilde{D}_1^{(1)}$, $\tilde{D}_2^{(1)} = \tilde{D}_{-1}^{(1)}$ has order ≥ -2 , and Lemma 4.5 gives ≥ -1 . The last statement follows as before. $\square$

6 Proofs of the theorems

Lemma 6.1 (Liouville). *A meromorphic function on $\mathbb{C}$ with the periods Λ and without poles is constant.*

Lemma 6.2 (proportionality). *Let $\mathcal{G} \in \{G, G_1\}$ and $S = s + \mathcal{G}$. Let F, H be meromorphic functions on $\mathbb{C}$ whose poles lie in S and are at most simple, such that $F(u + h) = F(u + 4K) = F(u)$, $F(u + 2iK') = -F(u)$, and $F(u + 2K) = F(u)$ if $\mathcal{G} = G$; and the same for H . If $H \neq 0$, then $\operatorname{Res}_s H \neq 0$ and $F = \kappa H$ with $\kappa = \operatorname{Res}_s F / \operatorname{Res}_s H$.*

Proof. For $\gamma \in \mathcal{G}$ the listed symmetries give $F(u + \gamma) = \epsilon(\gamma)F(u)$ and $H(u + \gamma) = \epsilon(\gamma)H(u)$ with the same sign $\epsilon(\gamma) = (-1)^n$, where $\operatorname{Im} \gamma = 2nK'$. Hence $\operatorname{Res}_{s+\gamma} F = \epsilon(\gamma) \operatorname{Res}_s F$, and likewise for H . If $\operatorname{Res}_s H = 0$, then H has no poles, is constant by Lemma 6.1, and changes sign under $u \mapsto u + 2iK'$, so $H \equiv 0$. Otherwise $F - \kappa H$ has no poles, is constant, and changes sign under $u \mapsto u + 2iK'$, so it vanishes. $\square$

Proposition 6.3 (positivity). *For every trajectory of the family, in counterclockwise order, the areas $A, A', \bar{A}_j, \bar{A}'_j, A_j^\dagger, A_j^{\dagger\prime}$ and $A_{j,\text{dual}}$ ($j = 1, 2$) are positive. In fact every term of their shoelace sums, centred as in Section 3, is positive.*

Proof. For A and $A_j^\dagger$ use Lemma 2.5(a), with $(P_{j,l}^\dagger - f_j) \times (P_{j,l+1}^\dagger - f_j) = (P_l - f_j) \times (P_{l+1} - f_j) / (|P_l - f_j|^2 |P_{l+1} - f_j|^2)$; for $A_{j,\text{dual}}$ use also (3); for A' and $A_j'^\dagger$ use Lemma 2.5(c) with $X = O$ and $X = f_j$. For $\bar{A}_j$: let e_l be the unit vector of the chord $P_l \rightarrow P_{l+1}$ and $p_l > 0$ the distance from f_j to its line. As f_j lies to the left of the chord (Lemma 2.5(a)), $Q_l^{(j)} - f_j = p_l e_l^\perp$, where $e_l^\perp$ is the right-hand normal, so $(Q_l^{(j)} - f_j) \times (Q_{l+1}^{(j)} - f_j) = p_l p_{l+1} e_l \times e_{l+1} > 0$ by Lemma 2.5(b). For $\bar{A}'_j$: f_j lies inside $\mathcal{E}$, so $Q_l'^{(j)} - f_j$ is a positive multiple of the outer normal $n(u_l)$, and Lemma 2.5(d) applies. $\square$

The positivity of $\bar{A}_j$ and $\bar{A}'_j$, term by term and for every winding number, is also proved by Kriebel [Kri26b, Theorem 1 and Section 4], for $\bar{A}'_j$ by the same argument; that of the terms of $A_j'^\dagger$ is (12) of [Kri26d], obtained there from an explicit factorisation.

