

Antipedal Areas of N -Periodics in the Elliptic Billiard: Row k_{401} Fails at the Centre and Holds at the Foci

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Abstract

Row k_{401} of Reznik, Garcia and Koiller’s list of experimentally found invariants of N -periodic billiard trajectories in an ellipse (record AMR-050-0018 of the UnsolvedMath collection) states that for $N \equiv 2 \pmod{4}$ and every point M the product $A'A_M^*$ is constant along the family; here A' is the signed area of the outer polygon, bounded by the tangents to the ellipse at the vertices, and A_M^* that of the antipedal polygon of the trajectory with respect to M . The row is false as printed: for the ellipse with semi-axes $8/5$ and 1 , two members of the convex 6-periodic family, certified in exact arithmetic, give $A'A_O^* = 801792/21875$ and $3241/100$ at the centre O , and for every ellipse with semi-axes $a > b$ the two 6-periodic members symmetric in both axes give values that differ by $4(a-b)^3(a+b)^5/(ab(2a+b)(a+2b))$. At the foci the row holds. We prove that for every $N \geq 3$ and every member of the convex N -periodic family $A_{f_1}^* + A_{f_2}^* = 2\kappa A$, where A is the signed area of the trajectory and κ an explicit rational function of a , b and the caustic parameter; the proof combines a side-by-side formula for antipedal areas, Stachel’s parametrisation by Jacobi elliptic functions and a residue computation. For even N the two focal areas are equal by central symmetry, so $A_{f_j}^* = \kappa A$, a factor that an earlier note of the author obtained by another argument, and with Chavez-Caliz’s theorem that $A'A$ is constant for even N the product $A'A_{f_j}^*$ is constant; it equals $16a^2b^2$ for $N = 4$ and $4a^2(4b^2 - a^2)$ for $N = 6$. The vanishing of κ accounts for the families with zero focal antipedal area found earlier, among them $N = 6$ with $a = 2b$, observed by Garcia and Reznik. Non-constancy at points other than the centre and the foci is only tested numerically. No printed statement, proof or refutation of k_{401} was found in the sources searched. This is an unrefereed note.

1 Introduction

1.1 The row

Let $\mathcal{E}$ be the ellipse $x^2/a^2 + y^2/b^2 = 1$ with $a > b > 0$, centre O and foci $f_1 = (c, 0)$, $f_2 = (-c, 0)$, $c = \sqrt{a^2 - b^2}$. A billiard trajectory in $\mathcal{E}$ whose chords are all tangent to a confocal ellipse $\mathcal{C}$, its *caustic*, closes after N reflections for one starting point if and only if it does so for every starting point (Poncelet’s porism), and the closed trajectories with a given caustic form a one-parameter family. In [RGK21] Reznik, Garcia and Koiller list quantities that they found experimentally to be constant along such families. The first row of their table of invariants of antipedal polygons [RGK21, v11, Table “Invariants of antipedal polygons”; Table 5 of the journal version] reads

code	invariant	value	which N	M	date	proven
k_{401}	$A'A_m^*$	?	$\equiv 2 \pmod{4}$	all	4/20	?

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Here A' is the signed area of the *outer polygon* P' of the N -periodic P , whose vertices are the intersections of the tangents to $\mathcal{E}$ at consecutive vertices of P , and A_m^* , which we write A_M^* , is the signed area of the *antipedal polygon* of P with respect to the point M , formed by the intersections of consecutive lines through the vertices P_i perpendicular to $P_i - M$ (Section 2). The corpus record AMR-050-0018 [ula26] asks to prove that for $N \equiv 2 \pmod{4}$ and every M the quantity $A'A_M^*$ is constant as P varies through the family.

The source explains the columns “invariant”, “value”, “which N”, “date” and “proven” of its tables, but not the column “M”, whose entries are O , “ f_1, f_2 ”, “ O, f_1, f_2 ” and “all”. We read “all” as “every point M of the plane”, as the record does; this reading is ours. The counterexample below is at $M = O$, which is covered by every reading that includes the centre.

1.2 Results

Theorem 1 (the row fails). *Let $a = \frac{8}{5}$ and $b = 1$. The billiard trajectories in $\mathcal{E}$ with the caustic $x^2/(96/65)^2 + y^2/(105/169) = 1$ form a convex 6-periodic family. Its members with a vertex at $(a, 0)$ and at $(0, b)$ satisfy*

$$A'A_O^* = \frac{801792}{21875} \quad \text{and} \quad A'A_O^* = \frac{3241}{100},$$

respectively. Hence $A'A_O^*$ is not constant along this family, and row k_{401} is false as printed. More generally, for every $a > b > 0$ the trajectories with caustic parameter $\lambda_6 = a^2b^2/(a+b)^2$ form a convex 6-periodic family, and its two members that are symmetric in both axes give values of $A'A_O^*$ that differ by $4(a-b)^3(a+b)^5/(ab(2a+b)(a+2b)) > 0$.

At the other points M that we tried, $A'A_M^*$ is not constant either, but this is only tested (Remark 3.2). At the foci the row is true, as a consequence of the following identity, which holds for every N .

Theorem 2 (focal antipedal areas). *Let $N \geq 3$ and $0 < \lambda < b^2$ be such that the billiard trajectories in $\mathcal{E}$ with the caustic $\mathcal{C} : x^2/(a^2 - \lambda) + y^2/(b^2 - \lambda) = 1$ form a convex N -periodic family, and put*

$$\kappa = \frac{a^2b^2(3b^2 - a^2) + \lambda(a^2 - b^2)^2}{2a^2b^2(b^2 - \lambda)}. \quad (1)$$

Then every member of the family satisfies $A_{f_1}^* + A_{f_2}^* = 2\kappa A$, where A is its signed area.

Corollary 3 (even N). *In Theorem 2 let N be even. Then every member of the family satisfies $A_{f_1}^* = A_{f_2}^* = \kappa A$, and $A'A_{f_1}^* = A'A_{f_2}^* = \kappa A'A$ is constant along the family. In particular row k_{401} holds for $M \in \{f_1, f_2\}$ and every $N \equiv 2 \pmod{4}$. The convex 4-periodic and 6-periodic families with the caustic parameters $a^2b^2/(a^2 + b^2)$ and $a^2b^2/(a + b)^2$ have*

$$A'A_{f_j}^* = 16a^2b^2 \quad \text{and} \quad A'A_{f_j}^* = 4a^2(4b^2 - a^2),$$

respectively. The focal antipedal area vanishes at one member if and only if it vanishes at every member, if and only if $\kappa = 0$, that is, $\lambda = a^2b^2(a^2 - 3b^2)/(a^2 - b^2)^2$; for the 6-periodic family this happens exactly when $a = 2b$.

For odd N only the sum of the two focal areas is proportional to A in our computations. The proof of Theorem 2 (Section 4) writes an antipedal area as a sum over the sides (Lemma 4.1), combines the two foci side by side with the focal properties of $\mathcal{C}$ (Lemma 4.2 and Proposition 4.3), turns the result into $\sum_{i=0}^{N-1} \varphi(v + 2i\Delta)$ for one elliptic function φ of the contact parameter v by Stachel’s parametrisation [Sta22, Theorem 2] (Lemmas 4.4 to 4.6), and shows by a residue computation that this sum vanishes when the coefficient of A is 2κ (Lemma 4.7 and Proposition 4.8). Corollary 3 then follows from the central symmetry of even periodic trajectories and Chavez-Caliz’s theorem [CC21, Theorem 6] (Section 5).

1.3 Credit and limits

The row k_{401} , the antipedal construction and the conventions are due to Reznik, Garcia and Koiller [RGK21]. The parametrisation used in Section 4 is Stachel’s [Sta22, Theorem 2]; we include a short proof of the form that we need. The central symmetry of even periodic trajectories is also Stachel’s [Sta21, Corollary 4.2(i)]. The equality $A_{f_1}^* = A_{f_2}^*$ for even N is the source’s row k_{607} , stated there for $N \equiv 0 \pmod{4}$ and proved for every even N , by the same symmetry, in the author’s unrefereed note [Fer26b]. The constancy of $A'A$ for even N is a theorem of Chavez-Caliz [CC21, Theorem 6]. The even- N identity $A_{f_j}^* = \kappa A$ is Lemma 4 of the author’s unrefereed note on row $k_{403,b}$ [Fer26c], with the same constant, proved there by pairing opposite sides. A zero-area antipedal polygon for $N = 6$, $a = 2b$ was noticed by Garcia and Reznik [GR21a], and the zero families for $N = 8, 10, 12, 16$ were found numerically in [Fer26b]. New, to our knowledge, are the disproof of k_{401} as printed, Theorem 2 for every N , odd N included, and the explicit focal product for $N = 6$; for $N = 4$ the values $\kappa = 2$ and $A'A = 8a^2b^2$, which give $A'A_{f_j}^* = 16a^2b^2$, are also in [Fer26c, Section 6]. No printed statement, proof or refutation of k_{401} was found in the sources searched (“Scope and priority” in Section 6); a search that finds nothing is not a proof of novelty. Non-constancy of $A'A_M^*$ at points other than O, f_1, f_2 is only tested numerically, and star families are not treated.

