

Outer-Pedal Area Products in Elliptic Billiards

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1 October 2026 (version 1.1)

Abstract

For a fixed confocal elliptic-caustic billiard family with least period congruent to two modulo four, we prove that the signed area of the outer tangent polygon times the signed area of its pedal polygon with respect to any fixed point is constant. Coprime star trajectories are included. The tangent-intersection polygon is an explicit affine image of a half-step shift of the billiard orbit. After complexification, its area and the outer-pedal area have disjoint possible simple-pole sets. Central pairing separates the pedal map into two area traces; two reversal symmetries provide the complementary zeros that make their product entire and elliptic. This proves invariant $k_{303,a}$ under an explicit least-period convention. For the center pedal, the same argument also covers every odd period, proving $k_{303,b}$ for all traversal lengths not divisible by four, including repeated traversals. We evaluate the six-periodic constant and give exact counterexamples to the unrestricted arbitrary-point repeated-list interpretation and to adding period four in the centered case. No absolute priority claim is made.

1 Statement and prior context

Let $a > b > 0$ and $0 < \lambda < b^2$. Fix the confocal ellipses

$$E : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad C : \frac{x^2}{a^2 - \lambda} + \frac{y^2}{b^2 - \lambda} = 1. \quad (1)$$

Let $P = (P_i)_{i \in \mathbb{Z}/N\mathbb{Z}}$ be a billiard orbit on E whose sides are tangent to C . Unless explicitly stated otherwise, $N \geq 3$ denotes its *least* positive period. Every admitted coprime winding number is allowed, including self-intersecting star trajectories. For a cyclic vector sequence X write

$$\mathcal{A}(X) = \frac{1}{2} \sum_i \det(X_i, X_{i+1}).$$

All areas in this paper are signed shoelace areas, not sums of the absolute areas of bounded faces.

Let V_i be the intersection of the tangents to E at P_i and P_{i+1} . For a fixed point $M \in \mathbb{R}^2$, let $Q_i(M)$ be its perpendicular foot on the tangent to E at P_i . These are the sidelines of the *outer tangent polygon* V , not the original billiard chords. Put

$$A' = \mathcal{A}(V), \quad B_M = \mathcal{A}(Q(M)).$$

Theorem 1. *If $N \equiv 2 \pmod{4}$, then for every fixed $M \in \mathbb{R}^2$ the product $A'B_M$ is constant as the initial vertex varies through the fixed-caustic family (1).*

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This is the outer-pedal area-product assertion $k_{303,a}$ of Reznik, Garcia and Koiller [1, 2, Table 4 and §3.4], under the stated least-period convention. Their table does not explicitly define minimal period; we make the qualification explicit rather than attribute it verbatim to the source. Section 8 shows that repeating a three-periodic trajectory to obtain six entries does not satisfy the unrestricted list-length version of the assertion. Hyperbolic and degenerate caustics are excluded.

Theorem 2. *For $M = O$, the product $A'B_O$ is constant whenever the closed traversal has length L not divisible by four. Here L need not be the least period: repeated traversals and all admitted star windings are included.*

Theorem 2 proves the adjacent centered assertion $k_{303,b}$ in the same source table, without an additional least-period qualification. Its proof is given in Section 6. The period-four obstruction there explains why the period restriction cannot simply be omitted. This extension is integrated into the present paper because it uses the same outer polygon and complementary-pole argument, not a separate discovery of the overlapping even-period case.

Chávez-Caliz [6, Theorem 6] proves a related product involving the original and outer tangent-polygon areas for even Poncelet periods. That theorem supplies a closely related complex pole-cancellation strategy, but its second polygon is not the outer-pedal polygon. Bialy and Tabachnikov [7, Theorem 4.1] study precisely the feet on boundary tangents for arbitrary fixed points, proving centroid and squared-distance invariants; those conclusions are different from the cyclic signed-area product here. We use the canonical billiard parametrization of Stachel [3, Theorem 2, pp. 11–12 of the preprint], and prove the additional tangent-intersection identity and pedal-area cancellations needed for Theorem 1.

2 The canonical parameter and the outer polygon

Set

$$\alpha = a^2, \quad \beta = b^2, \quad d = \alpha - \beta, \quad k^2 = \frac{d}{\alpha - \lambda}, \quad K = K(k), \quad K' = K(k'), \quad k' = \sqrt{1 - k^2}.$$

We suppress k in the Jacobi elliptic functions. Stachel's parametrization is

$$P(u) = (-a \operatorname{sn} u, b \operatorname{cn} u), \quad P_i(u) = P(u + ih), \quad h = 2\Delta = \frac{4\tau K}{N}, \quad \gcd(\tau, N) = 1. \quad (2)$$

Choose the orientation so that $0 < \tau < N/2$. Then $0 < \Delta < K$, and the caustic condition gives

$$\operatorname{sn}^2 \Delta = \frac{\lambda}{\beta}, \quad \operatorname{cn}^2 \Delta = \frac{\beta - \lambda}{\beta}, \quad \operatorname{dn}^2 \Delta = \frac{\alpha(\beta - \lambda)}{\beta(\alpha - \lambda)}. \quad (3)$$

The physical point period is $4K$, not $2K$. We work on the common elliptic period lattice $4K\mathbb{Z} + 4iK'\mathbb{Z}$. After complexification all dot products are bilinear, not Hermitian. The standard identity $\operatorname{dn}(u+2K) = \operatorname{dn} u$ is used throughout [4]; the opposite sign printed in a preliminary paragraph of Stachel's preprint is not used.