Proposition 6.4 ($N = 4$). *For $N = 4$, at the rhombus member of Lemma 2.6 and for both foci, $A = 2ab$, $A_j^\dagger = 2/(ab)$, $A_j'^\dagger = 1/(ab)$ and $A_j^* = 4ab$.*

Proof. For $f_1 = (-c, 0)$ the vectors $P_l - f_1$ are (c, b) , $(c - a, 0)$, $(c, -b)$, $(a + c, 0)$, with squares α , $(a - c)^2$, α , $(a + c)^2$; so the inverted vertices relative to f_1 are $(c, b)/\alpha$, $(-1/(a - c), 0)$, $(c, -b)/\alpha$, $(1/(a + c), 0)$, and $2A_1^\dagger = \frac{2b}{\alpha}(\frac{1}{a-c} + \frac{1}{a+c}) = \frac{4ab}{\alpha\beta}$. The vectors $V_l - f_1$ are $(c - a, \pm b)$ and $(a + c, \mp b)$, with squares $2a(a - c)$ and $2a(a + c)$, so the inverted vertices relative to f_1 are $(-1, b/(a - c))/(2a)$, $(-1, -b/(a - c))/(2a)$, $(1, b/(a + c))/(2a)$, and $(1, -b/(a + c))/(2a)$, and $2A_1'^\dagger = \frac{4b}{4\alpha}(\frac{1}{a-c} + \frac{1}{a+c}) = \frac{2ab}{\alpha\beta}$. The antipedal vertices are $(-a, b + s)$, $(-a, -b - s)$, $(a, -b + s)$, $(a, b - s)$ with $s = ac/b$, with area $4ab$. For f_2 use $F_2 = F_1$ (Lemma 3.3). $\square$

Proof of Theorem A. By Lemma 5.2, $A_1^\dagger$ and $A_1'^\dagger$ have at most simple poles, all in $S_1 = (K + iK' - \Delta) + G_1$, and by Lemma 3.3 they have the symmetries of Lemma 6.2 for $\mathcal{G} = G_1$; $A_1^\dagger \neq 0$ by Proposition 6.3. So $A_1'^\dagger = \kappa A_1^\dagger$, and $A_2'^\dagger(u) = A_1'^\dagger(u + 2K) = \kappa A_1^\dagger(u + 2K) = \kappa A_2^\dagger(u)$. By Proposition 6.3, $\kappa = A_1'^\dagger/A_1^\dagger > 0$, and for $N = 4$, $\kappa = \frac{1}{2}$ by Proposition 6.4. $\square$

The theorem holds for every N because the outer vertices $V_{\pm 1}$ become isotropic with respect to f_1 exactly at the phases at which two vertices of P coincide (Lemma 4.3, Lemma 2.3(v)); for the pedal and antipedal areas the pole sets depend on N modulo 4 (Table 2).

Proof of Theorems B to D. Let N be odd and let F_j be one of $A_j^\dagger$, $\bar{A}'_j$, $A_{j,\text{dual}}$, $\bar{A}_{j,\text{dual}}$. By Lemmas 5.2, 5.3 and 5.5 and Lemma 3.3, one of F_1 , F_2 has at most simple poles on S_1 and vanishes on S_2 , and the other one has at most simple poles on S_2 and vanishes on S_1 ; both are holomorphic elsewhere, and $S_1 \cap S_2 = \emptyset$. So $F_1 F_2$ is holomorphic on $\mathbb{C}$; it has the periods Λ (Lemma 3.3) and is constant by Lemma 6.1. The constants of Theorems B and C and of Theorem D(a) are positive by Proposition 6.3, and (3) gives the second form of Theorem D(a). Finally $A_1'^\dagger A_2'^\dagger = \kappa^2 A_1^\dagger A_2^\dagger$ by Theorem A. $\square$

Remark 6.5 (the dagger in row $k_{907,a}$). We also tested literal readings of the undefined symbol $A_{1,\text{dual}}^\dagger$: the area of the image of the dual polygon under the inversion in ω_1 , times $A_{2,\text{dual}}$; the product of the areas of the images of the two dual polygons under the inversions in ω_1 and ω_2 ; and the area of the pedal polygon of $P_1^\dagger$ with respect to f_1 (the reading of the source's symbol $A_{j,\text{ped}}^\dagger$), times $A_{2,\text{dual}}$. None of these products is constant for odd N (Section 7).