2 Definitions and conventions

The billiard. We write $u \times v = u_1v_2 - u_2v_1$ and $Ju = (-u_2, u_1)$ for vectors $u, v \in \mathbb{R}^2$. For $0 < \lambda < b^2$ let

$$\mathcal{C} = \mathcal{C}_\lambda : \frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1, \quad a'^2 = a^2 - \lambda, \quad b'^2 = b^2 - \lambda.$$

It is confocal with $\mathcal{E}$ ($a'^2 - b'^2 = c^2$), lies inside $\mathcal{E}$, and O, f_1, f_2 are interior points of $\mathcal{C}$ (as $c < a'$). A line $ux + vy = 1$ is tangent to $\mathcal{C}$ if and only if $a'^2u^2 + b'^2v^2 = 1$, and then it touches $\mathcal{C}$ at (a'^2u, b'^2v) . A *billiard trajectory* is a sequence $(P_k)_{k \in \mathbb{Z}}$ of points of $\mathcal{E}$ with $P_{k+1} \neq P_k$ such that at every P_k the normal of $\mathcal{E}$ bisects the angle between the chords P_kP_{k-1} and P_kP_{k+1} (the reflection law); its caustic is $\mathcal{C}$ if every line P_kP_{k+1} is tangent to $\mathcal{C}$. The two tangents from a point $X \in \mathcal{E}$ to $\mathcal{C}$ are symmetric with respect to the normal of $\mathcal{E}$ at X [Sta21, Lemma 2.1]; hence if one chord of a billiard trajectory is tangent to $\mathcal{C}$, so are all. We call a trajectory with caustic $\mathcal{C}$ an *N -periodic* if it returns to its first vertex after N reflections and not earlier, and we view it as the polygon $P = (P_1, \dots, P_N)$, indices modulo N . The source does not spell out a period convention; least period N is our convention. It plays no role in the counterexample, because a family is determined by its caustic and both members exhibited in Section 3 have six distinct vertices. By Lemma 4.4, either all members of a family are convex polygons or none is; we consider convex families and list their members counterclockwise. Reversing the labels changes the sign of every signed area used below, and none of our statements is affected by this.

Polygons. The signed area of a polygon $W = (W_1, \dots, W_N)$ is $S(W) = \frac{1}{2} \sum_{i=1}^N W_i \times W_{i+1}$, as in [RGK21]. The *outer polygon* P' of an N -periodic P has the vertices $P'_i = T_{P_i} \cap T_{P_{i+1}}$, where T_X is the tangent of $\mathcal{E}$ at X ; these intersections exist since no chord tangent to $\mathcal{C}$ passes through O . We put $A = S(P)$ and $A' = S(P')$. For a point M let ℓ_i be the line through P_i perpendicular to $P_i - M$, that is,

$$\ell_i = \{X : (X - P_i) \cdot (P_i - M) = 0\} \quad (P_i \neq M).$$

The *antipedal polygon* $\text{Ant}(P; M)$ has the vertices $Q_i^* = \ell_i \cap \ell_{i+1}$; it is defined if no two consecutive lines are parallel, that is, if M lies on no side line P_iP_{i+1} . We write $A_M^* = S(\text{Ant}(P; M))$. This is the construction of [RGK21, v11], where the antipedal polygon is formed by the intersections of rays shot

from the P_i along $(P_i - M)^\perp$, and A_m^* denotes its area in the tables and the list of symbols (the text writes A_m). Since every side line of an N -periodic is tangent to $\mathcal{C}$ and O , f_1, f_2 are interior points of $\mathcal{C}$, the polygons $\text{Ant}(P; O)$ and $\text{Ant}(P; f_j)$ are always defined. Numbering the vertices differently, for instance calling $\ell_{i-1} \cap \ell_i$ vertex i , relabels the polygon cyclically and does not change its signed area.

3 The counterexample

For $N = 6$ the family contains two members that are symmetric in both axes, one with a vertex on each axis, and everything about them can be written down. The caustic parameter

$$\lambda_6 = \frac{a^2 b^2}{(a+b)^2}$$

is classical (the source cites Ramírez-Ros [RR14] for explicit caustic parameters with $N = 3, 4, 6, 8$); Proposition 3.1 derives what we need from it directly.

Proposition 3.1. *Let $a > b > 0$, $\lambda = \lambda_6$, $s = \sqrt{b(2a+b)}$ and $t = \sqrt{a(a+2b)}$. Then $a'' = at/(a+b)$, $b'' = bs/(a+b)$, and the hexagons*

$$\begin{aligned} \text{I: } & (a, 0), (x_1, b''), (-x_1, b''), (-a, 0), (-x_1, -b''), (x_1, -b''), & x_1 &= \frac{a^2}{a+b}, \\ \text{II: } & (0, b), (-a'', y_2), (-a'', -y_2), (0, -b), (a'', -y_2), (a'', y_2), & y_2 &= \frac{b^2}{a+b}, \end{aligned}$$

are 6-periodics with caustic $\mathcal{C}_\lambda$, convex and listed counterclockwise. Their outer polygons have the vertices $(\pm a, \pm b^2/s)$, $(0, \pm b(a+b)/s)$, respectively $(\pm a^2/t, \pm b)$, $(\pm a(a+b)/t, 0)$, and their antipedal polygons with respect to O have the vertices

$$\begin{aligned} & \left(\pm a, \pm \frac{b^2 + ab - a^2}{s} \right), \left(0, \pm \frac{a^3 - a^2b + ab^2 + b^3}{bs} \right) \quad \text{for I,} \\ & \left(\pm \frac{a^2 + ab - b^2}{t}, \pm b \right), \left(\pm \frac{a^3 + a^2b - ab^2 + b^3}{at}, 0 \right) \quad \text{for II,} \end{aligned}$$

and

$$\begin{aligned} A'A_O^*(\text{I}) &= \frac{4a^2(a+2b)(a^3 - 2a^2b + 2ab^2 + 2b^3)}{b(2a+b)}, & A'A_O^*(\text{II}) &= \frac{4b^2(2a+b)(2a^3 + 2a^2b - 2ab^2 + b^3)}{a(a+2b)}, \\ A'A(\text{I}) = A'A(\text{II}) &= \frac{4a^2b^2(2a+b)(a+2b)}{(a+b)^2}, & A'A_O^*(\text{I}) - A'A_O^*(\text{II}) &= \frac{4(a-b)^3(a+b)^5}{ab(2a+b)(a+2b)}. \end{aligned} \tag{2}$$

Proof. From $a''^2 = a^2 - \lambda = a^3(a+2b)/(a+b)^2$ and $b''^2 = b^3(2a+b)/(a+b)^2$ we get a'' and b'' . The vertices lie on $\mathcal{E}$: $x_1^2/a^2 + b''^2/b^2 = (a^2 + b(2a+b))/(a+b)^2 = 1$, and similarly for II. The sides of I are the lines $y = \pm b''$, which touch $\mathcal{C}$ at $(0, \pm b'')$, and the images under the reflections in the axes of the line through $(a, 0)$ and (x_1, b'') , which is $ux + vy = 1$ with $u = 1/a$, $v = b/((a+b)b'')$. Since $a''^2u^2 + b''^2v^2 = a(a+2b)/(a+b)^2 + b^2/(a+b)^2 = 1$, this line touches $\mathcal{C}$, at $(a''^2/a, bb''/(a+b))$, a point of the side between $(a, 0)$ and (x_1, b'') because its ordinate lies strictly between 0 and b'' . For II, the sides $x = \pm a''$ touch $\mathcal{C}$ at $(\pm a'', 0)$, and the line through $(0, b)$ and $(-a'', y_2)$ is $ux + vy = 1$ with $u = -a/((a+b)a'')$, $v = 1/b$, which touches $\mathcal{C}$ at $(-aa''/(a+b), b''^2/b)$, strictly between the two vertices. Both hexagons are convex, since their vertices follow each other counterclockwise on $\mathcal{E}$. At each vertex the two sides are the two tangents from the vertex to $\mathcal{C}$; they are symmetric with respect

to the normal of $\mathcal{E}$ [Sta21, Lemma 2.1], and both sides run into the interior of $\mathcal{E}$, so the reflection law holds. Thus I and II are billiard trajectories with caustic $\mathcal{C}$, of least period 6; by Poncelet's porism (or Lemma 4.4) every trajectory with caustic $\mathcal{C}$ is a convex 6-periodic.

