

Focal Pedal-Antipedal Area Products in Elliptic Billiards

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1 October 2026

Abstract

We prove constancy of the signed area product of the pedal and antipedal polygons at either fixed focus of an elliptic billiard, when the least period is divisible by four and the confocal caustic is an ellipse. Convex and primitive star trajectories are included. The argument gives two explicit factors: the focal pedal area is a fixed multiple of the outer tangent-polygon area, and the focal antipedal area is a fixed multiple of the original area. Their product reduces the assertion to Chávez-Caliz's even-period area-product theorem. A velocity-coordinate telescope and a Jacobi quarter-period permutation provide the first factor; opposite-edge algebra provides the second. This establishes invariant $k_{403,b}$ in the usual genuine-period interpretation, with that convention stated explicitly. No absolute priority claim is made.

1 Statement and conventions

Fix $a > b > 0$, put $c^2 = a^2 - b^2$, and consider the boundary and caustic

$$E : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad C : \frac{x^2}{a^2 - \lambda} + \frac{y^2}{b^2 - \lambda} = 1, \quad 0 < \lambda < b^2. \quad (1)$$

Let $P = (P_1, \dots, P_N)$ be a periodic billiard trajectory on E tangent to C . Throughout, N denotes its *least* positive period. We allow every coprime winding number admitted by the elliptic caustic, including star trajectories. For any ordered polygon set

$$\mathcal{A}(V) = \frac{1}{2} \sum_i \det(V_i, V_{i+1}), \quad V_{N+1} = V_1.$$

All derived polygons retain trajectory order. Their areas are signed, not sums of absolute areas of bounded faces.

Write $A = \mathcal{A}(P)$ and let T be the area of the polygon formed by consecutive intersections of the tangents to E at P_i . For one fixed focus $F = (\pm c, 0)$, let p be the area of the pedal polygon whose vertices are the perpendicular feet from F to the original sidelines. Let q be the area of the antipedal polygon, whose defining sides are

$$(P_i - F) \cdot (X - P_i) = 0. \quad (2)$$

The conjecture $k_{403,b}$ in [1, 2, Table 5] asserts constancy of pq for $N \equiv 0 \pmod{4}$. Both factors concern the *original* orbit at the *same* focus. The source uses signed areas and expressly permits self-intersecting antipedals.

For compact notation put

$$\alpha = a^2, \quad \beta = b^2, \quad d = \alpha - \beta = c^2, \quad D = \alpha\beta, \quad s = \alpha + \beta, \quad J = (\alpha - \lambda)(\beta - \lambda).$$

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Theorem 1. *For every primitive billiard family in (1) with $N \equiv 0 \pmod{4}$, the same-focus signed pedal-antipedal product is constant. More precisely,*

$$p = KT, \quad q = RA, \quad pq = KR(AT), \quad (3)$$

where

$$K = \frac{\sqrt{J}(\beta - \lambda)}{\beta^2}, \quad R = \frac{D(3\beta - \alpha) + \lambda(\alpha - \beta)^2}{2D(\beta - \lambda)}. \quad (4)$$

All consecutive outer and focal-antipedal intersections are finite. The formulas include zero or negative antipedal area.

The source does not expressly expand “ N -periodic” into a minimum-period definition. Its genuine-family interpretation is used here, and is essential: an arbitrary repeated list of $4m$ vertices does not satisfy the quarter-period argument below. Section 6 records an exact control illustrating this distinction. Neither a hyperbolic caustic nor a degenerate endpoint is included in the theorem.

2 Reflection and the focal pedal area

Choose the orientation with the caustic on the left of every directed edge. Its center and both foci lie strictly inside it, so

$$\det(P_i, P_{i+1}) > 0, \quad \det(P_i - F, P_{i+1} - F) > 0. \quad (5)$$

The second inequality already proves finite consecutive antipedal intersections. Reversal changes every signed area’s sign and preserves the claimed identities.

At $P = (a \cos \theta, b \sin \theta)$ define

$$W = \beta \cos^2 \theta + \alpha \sin^2 \theta, \quad n = \frac{(b \cos \theta, a \sin \theta)}{\sqrt{W}}, \quad t = \frac{(-a \sin \theta, b \cos \theta)}{\sqrt{W}}.$$

Here n is the outward unit normal, t the positive unit tangent, and $\det(n, t) = 1$. The incoming and outgoing unit velocities are

$$v_- = ht + jn, \quad v_+ = ht - jn, \quad h = \sqrt{\frac{W - \lambda}{W}}, \quad j = \sqrt{\frac{\lambda}{W}}. \quad (6)$$

To see the normal-component formula directly, a chord with endpoint parameters $u - \delta, u + \delta$ has unit direction $(-a \sin u, b \cos u)/\sqrt{U}$, where $U = \alpha \sin^2 u + \beta \cos^2 u$. Confocal tangency gives $\sin^2 \delta = \lambda U/D$, so its squared normal component is $D \sin^2 \delta/(UW) = \lambda/W$. Its positive tangent component and reflection give (6).