Remark 6.6 (further products). The same argument gives, for odd N , the constancy of $\bar{A}_1 \bar{A}_2$ (row $k_{604,a}$ of [RGK21, v11], which is not among the records), of $A_{1,\text{pol}} A_{2,\text{pol}}$, the product for the polar polygons, which the source calls “dual” in one sentence (Remark 3.1), and of $A_1^\dagger A_2^\dagger$ (row $k_{903,a}$, [Fer26h]), by Lemmas 5.2 and 5.3. By Kriebel's theorem $\bar{A}'_j = \mu \bar{A}_j$ with a constant $\mu > 0$ common to both foci (row k_{606} , [Kri26b]), the constancy of $\bar{A}_1 \bar{A}_2$ is equivalent to Theorem C.

For every N , the functions $A_1^\dagger$, $A_{1,\text{pol}}$, $A_{1,\text{dual}}$ and, for $N \neq 4$, $\tilde{A}_{1,\text{dual}}$ have at most simple poles, all in S_1 , and the symmetries of Lemma 6.2 for $\mathcal{G} = G_1$ (Table 2, Lemma 3.3); so each of the last three is a constant multiple of $A_1^\dagger$, and the shift $u \mapsto u + 2K$ gives the same relation, with the same constant, for f_2 . Likewise $\bar{A}'_1$ and $A_2^\dagger$ have their poles in S_2 , so $\bar{A}'_1 = \gamma A_2^\dagger$ and $\bar{A}'_2 = \gamma A_1^\dagger$ with a constant $\gamma > 0$; this is the comparison made in [Kri26e, Section 5]. Hence Theorem C, Theorem D and the constancy of $A_{1,\text{pol}}A_{2,\text{pol}}$ are also consequences of row $k_{903,a}$ [Fer26h], as Theorem B is with Theorem A.

Proof of Theorem E. Positivity is Proposition 6.3. (i) Let N be even. By Lemma 5.4, A_1^* has at most simple poles, all in $\mathcal{P}_A = iK' + G$. By Lemma 3.3 it has the symmetries of Lemma 6.2 for $\mathcal{G} = G$, the $2K$ -periodicity because $A_1^*(u + 2K) = A_2^*(u) = A_1^*(u)$; and so has A , which is not identically zero (Lemma 5.1). So $A_1^* = \kappa^* A$, and $A_2^* = A_1^*$. (ii) Let $N \equiv 2 \pmod{4}$. By Lemma 5.3, the poles of $\bar{A}_1$ lie in S_2 , which equals $\mathcal{P}_A$ (Lemma 4.1(c)), and they are at most simple; the symmetries are as in (i), so $\bar{A}_1 = \kappa_p A$, and $\bar{A}_2 = \bar{A}_1$; $\kappa_p > 0$ by Proposition 6.3. (iii) Let $N \equiv 0 \pmod{4}$. If the ratio equals a constant C along the family, then $\kappa^* A = C \bar{A}_1$ on $\mathbb{R}$, hence on $\mathbb{C}$. If $\kappa^* \neq 0$, then $C \neq 0$, and $\bar{A}_1 = (\kappa^*/C)A$ would have poles at the points of $\mathcal{P}_A$ (Lemma 5.1), whereas its poles lie in S_2 , which is disjoint from $\mathcal{P}_A$ (Lemma 4.1(c)). So $\kappa^* = 0$. For $N = 4$, $\kappa^* = 2$ by Proposition 6.4. $\square$