The outer polygons: the tangent of $\mathcal{E}$ at (x_1, b'') is $x/(a+b) + yb''/b^2 = 1$, that is, $x + ys/b = a + b$, and it meets $x = a$ at $(a, b^2/s)$ and its mirror image in the y -axis at $(0, b(a+b)/s)$; the outer polygon of II follows in the same way. The antipedal lines with respect to O are $X \cdot P_i = |P_i|^2$. For I, ℓ_1 is $x = a$ and ℓ_2 is $x_1x + b''y = x_1^2 + b''^2$; they meet at ordinate $(x_1^2 - ax_1 + b''^2)/b'' = (b^3 + 2ab^2 - a^3)/((a+b)^2b''/b) = (b^2 + ab - a^2)/s$, using $b^3 + 2ab^2 - a^3 = (a+b)(b^2 + ab - a^2)$, and ℓ_2 meets its mirror image ℓ_3 at ordinate $(x_1^2 + b''^2)/b'' = (a^3 - a^2b + ab^2 + b^3)/(bs)$, using $a^4 + 2ab^3 + b^4 = (a+b)(a^3 - a^2b + ab^2 + b^3)$. The vertices for II are obtained in the same way. Each of these hexagons has the form $(\xi, \eta_1), (0, \eta_2), (-\xi, \eta_1), (-\xi, -\eta_1), (0, -\eta_2), (\xi, -\eta_1)$, or the same with the coordinates exchanged and the orientation kept, and the shoelace formula gives the area $2\xi(\eta_1 + \eta_2)$. Hence

$$\begin{aligned} A(\text{I}) &= \frac{2abs(2a+b)}{(a+b)^2}, & A'(\text{I}) &= \frac{2ab(a+2b)}{s}, & A_O^*(\text{I}) &= \frac{2a(a^3 - 2a^2b + 2ab^2 + 2b^3)}{bs}, \\ A(\text{II}) &= \frac{2abt(a+2b)}{(a+b)^2}, & A'(\text{II}) &= \frac{2ab(2a+b)}{t}, & A_O^*(\text{II}) &= \frac{2b(2a^3 + 2a^2b - 2ab^2 + b^3)}{at}, \end{aligned}$$

where $A(\text{I}) = 2b''(a + x_1)$ is the area of the rectangle $[-x_1, x_1] \times [-b'', b'']$ plus two triangles. With $s^2 = b(2a+b)$ and $t^2 = a(a+2b)$ this gives the first three formulas of (2). For the difference, after multiplication by $ab(2a+b)(a+2b)/4$ we need

$$a^3(a+2b)^2(a^3 - 2a^2b + 2ab^2 + 2b^3) - b^3(2a+b)^2(2a^3 + 2a^2b - 2ab^2 + b^3) = (a-b)^3(a+b)^5.$$

The first product is $a^8 + 2a^7b - 2a^6b^2 + 2a^5b^3 + 16a^4b^4 + 8a^3b^5$, the second is the same polynomial with a and b exchanged, and their difference $a^8 + 2a^7b - 2a^6b^2 - 6a^5b^3 + 6a^3b^5 + 2a^2b^6 - 2ab^7 - b^8$ is the expansion of $(a-b)^3(a+b)^5$. $\square$

Proof of Theorem 1. For $a = \frac{8}{5}$, $b = 1$ we have $\lambda_6 = \frac{64}{169}$, $a'' = \frac{96}{65}$, $b''^2 = \frac{105}{169}$, $s = \sqrt{105}/5$ and $t = \frac{12}{5}$, and Proposition 3.1 gives the data of Table 1. In particular $A'O^*$ takes the values $\frac{801792}{21875}$ and $\frac{3241}{100}$, whose difference is $\frac{371293}{87500} = 13^5/(2^2 \cdot 5^5 \cdot 7) \neq 0$. For every $a > b > 0$ the last formula of (2) is positive. $\square$

member	vertices	A	A'	A_O^*	$A'O_O^*$	$A'A_{f_j}^*$
I	$(\pm \frac{8}{5}, 0), (\pm \frac{64}{65}, \pm \frac{\sqrt{105}}{13})$	$\frac{336\sqrt{105}}{845}$	$\frac{96\sqrt{105}}{175}$	$\frac{2784\sqrt{105}}{4375}$	$\frac{801792}{21875}$	$\frac{9216}{625}$
II	$(0, \pm 1), (\pm \frac{96}{65}, \pm \frac{5}{13})$	$\frac{3456}{845}$	$\frac{28}{5}$	$\frac{463}{80}$	$\frac{3241}{100}$	$\frac{9216}{625}$

Table 1: The two members of Theorem 1 ($a = \frac{8}{5}$, $b = 1$, $\lambda = \frac{64}{169}$). The outer polygon of II has the vertices $(\pm \frac{16}{15}, \pm 1), (\pm \frac{26}{15}, 0)$, its antipedal polygon with respect to O the vertices $(\pm \frac{79}{60}, \pm 1), (\pm \frac{757}{480}, 0)$ (for I see Proposition 3.1). The focal antipedal areas are $A_{f_1}^* = A_{f_2}^* = \frac{32\sqrt{105}}{125}$ (I) and $\frac{2304}{875}$ (II), both equal to κA with $\kappa = \frac{338}{525}$; the foci are $(\pm \sqrt{39}/5, 0)$.

The exact program of this note (Section 6) generates both members from their first vertex by the reflection law, in exact arithmetic in $\mathbb{Q}(\sqrt{105}, \sqrt{39})$, and certifies every entry of Table 1: the vertices lie on $\mathcal{E}$, every side touches $\mathcal{C}$ at a point strictly inside the side, the reflection law holds at every vertex, the trajectory returns to its first vertex after six reflections and not earlier, and the hexagon is convex. Member II is rational, so its data can also be checked with fractions alone.

Remark 3.2 (other points and other N). The same computation reproduces two rows that the source lists as proved, at both members: $A'A = \frac{96768}{4225}$ (row k_{106} , [CC21]) and $A'/A'' = (ab/(a''b''))^2 = \frac{28561}{15120}$ (row k_{113}), where A'' is the area of the polygon of contact points. Reading the row with the antipedal polygon of P' instead of P does not save it: at $M = O$ the two members give $A'S(\text{Ant}(P'; O)) = \frac{271752}{6125}$ and $\frac{264992}{5625}$. Numerically, with families generated by the reflection law in 50-digit arithmetic, $A'A_M^*$ is not constant at $M = O$ and at $M = (0.31, 0.17)$ for $N = 3, \dots, 10$ and $a/b = 1.6, 2, 3, 1.15$; for instance, for $a = 1.6$, $b = 1$ its spread over six members is 4.21 and 4.16 for $N = 6$, and 0.0565 and 0.0551 for $N = 10$, against less than 10^{-45} at the foci. The first two verification runs (Section 6) found the same at three non-focal points, for N up to 18 in five ellipses. For odd N the spread at O is small, though far above the rounding level; in the same computations A_O^*/A is constant for odd N to 45 digits, while $A'A$ is not. The constancy of A_O^*/A for odd N follows from that of AA_O and of $A_OA_O^*$ (rows $k_{203,b}$ and $k_{403,a}$ of the source), where $A_O > 0$ is the signed area of the pedal polygon of P with respect to O ; these are proved in the unrefered notes [Fer26a] (for N odd or divisible by 4) and [Kri26] (for odd N). The other observations of this remark are not proved.

4 Proof of Theorem 2

4.1 Antipedal areas side by side

Lemma 4.1. *Let $W = (W_1, \dots, W_N)$ be a polygon with $W_{i+1} \neq W_i$ and let f be a point on no side line W_iW_{i+1} . For side i put $L_i = |W_{i+1} - W_i|$, $t_i = (W_{i+1} - W_i)/L_i$, $n_i = Jt_i$, $h_i = (f - W_i) \cdot n_i$, $s_i = (W_i - f) \cdot t_i$, $s'_i = (W_{i+1} - f) \cdot t_i$ and $T_i = \frac{1}{2}(W_i - f) \times (W_{i+1} - f) = \frac{1}{2}h_iL_i$. Then $\text{Ant}(W; f)$ is defined and*

$$S(\text{Ant}(W; f)) = \sum_{i=1}^N T_i \left(1 - \frac{s_i s'_i}{h_i^2}\right).$$

Here $h_i \neq 0$ is the signed distance of f from the line W_iW_{i+1} , and $s_i, s'_i = s_i + L_i$ are the abscissae of W_i, W_{i+1} on this line, measured from the foot of the perpendicular from f .

Proof. Put $w_i = W_i - f$, $d_i = |w_i| > 0$, $e_i = w_i/d_i$, and let Δ_i be the oriented angle from e_i to e_{i+1} , so that $e_{i+1} = \cos \Delta_i e_i + \sin \Delta_i J e_i$. Since $w_i \times w_{i+1} = 2T_i = h_i L_i \neq 0$, $\sin \Delta_i \neq 0$. In coordinates centred at f the antipedal line through W_i is $\ell_i = \{d_i e_i + \tau J e_i : \tau \in \mathbb{R}\}$, so consecutive lines are not parallel and $Q_i = Q_i^* - f$ is defined. Write $Q_i = d_i e_i + \beta_i J e_i$ and $Q_{i-1} = d_i e_i + \alpha_i J e_i$ (both lie on ℓ_i). The conditions $Q_i \cdot e_{i+1} = d_{i+1}$ and $Q_i \cdot e_i = d_i$, the latter written with $Q_i = d_{i+1} e_{i+1} + \alpha_{i+1} J e_{i+1}$ and $J e_{i+1} \cdot e_i = -\sin \Delta_i$, give

$$\beta_i = \frac{d_{i+1} - d_i \cos \Delta_i}{\sin \Delta_i}, \quad \alpha_{i+1} = \frac{d_{i+1} \cos \Delta_i - d_i}{\sin \Delta_i}.$$

As $(d_i e_i + \alpha J e_i) \times (d_i e_i + \beta J e_i) = d_i(\beta - \alpha)$, the signed area (which is translation invariant) is

$$\begin{aligned} S(\text{Ant}(W; f)) &= \frac{1}{2} \sum_i Q_{i-1} \times Q_i = \frac{1}{2} \sum_i d_i(\beta_i - \alpha_i) = \frac{1}{2} \sum_i (d_i \beta_i - d_{i+1} \alpha_{i+1}) \\ &= \sum_i \frac{2d_i d_{i+1} - (d_i^2 + d_{i+1}^2) \cos \Delta_i}{2 \sin \Delta_i}. \end{aligned}$$

