

Nonconstant Antipedal Area Ratios in Four-Periodic Elliptic Billiards

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1 October 2026

Abstract

For a periodic elliptic billiard, let A' be the area of the polygon formed by consecutive tangents to the ellipse, and A_M^* the area of the antipedal polygon of the original orbit with respect to a fixed point M . We give an exact counterexample to the claimed constancy of A'/A_M^* for periods divisible by four. With M at the center, we compute the ratio and its sharp range on a standard continuous four-periodic family. It is nonconstant for every noncircular ellipse. On the ellipse with semiaxes 4, 3, two primitive orbits with the same confocal caustic give the values 1 and 90000/113569. All polygons are finite, convex, and positively oriented. The contribution is a reproducible correction of the printed assertion, not a new billiard family or an absolute priority claim.

1 The assertion and area conventions

For an ordered counterclockwise orbit $P = (P_1, \dots, P_N)$ on an ellipse centered at O , write

$$\mathcal{A}(V_1, \dots, V_N) = \frac{1}{2} \sum_{i=1}^N \det(V_i, V_{i+1}), \quad V_{N+1} = V_1.$$

The outer polygon P' consists of the intersections of the ellipse tangents at consecutive orbit vertices. For a fixed point M , the antipedal line at P_i is

$$(P_i - M) \cdot (X - P_i) = 0. \tag{1}$$

The antipedal polygon P_M^* consists of consecutive intersections of these lines, when finite. Put $A' = \mathcal{A}(P')$ and $A_M^* = \mathcal{A}(P_M^*)$.

Section 3.5 and Table 5, pages 347–348, of [1] define this antipedal of the *original* orbit and list the assertion

$$k_{402} = A'/A_M^* \text{ is constant for } N \equiv 0 \pmod{4}, \quad M \text{ arbitrary and fixed.} \tag{2}$$

The denominator is unprimed: it is not the antipedal area of P' . The same expression occurs in Table 5 of [2]. The frozen dataset record AMR-050-0019 addresses this assertion. Neither table excludes $N = 4$ or $M = O$. The examples below have nonzero denominators and refute (2) within a single continuous fixed-caustic family.

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2 The four-periodic family

Fix $a > b > 0$ and define

$$H = \text{diag}(a^2, b^2), \quad D = a^2 b^2, \quad s = a^2 + b^2, \quad \lambda = D/s.$$

The outer ellipse is $X^\top H^{-1} X = 1$. For $t \in \mathbb{R}$ put

$$n = (\cos t, \sin t), \quad m = (-\sin t, \cos t), \quad h^2 = n^\top H n, \quad k^2 = m^\top H m, \quad d = n^\top H m,$$

where $h, k > 0$, and set $u = Hn/h$, $v = Hm/k$. In the positively oriented orthonormal coordinates (n, m) these become

$$H = \begin{pmatrix} h^2 & d \\ d & k^2 \end{pmatrix}, \quad u = (h, d/h), \quad v = (d/k, k), \quad h^2 + k^2 = s, \quad h^2 k^2 - d^2 = D. \quad (3)$$

This is a parametrization of standard simple four-periodic geometry; see [3, Proposition 4.7 and Appendix B.2–B.3]. We include the following certificate to make the fixed-family condition explicit.

Lemma 1. *The polygons $(u, v, -u, -v)$ form a continuous family of convex primitive four-periodic billiards. Their common confocal elliptic caustic has shape matrix $C = H - \lambda I$ and semiaxes $a^2/\sqrt{s}, b^2/\sqrt{s}$. Their outer tangent-polygon area is $A' = 4hk$.*

Proof. The vertices lie on the ellipse and their outward unit normals are $n, m, -n, -m$. Since $hk > |d|$, the identities

$$v - u = \frac{hk - d}{hk}(-h, k), \quad u + v = \frac{hk + d}{hk}(h, k)$$

give incoming and outgoing unit directions at u equal to $(h, k)/\sqrt{s}$ and $(-h, k)/\sqrt{s}$. They obey reflection in its tangent $x = h$. At v , reflection in $y = k$ changes $(-h, k)/\sqrt{s}$ to $(-h, -k)/\sqrt{s}$. Negation gives the other two reflections.

We have $0 < \lambda < b^2$, so C is positive definite. Its semiaxes satisfy $a_c^2 - b_c^2 = a^2 - b^2$. In the rotated coordinates define

$$w_+ = (k, h), \quad b_+ = hk + d, \quad w_- = (k, -h), \quad b_- = d - hk.$$

The first two edge lines are $w_\pm \cdot X = b_\pm$. Equation (3) gives $w_\pm^\top C w_\pm = b_\pm^2$, so they are caustic tangents, with contacts

$$\frac{Cw_+}{b_+} = u + \frac{k^2}{s}(v - u), \quad \frac{Cw_-}{b_-} = v + \frac{h^2}{s}(-u - v).$$

Both parameters lie in $(0, 1)$; the other contacts are their negatives. Also $\det(u, v) = D/(hk) > 0$, proving convex order and four distinct vertices, hence primitive period four. The formulas are continuous in t . Finally the consecutive ellipse tangents intersect at $(h, k), (-h, k), (-h, -k), (h, -k)$, whose signed area is $4hk$. $\square$