Remark 6.7 (rows $k_{403,b}$ and k_{404}). For $N \equiv 0 \pmod{4}$, $\bar{A}_1$ and A' have their poles in $S_2 = \mathcal{P}_A - \Delta$, at most simple and exactly there for A' (Lemmas 4.1, 5.1 and 5.3), so Lemma 6.2 gives $\bar{A}_j = \kappa' A'$. With $A_j^* = \kappa^* A$ and the theorem of Chavez-Caliz that AA' is constant for even N [CC21, Theorem 6], the product $A_j^* \bar{A}_j = \kappa^* \kappa' AA'$ is constant: this is row $k_{403,b}$, proved in [Fer26f] by these two factors. So the source's split between the product row $k_{403,b}$ ($N \equiv 0 \pmod{4}$) and the ratio row k_{404} ($N \equiv 2 \pmod{4}$) reflects whether the poles of $\bar{A}_j$ sit at those of A' or of A . For odd N , A_1^* has double poles and the ratio is not constant (Section 7).

7 Computations and reproducibility

All theorems, propositions and lemmas of this note are proved without computation. The computations test the statements on trajectories that are constructed without the parametrisation of Section 2, test the lemmas of Sections 2 to 5 at complex phases, give exact values for $N = 4$ and $N = 6$, and check that the statements fail where their hypotheses fail. The reproducibility package has five parts.

The program of this note (`check_note.py`, about 20 seconds; 56 checks, all passed). (i) Trajectories are built by the reflection law alone: from a point of $\mathcal{E}$ the first chord is the tangent to $\mathcal{C}$ in the counterclockwise direction, and every further chord is the mirror image of the previous one in the normal of $\mathcal{E}$; the caustic of a family (N, τ) is found by bisection on the advance of the eccentric angle after N reflections; every polygon is built by its elementary construction. In 50-digit arithmetic, for 48 families with $(a, b) \in \{(1.5, 1), (2.2, 1), (2.6, 1.7)\}$ and $(N, \tau) \in \{(3, 1), (4, 1), (5, 1), (5, 2), (6, 1), (7, 1), (7, 3), (8, 1), (8, 3), (9, 2), (9, 4), (10, 1), (10, 3), (11, 4), (12, 5), (14, 3)\}$, six members each, the closure, the tangency of the chords, the reflection law, the outer vertices and the statements of Remark 3.1(1), (2) hold to $1.1 \cdot 10^{-31}$. The largest relative spread over the members, for each row in its class of N , is $7 \cdot 10^{-49}$ (k_{404}), $3 \cdot 10^{-44}$ ($k_{605,a}$), $1 \cdot 10^{-41}$ ($k_{806,a}$, with the same constant for both foci to $9 \cdot 10^{-42}$), $1.2 \cdot 10^{-42}$ ($k_{904,a}$), and $1.3 \cdot 10^{-41}$, $1.2 \cdot 10^{-34}$ and $4.5 \cdot 10^{-33}$ ($k_{907,a}$ in reading R, in reading P, and for the polar product); the largest values occur in the star family $a = 2.2$, $(N, \tau) = (9, 4)$, where $b^2 - \lambda < 5 \cdot 10^{-10}$. Outside the classes the relative spreads are at least $3.5 \cdot 10^{-7}$ for the products (even N) and $7.9 \cdot 10^{-5}$ for the ratio of k_{404} (odd N and $N \equiv 0 \pmod{4}$). For $N = 4$, $\kappa = \frac{1}{2}$ and $\kappa^* = 2$ at all members of the three shapes. The ratio of row k_{404} for the 3-periodic listed twice is that of the 3-periodic, whose relative spread is at least 0.06. (ii) The rhombus of Lemma 2.6 with symbolic a, b (sympy): tangency, the reflection law, and Proposition 6.4 for both foci, exactly. (iii) For $a = 2, b = 1$,