Multiply numerator and denominator of the i -th term by $d_i d_{i+1}$ and use $d_i d_{i+1} \cos \Delta_i = w_i \cdot w_{i+1}$, $d_i d_{i+1} \sin \Delta_i = 2T_i$. In the frame (t_i, n_i) we have $w_i = s_i t_i - h_i n_i$ and $w_{i+1} = s'_i t_i - h_i n_i$, hence

$w_i \cdot w_{i+1} = h_i^2 + s_i s'_i$, $d_i^2 = h_i^2 + s_i^2$, $d_{i+1}^2 = h_i^2 + s_i'^2$, and the i -th term becomes

$$\begin{aligned} & \frac{2(h^2 + s^2)(h^2 + s'^2) - (2h^2 + s^2 + s'^2)(h^2 + ss')}{4T_i} \\ &= \frac{(s' - s)^2(h^2 - ss')}{4T_i} = \frac{L_i^2(h_i^2 - s_i s'_i)}{2h_i L_i} = T_i \left(1 - \frac{s_i s'_i}{h_i^2}\right), \end{aligned}$$

with $h = h_i$, $s = s_i$, $s' = s'_i$; the middle identity is checked by expanding. $\square$

Note that $\sum_i T_i = \frac{1}{2} \sum_i W_i \times W_{i+1} = S(W)$ for every point f , since the terms containing f cancel in the cyclic sum. The expression of the antipedal area as the sum of the terms $(2d_i d_{i+1} - (d_i^2 + d_{i+1}^2) \cos \Delta_i) / (2 \sin \Delta_i)$ is also used in [Fer26c]; the form $T_i(1 - s_i s'_i / h_i^2)$ is what lets us compare the two foci side by side.

4.2 The two foci on one side

From now on P is a member of a convex N -periodic family with caustic $\mathcal{C} = \mathcal{C}_\lambda$, listed counterclockwise. By Lemma 4.4(ii) below, every side runs in the counterclockwise sense of $\mathcal{C}$ along its tangent, so that $\mathcal{C}$, f_1 and f_2 lie to the left of every side.

Lemma 4.2. *Let a side of P go from P_i to P_{i+1} , with unit direction t , $n = Jt$, length $L = l + r$, where $l = |Q - P_i|$, $r = |P_{i+1} - Q|$ and Q is the point of contact with $\mathcal{C}$. Let $\rho_j = |Q - f_j|$, let $h^{(j)}$, $s^{(j)}$, $s'^{(j)}$, $T^{(j)}$ be the quantities of Lemma 4.1 for this side and the point f_j , and let $\omega \in (0, \pi)$ be the angle with $(f_1 - Q)/\rho_1 = \cos \omega t + \sin \omega n$. Then*

$$\begin{aligned} h^{(j)} &= \rho_j \sin \omega, \quad \rho_1 \rho_2 \sin^2 \omega = h^{(1)} h^{(2)} = b''^2, \\ \sum_{j=1}^2 T^{(j)} \frac{s^{(j)} s'^{(j)}}{h^{(j)2}} &= a'' L \left(\frac{\cos^2 \omega}{\sin \omega} - \frac{lr \sin \omega}{b''^2} \right), \quad \sum_{j=1}^2 T^{(j)} = a'' L \sin \omega. \end{aligned}$$

Proof. Q lies strictly between P_i and P_{i+1} , since Q is inside $\mathcal{E}$ and the side is the whole chord of $\mathcal{E}$ on its line. The foci of $\mathcal{C}$ are f_1, f_2 , and the normal of $\mathcal{C}$ at Q bisects the angle $f_1 Q f_2$ (reflection property of the ellipse $\mathcal{C}$). So the unit vectors $(f_1 - Q)/\rho_1$ and $(f_2 - Q)/\rho_2$ are mirror images in the normal line: their n -components are equal and their t -components opposite. Since the foci lie to the left of the side, the common n -component is positive; this defines $\omega \in (0, \pi)$, and $(f_2 - Q)/\rho_2 = -\cos \omega t + \sin \omega n$. Hence $h^{(j)} = (f_j - Q) \cdot n = \rho_j \sin \omega$. If $Q = (a'' \cos \theta, b'' \sin \theta)$, the tangent is $x \cos \theta / a'' + y \sin \theta / b'' = 1$, and the product of the distances of $(\pm c, 0)$ from it is

$$\frac{1 - c^2 \cos^2 \theta / a''^2}{\cos^2 \theta / a''^2 + \sin^2 \theta / b''^2} = \frac{b''^2 (a''^2 - c^2 \cos^2 \theta)}{b''^2 \cos^2 \theta + a''^2 \sin^2 \theta} = b''^2,$$

because $a''^2 - c^2 \cos^2 \theta = a''^2 \sin^2 \theta + b''^2 \cos^2 \theta$; this is $h^{(1)} h^{(2)}$.

Let $q_j = (f_j - Q) \cdot t$, so $q_1 = \rho_1 \cos \omega$, $q_2 = -\rho_2 \cos \omega$. With $P_i = Q - lt$ and $P_{i+1} = Q + rt$ we get $s^{(j)} = -l - q_j$, $s'^{(j)} = r - q_j$ and $s^{(j)} s'^{(j)} = q_j^2 + (l - r)q_j - lr$. Since $T^{(j)} / h^{(j)2} = L / (2h^{(j)})$,

$$\sum_j T^{(j)} \frac{s^{(j)} s'^{(j)}}{h^{(j)2}} = \frac{L}{2} \left(\sum_j \frac{q_j^2}{h^{(j)}} + (l - r) \sum_j \frac{q_j}{h^{(j)}} - lr \sum_j \frac{1}{h^{(j)}} \right),$$

and $\sum_j q_j^2 / h^{(j)} = (\rho_1 + \rho_2) \cos^2 \omega / \sin \omega = 2a'' \cos^2 \omega / \sin \omega$, $\sum_j q_j / h^{(j)} = \cot \omega - \cot \omega = 0$, $\sum_j 1 / h^{(j)} = (\rho_1 + \rho_2) / (\rho_1 \rho_2 \sin \omega) = 2a'' \sin \omega / b''^2$, using $\rho_1 + \rho_2 = 2a''$. Finally $\sum_j T^{(j)} = \frac{1}{2} L (h^{(1)} + h^{(2)}) = a'' L \sin \omega$. $\square$

Proposition 4.3. *For every $\mu \in \mathbb{R}$, with L_i, l_i, r_i, ω_i the quantities of Lemma 4.2 for side i ,*

$$A_{f_1}^* + A_{f_2}^* - 2(2 - \mu)A = -a'' \sum_{i=1}^N L_i \left(\frac{1}{\sin \omega_i} - \frac{l_i r_i \sin \omega_i}{b''^2} - \mu \sin \omega_i \right).$$

Proof. By Lemma 4.1 and $\sum_i T_i^{(j)} = A$, $A_{f_1}^* + A_{f_2}^* = 2A - \sum_i \sum_j T_i^{(j)} s_i^{(j)} s_i'^{(j)} / h_i^{(j)2}$, and by Lemma 4.2 the double sum equals $a'' \sum_i L_i (\cos^2 \omega_i / \sin \omega_i - l_i r_i \sin \omega_i / b''^2)$, while $2A = \sum_i \sum_j T_i^{(j)} = a'' \sum_i L_i \sin \omega_i$. Write $2A - 2(2 - \mu)A = (\mu - 1) a'' \sum_i L_i \sin \omega_i$ and use $\sin \omega + \cos^2 \omega / \sin \omega = 1 / \sin \omega$. $\square$

4.3 The parametrisation

Let $m = c/a'' \in (0, 1)$, $m' = \sqrt{1 - m^2} = b''/a''$, and let sn , cn , dn be the Jacobi elliptic functions of modulus m , with the complete integrals $K = K(m)$ and $K' = K(m')$ [DLM26, Chapter 22]. We use the following standard facts [DLM26, §§22.4, 22.8]. The functions sn , cn have the period $4K$ and $\text{sn}(u + 2K) = -\text{sn } u$, $\text{cn}(u + 2K) = -\text{cn } u$; sn^2 has the periods $2K, 2iK'$; dn has the periods $2K, 4iK'$ and $\text{dn}(u + 2iK') = -\text{dn } u$; cn and dn are even, sn is odd. The poles of sn , cn , dn are the points $2pK + (2q+1)iK'$, $p, q \in \mathbb{Z}$; they are simple, with $\text{dn } u \sim -i/(u - iK')$ and $\text{sn } u \sim 1/(m(u - iK'))$ at iK' . Moreover $\text{sn}(u + iK') = 1/(m \text{sn } u)$, $\text{cn}(u + iK') = -i \text{dn } u / (m \text{sn } u)$ and $\text{dn}(u + iK') = -i \text{cn } u / \text{sn } u$. For real u , $\text{sn } u = \sin \text{am } u$ and $\text{cn } u = \cos \text{am } u$, where am is increasing with $\text{am}(u + 2K) = \text{am } u + \pi$, and $\text{dn } u = (1 - m^2 \text{sn}^2 u)^{1/2} > 0$. Finally, the addition theorems give

$$\text{cn } x \text{cn } y + \text{sn } x \text{sn } y \text{dn}(x - y) = \text{cn}(x - y); \quad (3)$$

indeed, with $D' = 1 - m^2 \text{sn}^2 x \text{sn}^2 y$ the addition theorems give $\text{cn}(x - y) = (\text{cn } x \text{cn } y + \text{sn } x \text{sn } y \text{dn } x \text{dn } y) / D'$ and $\text{dn}(x - y) = (\text{dn } x \text{dn } y + m^2 \text{sn } x \text{sn } y \text{cn } x \text{cn } y) / D'$, and inserting the second turns the left side into $(\text{cn } x \text{cn } y + \text{sn } x \text{sn } y \text{dn } x \text{dn } y) / D'$.