Lemma 2. *If $S_r = \sum_i \sqrt{W_i - \lambda}/W_i^r$, then*

$$2DS_2 = sS_1. \quad (7)$$

Proof. Equation (6) gives

$$(v_-)_x(v_-)_y - (v_+)_x(v_+)_y = \frac{2\sqrt{\lambda}\sqrt{W - \lambda}(2D - sW)}{dW^2}.$$

The left side telescopes around the orbit: each outgoing velocity is the next incoming velocity. Since $d, \lambda > 0$, the asserted identity follows. No area invariance has been assumed. $\square$

Lemma 3. For a centrally paired orbit $P_{i+N/2} = -P_i$, the focal pedal area is

$$p = \frac{\sqrt{\lambda} \alpha (\beta - \lambda)}{\beta} S_1. \quad (8)$$

Proof. Take $F = (c, 0)$ and denote its feet on the incoming and outgoing sidelines at P by Q_-, Q_+ . Set $P_n = P \cdot n$, $P_t = P \cdot t$, $u_n = (P - F) \cdot n$ and $u_t = (P - F) \cdot t$. Orthogonal projection yields

$$Q_+ - Q_- = 2hj(u_nt + u_tn), \quad \frac{Q_+ + Q_-}{2} = P - u_th^2t - u_nj^2n.$$

Thus their *ordered* half determinant is

$$\frac{\det(Q_-, Q_+)}{2} = hj(P_nu_n - P_tu_t + u_t^2h^2 - u_n^2j^2). \quad (9)$$

For $X = \cos \theta$, $Y = \sin \theta$, these frame coordinates are

$$P_n = \frac{ab}{\sqrt{W}}, \quad P_t = -\frac{dXY}{\sqrt{W}}, \quad u_n = \frac{b(a - cX)}{\sqrt{W}}, \quad u_t = \frac{cY(a - cX)}{\sqrt{W}}.$$

Using $W = \beta + dY^2 = (a - cX)(a + cX)$, the bracket in (9) reduces to $(a - cX)[a(W - \lambda) + cX\lambda]/W$. Consequently

$$\frac{\det(Q_-, Q_+)}{2} = \sqrt{\lambda} \sqrt{W - \lambda} \frac{\alpha - \lambda + caX}{(a + cX)^2}. \quad (10)$$

This term can be negative; no unsigned-area decomposition is being used. Pairing P with $-P$ replaces X by $-X$. The pair contribution is

$$2\sqrt{\lambda} \sqrt{W - \lambda} \left(\frac{\alpha + \lambda}{W} - \frac{2\alpha\lambda}{W^2} \right).$$

Summing once over all vertices and applying (7) gives

$$p = \sqrt{\lambda} [(\alpha + \lambda)S_1 - 2\alpha\lambda S_2] = \frac{\sqrt{\lambda} \alpha (\beta - \lambda)}{\beta} S_1.$$

Reflection in the y -axis interchanges the foci and reverses both signed areas, giving the same formula for the other focus. $\square$

3 The outer area and a quarter-period permutation

Set $L = D - s\lambda$. For a vertex with boundary parameter θ , the two adjacent half-gap tangents $z = \tan \delta$ solve

$$(L + \lambda W)z^2 - 2\lambda d \cos \theta \sin \theta z - \lambda W = 0. \quad (11)$$

This follows by substituting $u = \theta + \delta$ in $D \sin^2 \delta = \lambda(\alpha \sin^2 u + \beta \cos^2 u)$. The leading coefficient satisfies

$$L + \lambda W \geq \alpha(\beta - \lambda) > 0,$$

so the roots have opposite signs and are finite. One is the forward half-gap tangent and the other is minus the backward one. Their difference is $2\sqrt{\lambda D} \sqrt{W - \lambda} / (L + \lambda W)$. The unit-circle tangent-side area formula, followed by the linear map $\text{diag}(a, b)$, therefore gives

$$T = D\sqrt{\lambda} \sum_i \frac{\sqrt{W_i - \lambda}}{L + \lambda W_i}. \quad (12)$$

This signed boundary formula applies in trajectory order also for stars. It also shows that consecutive outer vertices are finite.