$N = 6$, the hexagon with the vertices $(\pm 2, 0)$, $(\pm \frac{4}{3}, \pm \frac{\sqrt{5}}{3})$ is checked exactly to be a 6-periodic with $\lambda = \frac{4}{9}$; its focal antipedal polygons have six distinct vertices and area exactly 0, while $A = 20\sqrt{5}/9$ and $\bar{A}_j = 200\sqrt{5}/81$; so $\kappa^* = 0$ and $\kappa_p = 10/9$. At six members built by the reflection law, $|A_j^*/A| < 10^{-45}$. For $N \in \{4, 6, 8, 10, 12\}$ with $\tau = 1$, for $(N, \tau) \in \{(8, 3), (10, 3), (12, 5)\}$ and for eight values of a/b between 1.3 and 3, κ^* agrees at two members to $1.4 \cdot 10^{-33}$ and with (1) to $6 \cdot 10^{-34}$ (relative); for $\tau = 1$ it changes sign between $a/b = 1.9$ and 2 ($N = 6$; zero at 2), 1.8 and 1.9 ($N = 8, 10$), and 1.7 and 1.8 ($N = 12$). Root finding in a/b gives the families with $\kappa^* = 0$: $a/b = 1.85384730438182101462 \dots$ for $N = 8$ and $a/b = 1.77996941624483732972 \dots$ for $N = 12$, where $|A_j^*/A| < 5 \cdot 10^{-40}$, resp. $2 \cdot 10^{-33}$, at six members while A varies (relative spread $2 \cdot 10^{-4}$, resp. $1.3 \cdot 10^{-6}$); for a/b larger by 0.05 the ratio of row k_{404} varies. (iv) Proposition 6.3 for $\bar{A}_j$ on 16 star families (eight pairs (N, τ) with $N \leq 12$, $a/b \in \{1.5, 3\}$): every term of the shoelace sum is positive; the smallest value of $\bar{A}_j/A$ is $2.3 \cdot 10^{-12}$, so κ_p can be very small in star families. (v) The three products of Remark 6.5 vary (relative spreads at least $3.7 \cdot 10^{-9}$). (vi) With the parametrisation of Section 2 in 60-digit arithmetic: the identities of Lemmas 2.1, 2.3, 3.2 and 4.4 at random complex points (largest residual $4.1 \cdot 10^{-48}$); the identities of Theorems A to E at a complex phase; and the orders of all area functions at points of S_1 , S_2 and $\mathcal{P}_A$ in seven families ($N = 3, 4, 5, 6, 7, 8, 10$, convex and stars), which respect the bounds of Table 2; A_1^* has a double pole at $\mathcal{P}_A$ for odd N .

The programs with which the results were first obtained (original/): the Jacobi-form toolkit `ebtools.py`; `scan_rows.py` (the rows and the controls on 32 families, 50 digits); `local_checks.py` (orders and residue quotients at S_1 , S_2 , $\mathcal{P}_A$); `kappa_scan.py` (κ_p and κ^* for $N \equiv 2 \pmod{4}$); and `probe_k806.py` (the second root r_1 and the isotropic outer vertices); with their recorded outputs.

The programs of the verification runs, described in the next paragraph, with their recorded outputs, in `verification_run_A/` and `verification_run_B/`. All programs were re-run when the package was assembled, with the results recorded in `RERUN_LOG.txt`.

The programs of the verification run on the final text (independent_run_2/), with their recorded outputs: a geometry library in which trajectories are built by tangent stepping and every polygon by its elementary construction, with the circle at O_p located as the circle touched by the polar sides (`r2_rows.py`, 90 digits, 80 families); exact checks with sympy (`r2_exact.py`); the identities of Lemmas 2.1, 2.3, 3.2 and 4.4 and the orders of Table 2 at complex phases (`r2_complex.py`); and the constants of Theorem E (`r2_kstar.py`).