Since $0 < b''/b < 1$, there is a unique $\Delta \in (0, K)$ with $\text{cn } \Delta = b''/b$. Put $\sigma = \text{sn}^2 \Delta$, $\gamma = \text{cn}^2 \Delta$, $\delta = \text{dn } \Delta$; thus $\sigma + \gamma = 1$, $\delta^2 = 1 - m^2 \sigma$. Let

$$P(u) = (-a \text{sn } u, b \text{cn } u), \quad Q(u) = (-a'' \text{sn } u, b'' \text{cn } u) \quad (u \in \mathbb{R}).$$

These are bijections of $\mathbb{R}/4K\mathbb{Z}$ onto $\mathcal{E}$ and $\mathcal{C}$, both running counterclockwise, with eccentric anomaly $\text{am } u + \pi/2$.

Lemma 4.4 (Stachel's parametrisation). *(i) $a = a''\delta/\text{cn } \Delta$, $b = b''/\text{cn } \Delta$ and $\lambda = b''^2\sigma/\gamma$.*

(ii) For every real v the line through $P(v - \Delta)$ and $P(v + \Delta)$ is tangent to $\mathcal{C}$ at $Q(v)$, and $P(v + \Delta) - P(v - \Delta)$ is a positive multiple of $(-a'' \text{cn } v, -b'' \text{sn } v)$, the tangent vector of $\mathcal{C}$ at $Q(v)$ in the counterclockwise sense.

(iii) The vertices of a billiard trajectory with caustic $\mathcal{C}$ are $P_k = P(u_0 + 2\epsilon k\Delta)$, $k \in \mathbb{Z}$, for some $u_0 \in \mathbb{R}$ and $\epsilon \in \{1, -1\}$. It is periodic if and only if Δ/K is rational; if $\Delta/(2K) = \tau/N$ with coprime positive integers τ, N , its least period is N and its vertices go τ times around $\mathcal{E}$, and the polygon is convex if and only if $\tau = 1$. In particular a member of a convex N -periodic family, listed counterclockwise, has the vertices $P_i = P(u_0 + 2(i-1)\Delta)$, $i = 1, \dots, N$, its side from P_i to P_{i+1} touches $\mathcal{C}$ at $Q(v_i)$ with $v_i = u_0 + (2i-1)\Delta$, and $\Delta = 2K/N$.

Proof. (i) $b = b''/\text{cn } \Delta$ is the definition of Δ , and $\lambda = b^2 - b''^2 = b''^2(1 - \gamma)/\gamma$. Further $a''^2\delta^2/\gamma = (a''^2 - c^2\sigma)/\gamma$, while $a^2 = a''^2 + \lambda = (a''^2\gamma + b''^2\sigma)/\gamma = (a''^2 - c^2\sigma)/\gamma$.

(ii) The tangent of $\mathcal{C}$ at $Q(v)$ is $-x \operatorname{sn} v / a'' + y \operatorname{cn} v / b'' = 1$. For $w = v \pm \Delta$, inserting $P(w)$ and using (i) gives $(\delta \operatorname{sn} v \operatorname{sn} w + \operatorname{cn} v \operatorname{cn} w) / \operatorname{cn} \Delta$, which equals $\operatorname{cn}(v - w) / \operatorname{cn} \Delta = 1$ by (3), since $\operatorname{dn}(v - w) = \delta$ and $\operatorname{cn}(v - w) = \operatorname{cn} \Delta$. The points $P(v \pm \Delta)$ are distinct because $0 < 2\Delta < 4K$. By the addition theorems for $\operatorname{sn}(v \pm \Delta)$ and $\operatorname{cn}(v \pm \Delta)$, with $D = 1 - m^2 \operatorname{sn}^2 v \operatorname{sn}^2 \Delta > 0$, and by (i),

$$P(v + \Delta) - P(v - \Delta) = \frac{2 \operatorname{sn} \Delta \operatorname{dn} v}{D} (-a \operatorname{cn} v, -b \delta \operatorname{sn} v) = \frac{2 \operatorname{sn} \Delta \delta \operatorname{dn} v}{D \operatorname{cn} \Delta} (-a'' \operatorname{cn} v, -b'' \operatorname{sn} v),$$

and the factor is positive; $(-a'' \operatorname{cn} v, -b'' \operatorname{sn} v) = Q'(v) / \operatorname{dn} v$.

(iii) Let $P_0 = P(u_0)$. By (ii) the chords from $P(u_0)$ to $P(u_0 \pm 2\Delta)$ are tangent to $\mathcal{C}$ at the distinct points $Q(u_0 \pm \Delta)$, so they are the two tangents from P_0 to $\mathcal{C}$, and $P_1 = P(u_0 + 2\varepsilon\Delta)$ for some $\varepsilon = \pm 1$. If $P_{k-1} = P(u - 2\varepsilon\Delta)$ and $P_k = P(u)$, the reflection law and [Sta21, Lemma 2.1] show that the next chord is the other tangent from $P(u)$, which by (ii) ends at $P(u + 2\varepsilon\Delta)$. By induction, in both directions of k , $P_k = P(u_0 + 2\varepsilon k\Delta)$. As P is injective on $\mathbb{R}/4K\mathbb{Z}$, $P_k = P_0$ if and only if $2k\Delta \in 4K\mathbb{Z}$, which gives the statements on periodicity and the least period. For $\varepsilon = 1$ the eccentric anomaly increases at each step by $\operatorname{am}(u + 2\Delta) - \operatorname{am} u \in (0, \pi)$ and by $2\pi\tau$ over N steps, so the vertices follow each other counterclockwise and go τ times around $\mathcal{E}$; the polygon is convex if $\tau = 1$ and not convex if $\tau \geq 2$. Since Δ depends only on $\mathcal{C}$, this holds for all members alike. For a convex member listed counterclockwise, $\varepsilon = 1$, $\tau = 1$ and $\Delta = 2K/N$; the contact points follow from (ii). $\square$

Part (i), the tangency statement of (ii) and the vertex parameters $u_0 + 2k\Delta$ are Stachel's [Sta22, Theorem 2 and its proof]; in his proof the first contact point has the parameter Δ when the first vertex has the parameter 0. His part (2) states that $\Delta = 2\tau K/N$ gives N -periodic trajectories with turning number τ ; the converse used here, that a convex N -periodic family has $\Delta = 2K/N$, follows from his part (1) and the injectivity of P on $\mathbb{R}/4K\mathbb{Z}$, as in the proof of (iii).

4.4 One elliptic function

Lemma 4.5. *For the side with contact point $Q(v)$, v real, put $x = \operatorname{sn}^2 v$. Then*

$$\sin \omega = \frac{m'}{\operatorname{dn} v}, \quad L = \frac{2a''\delta\sqrt{\sigma/\gamma} \operatorname{dn}^2 v}{1 - m^2\sigma x}, \quad lr = \frac{\sigma a''^2 \operatorname{dn}^2 v (\delta^2 - m^2\gamma x)}{\gamma(1 - m^2\sigma x)}.$$

Proof. For $Q = Q(v)$ the focal distances are $\rho_{1,2} = a'' \pm c \operatorname{sn} v$, so $\rho_1 \rho_2 = a''^2 (1 - m^2 \operatorname{sn}^2 v) = a''^2 \operatorname{dn}^2 v$, and Lemma 4.2 gives $\sin^2 \omega = b''^2 / (a''^2 \operatorname{dn}^2 v)$. Let $\tilde{\tau} = (-a'' \operatorname{cn} v, -b'' \operatorname{sn} v)$, so $|\tilde{\tau}| = a'' \operatorname{dn} v$, and let $\mathcal{A} = \operatorname{diag}(a^{-2}, b^{-2})$; by Lemma 4.4(i), $a^{-2} = \gamma / (a''^2 \delta^2)$ and $b^{-2} = \gamma / b''^2$. The points $Q + \eta \tilde{\tau}$ of $\mathcal{E}$ are the roots of $\alpha \eta^2 + 2\beta \eta + \gamma_0 = 0$ with

$$\alpha = \tilde{\tau}^\top \mathcal{A} \tilde{\tau} = \frac{\gamma(1 - m^2\sigma x)}{\delta^2}, \quad \beta = \tilde{\tau}^\top \mathcal{A} Q = \frac{\gamma m^2 \sigma \operatorname{sn} v \operatorname{cn} v}{\delta^2}, \quad -\gamma_0 = 1 - Q^\top \mathcal{A} Q = \frac{\sigma(\delta^2 - m^2\gamma x)}{\delta^2},$$

where we used $1 - \delta^2 = m^2\sigma$ and $1 - \gamma = \sigma$. The roots have opposite signs, so $lr = -\gamma_0 |\tilde{\tau}|^2 / \alpha$ and $L = 2|\tilde{\tau}| \sqrt{\beta^2 - \alpha\gamma_0} / \alpha$. Expanding,

$$\begin{aligned} \delta^4 (\beta^2 - \alpha\gamma_0) / \gamma &= \gamma m^4 \sigma^2 x (1 - x) + \sigma (1 - m^2\sigma x) (\delta^2 - m^2\gamma x) \\ &= \sigma \delta^2 + m^2 \sigma x (\gamma m^2 \sigma - \gamma - \sigma \delta^2) = \sigma \delta^2 (1 - m^2 x), \end{aligned}$$

because $\gamma m^2 \sigma - \gamma = -\gamma \delta^2$ and $\gamma + \sigma = 1$. So $\beta^2 - \alpha\gamma_0 = \gamma \sigma \operatorname{dn}^2 v / \delta^2$, and the formulas for L and lr follow. $\square$