3 Antipedal area and its exact variation

Lemma 2. *For $\Delta = \det(u, v) > 0$, the central parallelogram $(u, v, -u, -v)$ has a finite convex counterclockwise antipedal at O , of area*

$$A_O^* = \frac{4|u|^2|v|^2}{\Delta}.$$

Proof. Put $U = |u|^2$ and $V = |v|^2$. The four defining lines are $u \cdot X = U$, $v \cdot X = V$, $u \cdot X = -U$, $v \cdot X = -V$. The invertible linear map $X \mapsto (u \cdot X, v \cdot X)$ has determinant $\Delta > 0$ and sends their consecutive intersections to $(U, V), (-U, V), (-U, -V), (U, -V)$. This rectangle has area $4UV$. Dividing by the determinant proves the formula and the geometric claims. $\square$

Theorem 3. *On the family of Lemma 1, with $M = O$, the ratio is*

$$R(t) = \frac{A'}{A_O^*} = \frac{D(D + d^2)}{D^2 + s^2 d^2}. \quad (4)$$

Its sharp range is

$$\frac{Ds^2}{(s^2 - 2D)^2} \leq R(t) \leq 1. \quad (5)$$

In particular it is nonconstant for every noncircular ellipse $a > b > 0$.

Proof. The two lemmas give $R = D/(|u|^2|v|^2)$. Cayley–Hamilton yields $H^2 = sH - DI$, hence

$$|u|^2 = s - D/h^2, \quad |v|^2 = s - D/k^2, \quad |u|^2|v|^2 = \frac{D^2 + s^2 d^2}{D + d^2}.$$

This proves (4). As a function of $z = d^2$, its derivative is

$$\frac{D^2(D - s^2)}{(D^2 + s^2 z)^2} < 0.$$

Now $d = (b^2 - a^2) \sin t \cos t$, so z ranges continuously from 0 to $(s^2 - 4D)/4$ as t runs from 0 to $\pi/4$. Substitution gives (5). The endpoints differ because

$$(s^2 - 2D)^2 - Ds^2 = (s^2 - D)(s^2 - 4D) > 0.$$

The circle limit $a = b$ gives the constant value 1, as expected. $\square$

4 Two rational witnesses

Take $a = 4, b = 3$. The common outer ellipse and caustic are respectively

$$x^2/16 + y^2/9 = 1, \quad x^2/(256/25) + y^2/(81/25) = 1.$$

At $t = 0$, the orbit is $(4, 0), (0, 3), (-4, 0), (0, -3)$. Its outer and antipedal polygons both have vertices $(4, 3), (-4, 3), (-4, -3), (4, -3)$. Thus $A' = A_O^* = 48$ and $R = 1$.

At $t = \pi/4$, the orbit is

$$(16/5, 9/5), (-16/5, 9/5), (-16/5, -9/5), (16/5, -9/5).$$

Its outer vertices are $(0, 5), (-5, 0), (0, -5), (5, 0)$, giving $A' = 50$. Each orbit vertex has squared norm $337/25$. Solving the four lines (1) at $M = O$ gives the antipedal vertices

$$(0, 337/45), (-337/80, 0), (0, -337/45), (337/80, 0).$$

Consequently

$$A_O^* = 113569/1800, \quad R = 90000/113569, \quad 1 - R = 23569/113569 > 0.$$

Lemma 1 places both witnesses in one connected family, not merely on the same ellipse. All relevant polygons are finite, convex and positively oriented. Reversing traversal changes both area signs and preserves the ratio; using absolute areas also leaves the counterexample unchanged. No zero denominator or repeated shorter orbit is used. Moreover the defining line intersections and their nonzero areas vary continuously with M near O . The strict inequality therefore persists throughout some neighborhood of O .

5 Attribution and verification scope

The simple four-periodic family, its rectangular and axial-rhombus members, its caustic and its outer-rectangle geometry are established material; see [3, Propositions 4.7, 4.9 and Appendix B]. Our earlier note [4] uses the same family to test two *different* source expressions. Here the antipedal-area lemma gives the variation of the distinct printed ratio A'/A_M^* . We do not claim a new orbit construction, a refutation of every table entry, or absolute priority. A bounded source and literature search did not locate a correction of this exact row; absence from that search is not proof of novelty.

The supplementary dependency-free Python calculations reconstruct the two witnesses using exact rational arithmetic. They check ellipse membership, reflection, interior caustic tangency, consecutive line intersections, convex orientation, areas and traversal reversal. A separately written checker provides an additional computational check. These finite computations supplement the analytic family proof; they are not proof-assistant verification. This unrefereed note was developed with AI assistance in reasoning, source search, exact computation and drafting, and has not undergone independent human peer review.

References

- [1] D. Reznik, R. Garcia and J. Koiller, *Fifty New Invariants of N -Periodics in the Elliptic Billiard*, Arnold Mathematical Journal **7** (2021), 341–355. [doi:10.1007/s40598-021-00174-y](https://doi.org/10.1007/s40598-021-00174-y).
- [2] D. Reznik, R. Garcia and J. Koiller, *Eighty New Invariants of N -Periodics in the Elliptic Billiard*, arXiv:2004.12497v11, 29 October 2020. [Version-specific source](#).
- [3] R. Garcia and D. Reznik, *Exploring self-intersected N -periodics in the elliptic billiard*, Annales Mathematicae et Informaticae, online 22 February 2022. [doi:10.33039/ami.2022.02.001](https://doi.org/10.33039/ami.2022.02.001).
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