Stachel's canonical parametrization [4, Theorem 2] is

$$P(u) = (-a \operatorname{sn} u, b \operatorname{cn} u), \quad k^2 = \frac{d}{\alpha - \lambda}, \quad u_i = u_0 + \frac{4i\tau K_0}{N}, \quad (13)$$

where K_0 is the real complete quarter period and $\gcd(N, \tau) = 1$. For even N , τ is odd and the index shift $N/2$ negates the vertices, justifying Lemma 3. For $N \equiv 0 \pmod{4}$, the index shift $N/4$ adds τK_0 . Since $W = \alpha - d \operatorname{sn}^2 u$, the standard squared Jacobi quarter-period identity yields

$$W' = \frac{\lambda W + L}{W - \lambda}, \quad W' - \lambda = \frac{J}{W - \lambda}, \quad L + \lambda W' = \frac{JW}{W - \lambda}. \quad (14)$$

All real denominators and square roots here are positive. The index shift permutes the vertices, so (12) can be evaluated at the W'_i . It becomes

$$T = \frac{D\sqrt{\lambda}}{\sqrt{J}} S_1. \quad (15)$$

Combining with (8) gives $p = KT$ with K as in (4). An arbitrary padded list does not supply this quarter-period permutation.

4 An opposite-edge formula for the antipedal

Translate the origin to F . Consecutive antipedal sides then have equations $u \cdot Z = |u|^2$ and $v \cdot Z = |v|^2$, where $u = P - F$, $v = Q - F$. Collecting adjacent terms in the signed shoelace sum expresses the antipedal area as $q = \sum_i E(P_i - F, P_{i+1} - F)$, with

$$E(u, v) = \frac{|u|^2 |v|^2 - (|u|^2 + |v|^2)(u \cdot v)/2}{\det(u, v)}. \quad (16)$$

Indeed, if $r = |u|$, $r' = |v|$ and η is their oriented normal-angle increment, the grouped term is $rr' \csc \eta - (r^2 + r'^2) \cot \eta/2$. It follows by writing the two intersection coordinates along each oriented side and collecting half their determinants. The derivation does not require a convex antipedal.

Lemma 4. *For any centrally paired elliptic-caustic trajectory, $q = RA$ with R given by (4).*

Proof. Write an edge's endpoint parameters as $t - \delta, t + \delta$, and set

$$U = \alpha - d \cos^2 t, \quad C_0 = \cos \delta, \quad S_0 = \sin \delta, \quad z = c \cos t.$$

Then $S_0^2 = \lambda U/D$, $z^2 = \alpha - U$ and

$$\det(P, Q) = 2abS_0C_0, \quad \det(P - F, Q - F) = 2bS_0(aC_0 - z).$$

Use the focal-distance formula $|P - F| = a - c \cos(t - \delta)$ in (16). Its numerator divided by $2S_0^2$ is

$$H(c) = (2\beta - U)(a - zC_0)^2 + U(U - \beta)S_0^2.$$

The opposite edge has c replaced by $-c$. Combining the two fractions and dividing by $\det(P, Q)$ gives

$$\begin{aligned} & \frac{E(P - F, Q - F) + E(-P - F, -Q - F)}{\det(P, Q)} \\ &= \frac{H(c)/(aC_0 - z) + H(-c)/(aC_0 + z)}{2a\beta C_0} \\ &= \frac{(2\beta - U)(\alpha + z^2 C_0^2 - 2z^2) + U(U - \beta)(1 - C_0^2)}{\beta(\alpha C_0^2 - z^2)} \\ &= 2 - \frac{(D - s\lambda)U}{D(\beta - \lambda)}. \end{aligned} \quad (17)$$

For the last simplification substitute $z^2 = \alpha - U$ and $C_0^2 = 1 - \lambda U/D$: the numerator becomes $U[2(\beta - \lambda) - U(D - s\lambda)/D]$ and the denominator $U(\beta - \lambda)$. All canceled normal determinants are nonzero; in particular

$$(aC_0 - z)(aC_0 + z) = \frac{U(\beta - \lambda)}{\beta} > 0.$$

The elementary identity

$$\det(P_i, P_{i+1}) \cos(2t_i) = \frac{ab}{2} [\sin(2\theta_{i+1}) - \sin(2\theta_i)]$$

telescopes over the closed polygon. Since $U_i = s/2 - d \cos(2t_i)/2$, summing over one representative of each opposite-edge pair gives

$$\sum_{\text{pairs}} \det(P_i, P_{i+1}) = A, \quad \sum_{\text{pairs}} \det(P_i, P_{i+1}) U_i = \frac{s}{2} A.$$