For $u \in \mathbb{C}$ let $x = \operatorname{sn}^2 u$ and, for a real parameter μ ,

$$\varphi(u) = 2\delta \operatorname{dn} u \left(\frac{(1 - m^2 x)(\gamma - \sigma \delta^2)}{\gamma(1 - m^2 \sigma x)^2} - \frac{\mu m'^2}{1 - m^2 \sigma x} \right). \quad (4)$$

Lemma 4.6. *For every side, with contact point $Q(v)$, $L(1/\sin \omega - lr \sin \omega/b'^2 - \mu \sin \omega) = (a''\sqrt{\sigma/\gamma/m'})\varphi(v)$. Consequently, for every member of a convex N -periodic family,*

$$A_{f_1}^* + A_{f_2}^* - 2(2 - \mu)A = -\frac{a''^2}{m'}\sqrt{\frac{\sigma}{\gamma}}S(v_1), \quad S(v) = \sum_{i=0}^{N-1} \varphi(v + 2i\Delta). \quad (5)$$

Proof. By Lemma 4.5 and $b'' = a''m'$, the bracket equals $(\operatorname{dn}^2 v (1 - \sigma(\delta^2 - m^2\gamma x)/(\gamma(1 - m^2\sigma x))) - \mu m'^2)/ (m' \operatorname{dn} v)$, and $1 - \sigma(\delta^2 - m^2\gamma x)/(\gamma(1 - m^2\sigma x)) = (\gamma - \sigma\delta^2)/(\gamma(1 - m^2\sigma x))$. Multiplying by L gives the first claim, with $\operatorname{dn}^2 v = 1 - m^2x$. The second follows from Proposition 4.3 and Lemma 4.4(iii), since the contact parameters of the sides are $v_1 + 2i\Delta$, $i = 0, \dots, N-1$. $\square$

4.5 Residues

Let $\Lambda = 2K\mathbb{Z} + 4iK'\mathbb{Z}$, $x_0 = 1/(m^2\sigma) > 1$, and $p_{\pm} = \pm\Delta + iK'$.

Lemma 4.7. *Let $N \geq 3$ and $\Delta = 2K/N$. The function φ is meromorphic on $\mathbb{C}$, even, Λ -periodic, and $\varphi(u + 2iK') = -\varphi(u)$. Its poles lie in $\{p_+, p_-, p_+ + 2iK', p_- + 2iK'\} + \Lambda$ and have order at most two. If $(c_2/(u - p_+)^2 + c_1/(u - p_+))$ is the principal part at p_+ , the principal parts at p_- , $p_+ + 2iK'$ and $p_- + 2iK'$ are given by the pairs $(-c_2, c_1)$, $(-c_2, -c_1)$ and $(c_2, -c_1)$. The residue c_1 vanishes for*

$$\mu = \mu^* := \frac{(\gamma - \sigma\delta^2)(2\delta^2 - \gamma(1 + \delta^2))}{2\gamma\delta^2\sigma m'^2} = \frac{(\gamma - \sigma\delta^2)(m'^2 + \delta^2)}{2\gamma\delta^2 m'^2}.$$

Proof. Parity, periodicity and antiperiodicity follow from those of dn and sn^2 . The singularities of φ are among the poles iK' , $3iK'$ (modulo Λ) of dn and sn^2 and the zeros of $1 - m^2\sigma \operatorname{sn}^2 u$. At iK' , $x \rightarrow \infty$ with a double pole; both terms in the bracket of (4) are $O(1/x) = O((u - iK')^2)$, so the bracket has a zero of order at least two and $\varphi = O(u - iK')$ is holomorphic there; likewise at $3iK'$, by antiperiodicity. Next, $\operatorname{sn}^2(\Delta + iK') = 1/(m^2\sigma) = x_0$. The function sn^2 is elliptic of order two for the lattice $2K\mathbb{Z} + 2iK'\mathbb{Z}$ and even, so the solutions of $\operatorname{sn}^2 u = x_0$ are $u \equiv \pm(\Delta + iK')$ modulo $2K\mathbb{Z} + 2iK'\mathbb{Z}$; they are two distinct points, since $2\Delta \notin 2K\mathbb{Z}$, hence simple. Modulo Λ they are the four points $p_{\pm}$, $p_{\pm} + 2iK'$ (note $-p_+ = p_- - 2iK'$). At p_+ the values $\operatorname{sn} p_+ = 1/(m \operatorname{sn} \Delta)$, $\operatorname{cn} p_+ = -i\delta/(m \operatorname{sn} \Delta)$, $\operatorname{dn} p_+ = -i \operatorname{cn} \Delta / \operatorname{sn} \Delta$ are finite and nonzero, so $dx/du = 2 \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u \neq 0$ there, $1 - m^2\sigma x$ has a simple zero, and φ has at most a double pole; the same holds at the other three points. If $\varphi = c_2/(u - p)^2 + c_1/(u - p) + O(1)$ near p , then $\varphi(u) = \varphi(-u)$ has the principal part $(c_2, -c_1)$ at $-p$, and $\varphi(u) = -\varphi(u - 2iK')$ has the principal part $(-c_2, -c_1)$ at $p + 2iK'$. Applied to $p = p_+$ this gives $(c_2, -c_1)$ at $-p_+$, hence $(-c_2, c_1)$ at $-p_+ + 2iK' = p_-$, $(-c_2, -c_1)$ at $p_+ + 2iK'$, and $(c_2, -c_1)$ at $p_- + 2iK' \equiv -p_+$.

For the residue we use x as a local coordinate near p_+ . Write $\varphi = 2\delta \operatorname{dn} u \psi(x)$ with $\psi(x) = A_0(x)/(1 - m^2\sigma x)^2 + A_1/(1 - m^2\sigma x)$, $A_0(x) = (\gamma - \sigma\delta^2)(1 - m^2x)/\gamma$, $A_1 = -\mu m'^2$. Since $du = dx/(2 \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u)$, $\varphi du = \delta \psi(x) w(x) dx$ with $w = 1/(\operatorname{sn} u \operatorname{cn} u)$, a holomorphic and nonvanishing function of x near x_0 with $w^2 = 1/(x(1 - x))$; hence $w'/w = (2x - 1)/(2x(1 - x))$. Residues of differentials do not depend on the coordinate, and $1 - m^2\sigma x = -m^2\sigma(x - x_0)$, so

$$c_1 = \delta \operatorname{Res}_{x=x_0} \psi w = \delta \left(\frac{(A_0 w)'(x_0)}{m^4 \sigma^2} - \frac{A_1 w(x_0)}{m^2 \sigma} \right),$$

which vanishes if and only if $A'_0(x_0) + A_0(x_0)(w'/w)(x_0) = m^2\sigma A_1$. Here $A'_0(x_0) = -m^2(\gamma - \sigma\delta^2)/\gamma$, $A_0(x_0) = -(\gamma - \sigma\delta^2)/\sigma$ and, by $m^2\sigma - 1 = -\delta^2$ and $2 - m^2\sigma = 1 + \delta^2$, $(w'/w)(x_0) = -m^2\sigma(1 + \delta^2)/(2\delta^2)$. The condition becomes $(\gamma - \sigma\delta^2)((1 + \delta^2)/(2\delta^2) - 1/\gamma) = -\sigma\mu m'^2$, that is $\mu = \mu^*$. The second expression for μ^* follows from $2\delta^2 - \gamma(1 + \delta^2) = (\delta^2 - 1) + \sigma(1 + \delta^2) = \sigma(m'^2 + \delta^2)$. $\square$

Proposition 4.8. *For $\mu = \mu^*$ the function S of (5) vanishes identically.*

Proof. S is meromorphic and Λ -periodic, and since $2N\Delta = 4K \in \Lambda$ its index i may be taken in $\mathbb{Z}/N\mathbb{Z}$. Let v be a pole candidate with, say, $v + 2i_1\Delta \equiv p_+$ modulo Λ (if $v + 2i\Delta \equiv p_-$, then $v + 2(i+1)\Delta \equiv p_+$). The term i is singular at v if $v + 2i\Delta$ is congruent to p_+ , that is $2(i - i_1)\Delta \in \Lambda$, or to $p_- = p_+ - 2\Delta$, that is $2(i - i_1 + 1)\Delta \in \Lambda$; it cannot be congruent to $p_{\pm} + 2iK'$, because the difference would be a real number minus $2iK'$, while the imaginary parts of the elements of $\mathbb{R} + \Lambda$ lie in $4K'\mathbb{Z}$. Now $2k\Delta = 4kK/N \in \Lambda$ if and only if $N \mid 2k$, and the numbers of $i \in \mathbb{Z}/N\mathbb{Z}$ with $N \mid 2(i - i_1)$ and with $N \mid 2(i - i_1 + 1)$ are equal (1 for odd N , 2 for even N). By Lemma 4.7 the double parts c_2 and $-c_2$ cancel, and the simple part is a multiple of $c_1 = 0$. The points over $p_+ + 2iK'$ are treated in the same way, with the pairs $(-c_2, -c_1)$ and $(c_2, -c_1)$. So S is an elliptic function without poles, hence constant, and $S(v + 2iK') = -S(v)$ forces the constant to be 0. $\square$