Verification. The results were first obtained in a written version containing Theorems A to E with proofs by the method of Sections 4 to 6, and with the programs in `original/`. Two independent verification runs, both AI-assisted, followed; each wrote its own programs without using those of the first write-up. The first run compared the rows and the definitions with the tex source of [RGK21, v11]; re-derived every lemma, the list of poles and the five theorems; constructed trajectories by tangent stepping without Jacobi functions (59 families, 50 digits) and measured the orders of the area functions at S_1 , S_2 and $\mathcal{P}_A$ (8 families, 90 digits); found the positivity argument for the focal pedal areas, the case $N = 4$ at the rhombus and the families with $\kappa^* = 0$ for $N = 8$ and 12; and tested the daggered readings of row $k_{907,a}$. The second run constructed trajectories by the reflection law (112 families, N up to 18); checked the lemmas at random complex phases and the orders and residues (11 families, 80 digits); showed that its programs detect wrong inputs (caustics, pedal points, classes of N); searched for families with $\kappa_p = 0$ (208 families with $N \equiv 2 \pmod{4}$ up to $N = 30$; none); re-ran the original programs, whose recorded outputs were reproduced; and checked the bibliography. Neither run found a gap in the proofs. They required eleven corrections, all of which were made. The mathematical ones are the statement on row k_{404} for $N \equiv 0 \pmod{4}$ (Theorem E(iii)), the positivity of the focal pedal areas in place of a possible exceptional set of row k_{404} , the poles of A' at S_1 (Table 2), the full zero set of $d_l \times d_{l+1}$ (Lemma 4.4) and the parity argument for $N = 4$ in Lemma 5.2; the others

concern references, the reading of the dagger in Theorem D, the hypotheses in each statement and the example $N = 6$, $a = 2b$, and are listed in the verification report of the package. When this note was written, the closed form of Lemma 4.4, the positivity of $A_j^\dagger$, $A_j'^\dagger$ and $\bar{A}_j'$ for every winding number (which gives $\kappa > 0$ and the positivity of the products), the polar product, Proposition 6.4 and the program `check_note.py` were added, and the search of the deposits was repeated. This search found [Kri26d, Kri26e, Fer26a], and the reading of the deposited sources showed that [Fer26d], [Fer26f] and [Fer26b] contain the case $N = 4$ of Theorem A, part (i) of Theorem E and the families with $\kappa^* = 0$; the statements of novelty were restricted accordingly.

A further independent verification run, also AI-assisted, examined the final text. It read the five rows and the definitions again in the tex source of [RGK21, v11]; read the deposits [Kri26d, Kri26e, Kri26b] in full and compared their statements, hypotheses and methods with Theorems B and C and Proposition 6.3; and re-read the cited parts of [Fer26d, Fer26f, Fer26g, Fer26b, Fer26a], [Sta22, Theorem 2], [Sta21, Corollary 4.2], [CC21, Theorem 6] and [DLM26]. It re-derived the parts added when the note was written (Lemma 2.5, Proposition 6.3, the closed form of Lemma 4.4, the polar product, Proposition 6.4, Theorem E(iii)) and then every other lemma and the five theorems line by line. With its own programs it tested the rows on 80 families at 90 digits (four ellipses, twenty pairs (N, τ) with $N \leq 15$, stars included; largest relative spread in the classes $6 \cdot 10^{-47}$, outside them at least $3.5 \cdot 10^{-7}$), with the circle at O_p located geometrically, and found every term of the shoelace sums of Proposition 6.3 positive; recomputed exactly the rhombus, the hexagon and the worked examples printed in [Kri26d, Kri26e, Kri26b, Fer26d], which shows that Kriebel's areas are the areas $A_j^\dagger$ and $\bar{A}_j'$ of this note; checked Lemmas 3.2 and 4.4 and Table 2 at complex phases in ten families; located the families with $\kappa^* = 0$ for $N = 6, 8, 10, 12$; and re-ran the package from its archive, whose recorded outputs were reproduced. It found no mathematical error. It required corrections of the credits (the content and the method of [Kri26d, Kri26e], the positivity proved in [Kri26b, Kri26d], Theorem E(iii) as a consequence of [Fer26f] and [CC21], and the reduction of Theorems C and D to row $k_{903,a}$ in Remark 6.6), of four numerical bounds of this section and of one bibliographic entry; all were made. No proof depends on a computation.