The tangent to E at $P(v)$ is $n(v) \cdot X = 1$, where

$$n(v) = \left(-\frac{\operatorname{sn} v}{a}, \frac{\operatorname{cn} v}{b} \right). \quad (4)$$

Write $s_0 = \operatorname{sn} \Delta$, $c_0 = \operatorname{cn} \Delta$, $e_0 = \operatorname{dn} \Delta$.

Lemma 3. *The intersection of the tangents at $P(t - \Delta)$ and $P(t + \Delta)$ is*

$$V(t) = \left(-\frac{ae_0}{c_0} \operatorname{sn} t, \frac{b}{c_0} \operatorname{cn} t \right). \quad (5)$$

Consequently, for $A(u) = \mathcal{A}((P(u + ih))_i)$,

$$A'(u) = \frac{e_0}{c_0^2} A(u + \Delta). \quad (6)$$

Proof. Put $s = \operatorname{sn} t$, $c = \operatorname{cn} t$, $e = \operatorname{dn} t$ and $D_t = 1 - k^2 s^2 s_0^2$. The addition formulas [5] give

$$\operatorname{sn}(t \pm \Delta) = \frac{sc_0 e_0 \pm s_0 ce}{D_t}, \quad \operatorname{cn}(t \pm \Delta) = \frac{cc_0 \mp ss_0 ee_0}{D_t}.$$

Substitution of (5) in either tangent equation gives

$$n(t \pm \Delta) \cdot V(t) = \frac{s^2 e_0^2 + c^2}{D_t} = 1.$$

For a real nondegenerate billiard chord the two tangents are distinct; the displayed point is their actual intersection. The formula supplies its meromorphic continuation through any removable complex exceptions. Since $V_i = V(u + \Delta + ih)$, the determinant of the fixed diagonal linear map in (5) is e_0/c_0^2 , which proves (6). In particular the half-step shift cannot be omitted. $\square$

3 The pedal map and central pairing

In this and the next two sections we prove Theorem 1 and assume $N \equiv 2 \pmod{4}$.

Direct perpendicular projection onto the tangent in (4) gives

$$\begin{aligned} Q_M(u) &= M + \frac{(1 - M \cdot n(u))n(u)}{n(u) \cdot n(u)}, \\ W(u) &= \alpha - d \operatorname{sn}^2 u, \quad n(u) \cdot n(u) = \frac{W(u)}{\alpha\beta}. \end{aligned} \quad (7)$$

Introduce the meromorphic vector functions

$$U(u) = \frac{n(u)}{n(u) \cdot n(u)}, \quad R_M(u) = \frac{(M \cdot n(u))n(u)}{n(u) \cdot n(u)}.$$

Thus $Q_M - M = U - R_M$. Translation does not change cyclic signed area: its extra determinant terms telescope. As N is even and τ is odd, (2) gives

$$P(u + (N/2)h) = -P(u), \quad U(u + (N/2)h) = -U(u), \quad R_M(u + (N/2)h) = R_M(u).$$

Therefore the mixed determinants in $\mathcal{A}(U - R_M)$ cancel in opposite pairs, and

$$B_M(u) = \mathcal{A}((U(u + ih))_i) + \mathcal{A}((R_M(u + ih))_i). \quad (8)$$

The two negative signs in the quadratic R_M term cancel. Identity (8) does not assume that the pedal polygon itself is centrally symmetric for $M \neq 0$.

4 All possible complex poles

The common Jacobi pole positions are

$$z_p = 2mK + (2n+1)iK', \quad m, n \in \mathbb{Z}.$$

They are simple. At the representative $z_p = iK'$, the standard shift formulas are

$$\operatorname{sn}(iK' + w) = \frac{1}{k \operatorname{sn} w}, \quad \operatorname{cn}(iK' + w) = -\frac{i \operatorname{dn} w}{k \operatorname{sn} w}. \quad (9)$$

Both are odd in w . Real and imaginary half-period translations only change coordinate signs, so P and n are odd about every z_p . Put $Z = z_p + K$. At the base representative $Z = K + iK'$,

$$\operatorname{sn}(Z + w) = \frac{\operatorname{dn} w}{k \operatorname{cn} w}, \quad \operatorname{cn}(Z + w) = -\frac{ik'}{k \operatorname{cn} w}. \quad (10)$$

Hence P and n are even about every translated Z . These are meromorphic identities, not just identities on the real parameter line; see [4, 5].