Proof of Theorem 2. Take $\mu = \mu^*$ in (5). Every member has a real v_1 (Lemma 4.4(iii)), and for real arguments the denominators in Lemmas 4.5 and 4.6 are positive. By Proposition 4.8, $A_{f_1}^* + A_{f_2}^* = 2(2 - \mu^*)A$. It remains to express $2 - \mu^*$ through a, b, λ . By Lemma 4.4(i) and $\text{cn } \Delta = b''/b$,

$$\gamma = \frac{b''^2}{b^2}, \quad \sigma = \frac{\lambda}{b^2}, \quad \delta^2 = \frac{a^2 b''^2}{a''^2 b^2}, \quad m'^2 = \frac{b''^2}{a''^2}.$$

Hence $\gamma - \sigma\delta^2 = b''^2(a''^2 b^2 - a^2 \lambda)/(a''^2 b^4)$ with $a''^2 b^2 - a^2 \lambda = a^2 b^2 - \lambda(a^2 + b^2)$, $m'^2 + \delta^2 = b''^2(a^2 + b^2)/(a''^2 b^2)$ and $2\gamma\delta^2 m'^2 = 2a^2 b''^6/(a''^4 b^4)$, so that

$$\mu^* = \frac{(a^2 + b^2)(a^2 b^2 - \lambda(a^2 + b^2))}{2a^2 b^2(b^2 - \lambda)}, \quad 2 - \mu^* = \frac{4a^2 b^2(b^2 - \lambda) - (a^2 + b^2)(a^2 b^2 - \lambda(a^2 + b^2))}{2a^2 b^2(b^2 - \lambda)},$$

and the numerator equals $a^2 b^2(3b^2 - a^2) + \lambda((a^2 + b^2)^2 - 4a^2 b^2) = a^2 b^2(3b^2 - a^2) + \lambda(a^2 - b^2)^2$. Thus $2 - \mu^* = \kappa$. $\square$

Remark 4.9 (the hypotheses). Convexity enters only through $\Delta = 2K/N$ in Lemma 4.7 and Proposition 4.8 and through the orientation used in Lemma 4.2; the elliptic caustic enters through $0 < m < 1$ and the position of the foci inside $\mathcal{C}$. We do not state Theorem 2 for the star families, where $\Delta = 2\tau K/N$ with $\tau \geq 2$ coprime to N , although the first verification run found the identity numerically for five of them (Section 6). For odd N the two focal areas differ; for instance, for $a = 1.6$, $b = 1$, $N = 3$ the ratio $A_{f_1}^*/A$ ranges along the family from $9.670\dots$, at the member with a vertex at $(-a, 0)$, to $21.347\dots$, at the member with a vertex at $(a, 0)$, while $(A_{f_1}^* + A_{f_2}^*)/(2A) = \kappa = 15.508\dots$ is constant. For $a = b$ the formula gives $\kappa = b^2/b''^2 = 1/\cos^2(\pi/N)$, the value for the regular N -gon. The expression for $2 - \mu^*$ obtained in the proof above also appears in the proof of [Fer26c, Lemma 4].

5 Even N

Proof of Corollary 3. Let N be even. By Lemma 4.4(iii), $\Delta = 2K/N$, so raising the index by $N/2$ adds $N\Delta = 2K$ to the parameter: $P_{i+N/2} = P(u_0 + 2(i-1)\Delta + 2K) = -P_i$, because sn and cn change sign under $u \mapsto u + 2K$. This central symmetry is Stachel's [Sta21, Corollary 4.2(i)]. The half-turn $X \mapsto -X$ maps the antipedal line $\{X : (X - P_i) \cdot (P_i - f_1) = 0\}$ of f_1 at P_i onto $\{Y : (Y + P_i) \cdot (P_i - f_1) = 0\} = \{Y : (Y - P_{i+N/2}) \cdot (P_{i+N/2} - f_2) = 0\}$, the antipedal line of $f_2 = -f_1$ at $P_{i+N/2}$. So $\text{Ant}(P; f_2)$ is the image of $\text{Ant}(P; f_1)$ under the half-turn, with the labels shifted by $N/2$; neither operation changes the signed area, and $A_{f_2}^* = A_{f_1}^*$. By Theorem 2, $A_{f_j}^* = \kappa A$. This is the argument of [Fer26b] for the equality, and $A_{f_j}^* = \kappa A$ is [Fer26c, Lemma 4].

Chavez-Caliz [CC21, Theorem 6] proves that for two concentric ellipses in general position that admit a one-parameter family of Poncelet n -gons with n even, $A(P)A(Q)$ is constant along the family,

where P is inscribed in the outer ellipse, its sides touch the inner one, and Q is formed by the tangents to the outer ellipse at the vertices of P . Here the outer ellipse is $\mathcal{E}$, the inner one $\mathcal{C}$, the Poncelet polygons are the N -periodics (Lemma 4.4) and $Q = P'$. General position means in [CC21] that the two conics meet transversally in $\mathbb{C}P^2$. The common points of $\mathcal{E}$ and $\mathcal{C}$ are the four points $(\pm x_0, \pm y_0)$ with $x_0^2 = a^2(a^2 - \lambda)/c^2 > 0$ and $y_0^2 = -b^2(b^2 - \lambda)/c^2 < 0$; at each of them, (x, y) say, the gradients $(2x/a^2, 2y/b^2)$ and $(2x/(a^2 - \lambda), 2y/(b^2 - \lambda))$ have the determinant $4xy\lambda c^2/(a^2 b^2(a^2 - \lambda)(b^2 - \lambda)) \neq 0$, so the tangents differ; and there is no common point at infinity, since the determinant of the system $x^2/a^2 + y^2/b^2 = 0 = x^2/(a^2 - \lambda) + y^2/(b^2 - \lambda)$ in (x^2, y^2) is $\lambda c^2/(a^2 b^2(a^2 - \lambda)(b^2 - \lambda)) \neq 0$. (The same check is made in [Fer26c], in the section on the published area theorem.) Hence $A'A$ is constant, and so is $A'A_{f_j}^* = \kappa A'A$.

For $N = 4$ let $\lambda_4 = a^2 b^2/(a^2 + b^2)$. The rhombus $(a, 0), (0, b), (-a, 0), (0, -b)$ is a convex 4-periodic with caustic $\mathcal{C}_{\lambda_4}$: its side $x/a + y/b = 1$ satisfies $a''^2/a^2 + b''^2/b^2 = 2 - \lambda_4(a^{-2} + b^{-2}) = 1$ and touches $\mathcal{C}_{\lambda_4}$ at $(a''/a, b''/b)$, inside the side; the other sides are its mirror images, and the reflection law follows as in the proof of Proposition 3.1. Its area is $2ab$ and its outer polygon is the rectangle with vertices $(\pm a, \pm b)$, so $A'A = 8a^2 b^2$. In (1), $b^2 - \lambda_4 = b^4/(a^2 + b^2)$ and the numerator is $4a^2 b^6/(a^2 + b^2)$, so $\kappa = 2$ and $A'A_{f_j}^* = 16a^2 b^2$. For $N = 6$, $b^2 - \lambda_6 = b^3(2a + b)/(a + b)^2$ and the numerator of (1) is $2a^2 b^3(2b - a)$, so

$$\kappa = \kappa_6 = \frac{(a + b)^2(2b - a)}{b^2(2a + b)}, \quad A'A_{f_j}^* = \kappa_6 \frac{4a^2 b^2(2a + b)(a + 2b)}{(a + b)^2} = 4a^2(4b^2 - a^2)$$

by (2). Finally, a convex member listed counterclockwise has $A > 0$, so $A_{f_j}^* = \kappa A$ vanishes at one member if and only if $\kappa = 0$, and then at every member; $\kappa = 0$ means $\lambda(a^2 - b^2)^2 = a^2 b^2(a^2 - 3b^2)$, and $\kappa_6 = 0$ means $a = 2b$. (For $a = \frac{8}{5}, b = 1$: $\kappa_6 = \frac{338}{525}$, $A'A_{f_j}^* = \frac{9216}{625}$, as in Table 1.) $\square$

Remark 5.1 (rows k_{607} and the zero families). The equality $A_{f_1}^* = A_{f_2}^*$ is the source's row k_{607} , which the source states for $N \equiv 0 \pmod{4}$; it holds for every even N , as proved in [Fer26b]. That note also reports families of even N on which the common focal antipedal area vanishes: $N = 6$ at $b/a = \frac{1}{2}$, and $N = 8, 10, 12, 16$ at $b/a = 0.53941875236237826881\dots, 0.55434860173007045619\dots, 0.56180740571918263508\dots, 0.56883137603733991857\dots$. By Corollary 3 these are exactly the families with $\kappa = 0$. For $N = 6$ this is the exact statement $\kappa_6 = 0 \iff a = 2b$; for $N = 8, 10, 12, 16$ the exact program of this note solves $\kappa = 0$ numerically ($a = 1$, caustic parameter by bisection on the closure condition) and reproduces all twenty printed digits. For $N = 4$, $\kappa = 2$ never vanishes. The case $N = 6$, $a = 2b$ was noticed by Garcia and Reznik: their list of videos contains one titled “N=6 zero-area antipedal @ a/b=2” [GR21a]. The title does not name the point M ; for $a = 2b$ the area A_O^* of Proposition 3.1 is not zero, while $A_{f_1}^* = A_{f_2}^* = 0$.