Scope and priority. The note proves rows $k_{806,a}$, $k_{904,a}$, $k_{605,a}$, $k_{907,a}$ and k_{404} of [RGK21] for every ellipse, every confocal elliptic caustic whose trajectories have least period N in the class of the row, and every winding number; row $k_{907,a}$ is proved for the row read without its undefined dagger, in two readings of “dual” (and for the polar product). Hyperbolic caustics are not treated, and closed walks that repeat a trajectory of smaller least period are not N -periodics in our sense. The rows and the experiments are due to Reznik, Garcia and Koiller [RGK21]; the parametrisation is Stachel's [Sta22], and so is the central symmetry of even periodics [Sta21]; the constancy of AA' for even N is Chavez-Caliz's [CC21]. The case $N = 4$ of Theorem A is in [Fer26d]; rows $k_{904,a}$ and $k_{605,a}$ were first proved by Kriebel [Kri26d, Kri26e], before this note was written (with repeated traversals and the circular billiard, and with the positivity in [Kri26d]), and Theorems B and C are independent proofs, for which no novelty is claimed; part (i) of Theorem E is Lemma 4 of [Fer26f] (see also [Fer26a]), part (ii) is a case of [Fer26g, Theorem B], the positivity of the focal pedal areas of P and P' is proved in [Kri26b] and that of the terms of $A_j^\dagger$ in [Kri26d], and the families with $\kappa^* = 0$ were found in [Fer26b] (for $N = 6$, see [GR21]). To our knowledge new are Theorem A for $N \neq 4$, Theorem D (which also follows from row $k_{903,a}$ by Remark 6.6), the ratio statement of row k_{404} as such, and Theorem E(iii) for $N \geq 8$ (which also follows from Theorem 1 of [Fer26f] and [CC21, Theorem 6]). No printed or deposited proof of row $k_{806,a}$ for $N \neq 4$, of row $k_{907,a}$ or of the ratio statement of row k_{404} was found in the sources searched: the source in its arXiv and journal versions; the arXiv listings found by queries on elliptic billiards, on antipedal, inversive, polar and dual polygons, and on the works of the proposers and of Stachel, among them [GHR26], which treats triangles only; the works citing [RGK21]

in OpenAlex (19) and Semantic Scholar (31); zbMATH Open and Crossref; and the Zenodo deposits that DataCite and Zenodo’s own search list for queries on billiards (the last searches on 15 October 2026, 20:02 UTC). Of these deposits, [Fer26e] treats row k_{805} ($A/A_j^\dagger$ for $N \equiv 2 \pmod{4}$), [Fer26c] the cosine sums of focus-inversive polygons, [Fer26i] the outer antipedal areas, [Kri26a] a correction of row k_{108} , [Kri26f] the outer pedal area centroid (row k_{307}), and [Kri26c] row $k_{403,a}$; none of them covers the five rows. A search that finds nothing is not a proof of novelty, and we make no priority claim.

What was read: [RGK21] in the tex source of arXiv v11 (definitions, the tables containing the five rows, the caption on polar and dual polygons, the table of symbols), and the numbering scheme and Tables 5 and 7 of the journal version in a text extraction of the journal’s file; Theorem 6 of [CC21] in arXiv v1; [Fer26d] and Theorem 1 and Lemma 4 of [Fer26f] in their deposited sources, and the relevant sections of [Fer26i, Fer26b, Fer26g, Fer26a]; the list of videos of [GR21] in arXiv v3; and the abstracts of [RGR21, GHR26] and of the other deposits. The deposits [Kri26d, Kri26e, Kri26b] were read in full (version 1.0) by the last verification run, which also read Theorem 2 of [Sta22], Corollary 4.2(i) of [Sta21] and the proposition of [RGH22] on $A_1^\dagger A_2^\dagger$ for $N = 3$ in arXiv v2, as the first verification run had done. The facts (J1)–(J4) are quoted from [DLM26], whose section and table numbers were checked by the last verification run.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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