Summing (17) yields

$$q = \left[2 - \frac{s(D - s\lambda)}{2D(\beta - \lambda)} \right] A = RA.$$

No division by a total signed area occurs. Replacing c by $-c$ gives the same result for the other focus. $\square$

5 The published area theorem and its hypotheses

Chávez-Caliz proves that AT is constant for even-period Poncelet families between concentric ellipses in general position [3, Theorem 6, p. 104]. Her area definition is the signed shoelace formula [3, p. 93]; the theorem is not restricted to convex polygons or winding one. Its tangent polygon is precisely the outer polygon used in (12).

For (1), general position is automatic. The complex projective conics have four distinct finite intersections with

$$x^2 = \frac{\alpha(\alpha - \lambda)}{\alpha - \beta}, \quad y^2 = -\frac{\beta(\beta - \lambda)}{\alpha - \beta}.$$

Their coordinates are nonzero, and the determinant of their affine gradients equals

$$\frac{4xy\lambda(\alpha - \beta)}{\alpha\beta(\alpha - \lambda)(\beta - \lambda)} \neq 0.$$

There is no common point at infinity, since the two leading diagonal quadratic forms have different coefficient ratios. Hence all four intersections are transverse. No in-domain limiting extension of the cited theorem is required.

It follows that AT is constant. Equations (8) and (15) give $p = KT$, and Lemma 4 gives $q = RA$. Since K, R depend only on the fixed ellipse and caustic, Theorem 1 follows. $\square$

6 Checks, period scope and provenance

For the standard primitive four-periodic family, $\lambda = D/s$, $R = 2$, $K = D/s^2$ and $AT = 8D$, so

$$pq = \frac{16D^2}{s^2}.$$

For $a = 5, b = 3$, both the axial diamond and the rectangular member give $pq = 202500/289$, although their two factors differ. The supplement includes exact rational/quadratic-field geometry checks, including crossed antipedals, and coefficientwise polynomial checks of the displayed identities. Rounded higher-period experiments are corroboration only; no finite set of phases supplies the general proof.

Remark 5. *Least period is a substantive hypothesis. In the same ellipse $a = 5, b = 3$ with $\lambda = 225/64$, exact primitive six-periodic trajectories starting at boundary half-angle parameters 0 and $1/3$ have focal products*

$$\frac{823875}{1024} \quad \text{and} \quad \frac{1390594355}{1721344},$$

respectively. The supplementary exact checker verifies closure, reflection, caustic tangency and both focal areas. Traversing each twice creates a 12-entry list and multiplies its product by four, without changing its least period of six. This refutes a padded-list extension, not the genuine-period assertion proved above. Conversely, repeating an orbit whose least period is divisible by four preserves constancy and scales the product by the square of the repetition count.

The ellipse-billiard construction, Jacobi parametrization and even outer-area-product theorem are prior work and are credited explicitly. The contribution here is the focal pedal bridge and antipedal factor, with their application to the stated invariant and precise period scope. A bounded primary-literature search did not identify this exact conclusion, but does not establish originality or absolute priority.

This manuscript is an unrefereed preprint. AI assistance was used in derivation, internal checking, coding and manuscript preparation. There has been no independent human peer review or formal proof-assistant verification. The written argument and cited theorem hypothesis map, not numerical experiments, establish the result.

References

- [1] D. Reznik, R. Garcia and J. Koiller, Fifty new invariants of N -periodics in the elliptic billiard, *Arnold Math. J.* **7** (2021), 341–355. [doi:10.1007/s40598-021-00174-y](https://doi.org/10.1007/s40598-021-00174-y).
- [2] D. Reznik, R. Garcia and J. Koiller, Eighty new invariants of N -periodics in the elliptic billiard, [arXiv:2004.12497v11](https://arxiv.org/abs/2004.12497v11) (2020).
- [3] A. C. Chávez-Caliz, More about areas and centers of Poncelet polygons, *Arnold Math. J.* **7** (2021), 91–106. [doi:10.1007/s40598-020-00154-8](https://doi.org/10.1007/s40598-020-00154-8).
- [4] H. Stachel, On the motion of billiards in ellipses, *Eur. J. Math.* **8** (2022), 1602–1622. [doi:10.1007/s40879-021-00524-2](https://doi.org/10.1007/s40879-021-00524-2). Parametrization cited from Theorem 2, pp. 11–12, of [arXiv:2105.03624v2](https://arxiv.org/abs/2105.03624v2) (2021).