6 Computations and reproducibility

The proofs in this note do not depend on computations. The computations certify the counterexample a second time, test the displayed formulas and the main identities on billiard trajectories generated directly by the reflection law, and produced the numbers quoted in Remarks 3.2, 4.9 and 5.1. The reproducibility package has four parts.

The exact program of this note, check_note.py (Python with `sympy` and `mpmath`; about six seconds), has five parts. (i) The two members of Theorem 1 are generated from their first vertex by the reflection law in exact arithmetic in $\mathbb{Q}(\sqrt{105}, \sqrt{39})$ with exact sign tests; every defining property (vertices on $\mathcal{E}$, tangency with interior contact, reflection law, return after six reflections and not earlier, convexity), every entry of Table 1 and the values in Remark 3.2 are certified. (ii) Proposition 3.1 for every ellipse, $A_{f_j}^* = \kappa_6 A$ at both members for every ellipse (with $a = u^2 + 1$, $b = 2u$, so that the

foci are rational), the closed forms of Corollary 3, the identity $\kappa = 2 - \mu^*$ and the general-position computation, in exact symbolic arithmetic. (iii) Theorem 2 for $N = 3, \dots, 10$ on the ellipses with $a/b = 1.6, 2, 3, 1.15$, six members each, in 50-digit arithmetic, with the caustic parameter found by bisection on the closure condition (largest relative deviation $4 \cdot 10^{-47}$); Corollary 3 for even N ; and the controls of Remarks 3.2 and 4.9. (iv) The displayed formulas of Lemmas 4.1, 4.2 and 4.4 to 4.6 and Proposition 4.3, side by side, for $N = 3, \dots, 7$ (errors below 10^{-46}), and $S(v)$ of Proposition 4.8 at complex points, with the control that $S \neq 0$ for $\mu = \mu^* + 0.1$. (v) The aspect ratios of Remark 5.1. All 116 checks pass. *The programs with which the results were first obtained* (`original/`), with their recorded outputs, include a second exact certificate, the whole 4- and 6-periodic families for every ellipse in exact rational-function arithmetic, and a 50-digit sweep over five ellipses, $N \leq 18$ and five points M . *The programs of the first two independent verification runs* (`verification_run_A/`, `verification_run_B/`) and *those of the third* (`independent_run_2/`) come with their recorded outputs; the range of $A_{f_1}^*/A$ for $N = 3$ in Remark 4.9 was computed by the third. The file `README.md` of the package, and that of each folder, give the command lines, the running times and what each program checks.

Verification. The results were first obtained, with proofs and with the programs in `original/`. Three independent verification runs, all AI-assisted, followed; each wrote its own programs. The first two examined the results before this note was written; the third examined the final text. The first run re-derived every lemma and the proof of the focal identity line by line, proved the parametrisation of Lemma 4.4 from (3) without the literature, recomputed the counterexample in exact arithmetic with the standard library only, computed the difference of Proposition 3.1 and the identity $2 - \mu^* = \kappa$ symbolically, and tested Theorem 2 on trajectories computed directly from the reflection law for $N = 3, \dots, 10$ and 14 in three ellipses, the principal parts of φ by contour integrals, and the zero families of [Fer26b]; it also tested five star families (Remark 4.9). The second run generated both members of Theorem 1 from their first vertices by the reflection law in two exact implementations of its own, checked that perturbed inputs are rejected, computed the two symmetric members for every ellipse symbolically, recomputed the residue, swept $N = 3, \dots, 18$ in five ellipses at five points M in 50-digit arithmetic, tested each identity of the proof on trajectories computed from the reflection law, compared the cited statements with the sources, and repeated the literature search. Both runs confirmed the disproof and the focal identity; neither found a mathematical error. They required eleven corrections, all of which were made: the general-position hypothesis of [CC21, Theorem 6] is verified; the converse direction of the parametrisation is proved (Lemma 4.4); [Fer26b], [RR14] and the observation of [GR21a] are credited, and the source’s row k_{607} is described as stated for $N \equiv 0 \pmod{4}$; the reading of the column “M” and the least-period convention are presented as ours; Stachel’s corollary is cited with the numbering of the arXiv version; statements on novelty are restricted to the sources searched; and the package records the argument list of the original sweep. When this note was written, the vertex lists and the proof of Proposition 3.1, the complete proof of Lemma 4.4 with identity (3), the closed-form computation of $2 - \mu^*$ in the proof of Theorem 2 and the program `check_note.py` were added.

The third run examined the text of the note and the package line by line, as if nothing had been checked, and these additions first. It read the cited parts of [RGK21, v11], of Table 5 of its journal version and of [Sta22, Sta21, CC21, Fer26c, Fer26b, GR21a] again, re-derived every proof, and recomputed the statements with programs of its own: both members of Theorem 1 from their first vertices by the reflection law, in exact arithmetic with the standard library only, with every entry of Table 1; Proposition 3.1 and the closed forms of Corollary 3 for every ellipse, the two members again being generated by the reflection law, in exact symbolic arithmetic; the residue of Lemma 4.7 by a series expansion and the identity $2 - \mu^* = \kappa$; the principal parts of φ and $S(v)$ at complex points;

Theorem 2 and Corollary 3 on trajectories computed from the reflection law in 60-digit arithmetic ($N = 3, \dots, 10$, four ellipses), and every displayed formula of Section 4 on such trajectories; and the aspect ratios of Remark 5.1 by two methods, to all twenty printed digits. It repeated the literature search and re-ran every program of the package from the archive; all recorded outputs were reproduced. It found no mathematical error. The corrections it required were made: in Remark 4.9 the range of $A_{f_1}^*/A$ for $N = 3$, for which the spread over six members had been given; in Remark 3.2 the credit to [Fer26a, Kri26], from which the constancy of A_O^*/A for odd N follows; the restriction of the novelty statement to the product for $N = 6$, the values for $N = 4$ being in [Fer26c]; this record of the verification runs; and two wordings, the attribution after Lemma 4.4 and the determinant in Section 5. The statements as printed are tested by the exact program of this note and by the programs of the third run.

Scope and priority. The note settles row k_{401} as printed: it is false (Theorem 1), already at the centre for $N = 6$ in every ellipse, and it holds at the foci for every even N (Corollary 3). Its failure at points other than the centre is only tested numerically (Remark 3.2); star families and hyperbolic caustics are not treated. The row and its conventions are due to Reznik, Garcia and Koiller [RGK21]. The parametrisation is Stachel’s [Sta22, Theorem 2], the central symmetry and the reflection property [Sta21, Lemma 2.1] are also printed by him, and the constancy of $A'A$ is Chavez-Caliz’s [CC21, Theorem 6]. The identity $A_{f_j}^* = \kappa A$ for even N is Lemma 4 of the author’s unrefereed note [Fer26c]; the equality $A_{f_1}^* = A_{f_2}^*$ for every even N and the zero families are in the author’s unrefereed note [Fer26b]; the zero for $N = 6$, $a = 2b$ was observed by Garcia and Reznik [GR21a]. To our knowledge, the disproof of k_{401} as printed, Theorem 2 for every N , odd N included, and the focal product for $N = 6$ in Corollary 3 are not in print; for $N = 4$ the values $\kappa = 2$ and $A'A = 8a^2b^2$ are in [Fer26c, Section 6].

What was read: [RGK21] in the arXiv version v11 (tables, definitions, list of symbols, figure of antipedal polygons) and Table 5 of the journal version; [Sta22, Theorem 2] with its proof and [Sta21, Lemma 2.1 and Section 4], in the arXiv versions cited; in the arXiv version of [CC21], the definition of general position and Theorem 6; [Fer26c] completely; the sections of [Fer26b] on central symmetry and on zero families; the main theorems of [Fer26a, Kri26] with their conventions; the table of videos of [GR21a] (the videos were not watched). [RR14] was not read; what we say about it is taken from the source. The proofs of the cited theorems were not checked; the parametrisation is proved here.

The searches were made on 15 October 2026: queries to the arXiv interface (for antipedal polygons, for invariants of elliptic billiards, for the authors of the source and for Stachel), the 19 works that cite the journal version of [RGK21] in OpenAlex (titles and abstracts) and OpenAlex full-text searches for antipedal polygons in billiards, Crossref, zbMATH Open, the Zenodo deposits that mention antipedal polygons (11 on that day) or elliptic billiards, the survey [GR21b], and a few web searches; Semantic Scholar was not available. The third verification run repeated the queries to arXiv, OpenAlex, Crossref, zbMATH Open and Zenodo on the same day. No printed statement, proof or refutation of k_{401} was found in these sources; the closest works found are [Fer26c, Fer26b] and other unrefereed notes of 2026 on sibling rows, none of which states, proves or refutes k_{401} . A search that finds nothing is not a proof of novelty.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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