At a Jacobi pole the leading coefficients of sn and cn are nonzero and have ratio $\operatorname{cn}/\operatorname{sn} = \pm i$. Thus $n \cdot n$ has a nonzero double-pole coefficient proportional to $1/\alpha - 1/\beta$. It follows that U is holomorphic there with a zero and R_M is holomorphic there. Consequently Q_M has no pole at a Jacobi pole.

Lemma 4. *The possible poles of A' and B_M are, respectively,*

$$z_p - \Delta - jh \quad \text{and} \quad Z - \Delta - jh, \quad j \in \mathbb{Z}. \quad (11)$$

For $N \equiv 2 \pmod{4}$ these sets are disjoint, and every possible pole of either area function is at most simple.

Proof. The only remaining possible poles of Q_M are zeros of W . They are exactly

$$Z - \Delta, \quad Z + \Delta \pmod{2K\mathbb{Z} + 2iK'\mathbb{Z}}. \quad (12)$$

Indeed the quarter-period formula and (3) give

$$\operatorname{sn}^2(K \pm \Delta) = \frac{\operatorname{cn}^2 \Delta}{\operatorname{dn}^2 \Delta} = \frac{\alpha - \lambda}{\alpha};$$

the imaginary shift then gives $\operatorname{sn}^2(Z \pm \Delta) = \alpha/d$. The two roots are distinct since $0 < \Delta < K$, and exhaustive because sn^2 has order two on the indicated smaller lattice. At either root,

$$\operatorname{sn}^2 u = \frac{\alpha}{d}, \quad \operatorname{cn}^2 u = -\frac{\beta}{d}, \quad \operatorname{dn}^2 u = -\frac{\lambda}{\alpha - \lambda}.$$

All are nonzero. Thus $W' = -2d \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u \neq 0$, and the foot map has at most simple poles at these roots.

Formula (6) gives the first pole set in (11). A determinant in A cannot have both adjacent vertices at Jacobi poles unless $h \in 2K\mathbb{Z}$. That would require $N \mid 2\tau$, impossible for a genuine period $N \geq 6$. Hence A' has at most simple poles. Several nonadjacent vertices can have coincident simple poles, but adding their contributions cannot increase the order.

Taking the cyclic trace of (12) gives the second set in (11); the two signs differ by $h = 2\Delta$. A collision of the two area pole sets would require

$$K \in h\mathbb{Z} + 2K\mathbb{Z}, \quad \text{equivalently} \quad 4\tau j = N(2\ell + 1)$$

for some integers j, ℓ . The left side is divisible by four, whereas the right side is two modulo four. Thus the sets are disjoint on the full complex lattice.

It remains to bound the order of B_M . At the adjacent pair $Z - \Delta, Z + \Delta$, evenness in (10) gives $n(Z - \Delta) = n(Z + \Delta)$. The residues of the two feet are parallel to this common normal, so the double-pole coefficient of their determinant is zero. Translated pairs satisfy the same statement, possibly with opposite normal signs. The only additional adjacent coincidence could require $4\Delta \equiv 0 \pmod{2K}$. In $0 < \Delta < K$ this means $\Delta = K/2$, so $\tau/N = 1/4$, incompatible with the primitive period under consideration. These are all possible adjacent coincidences. Each determinant in the pedal area therefore has at most a simple pole, as does their sum. $\square$

5 Complementary zeros and the proof of the theorem

Both area traces are h -periodic by cyclic reindexing. First consider a possible outer-area pole $u = z_p - \Delta$. Every foot is holomorphic there by Lemma 4. Its normal is

$$n_j = n(z_p + (j - \frac{1}{2})h).$$

Oddness about z_p and the period $Nh = 4\tau K$ give

$$n_{1-j} = -n_j, \quad U_{1-j} = -U_j, \quad (R_M)_{1-j} = (R_M)_j.$$

The permutation $j \mapsto 1 - j$ reverses cyclic order. Reversal changes the sign of signed area, while multiplying every vector by -1 does not. Thus

$$\mathcal{A}(U) = -\mathcal{A}(U) = 0, \quad \mathcal{A}(R_M) = -\mathcal{A}(R_M) = 0.$$

By (8), $B_M(z_p - \Delta) = 0$. This holomorphic zero removes the possible simple pole of A' in the product.

Next consider a possible pedal-area pole $u = Z - \Delta$. Equation (6) gives

$$A'(Z - \Delta) = \frac{e_0}{c_0^2} A(Z).$$

All the vertices $P(Z + jh)$ are finite: a Jacobi pole among them would require the excluded relation $K \in h\mathbb{Z} + 2K\mathbb{Z}$. By evenness about Z ,

$$P(Z - jh) = P(Z + jh).$$

The permutation $j \mapsto -j$ reverses this polygon without changing its vertices. Therefore $A(Z) = -A(Z) = 0$. This is a holomorphic zero of A' , cancelling the possible simple pole of B_M in the product.

These arguments apply at every translated pole. Half-period translations only multiply vector coordinates by signs and preserve the respective oddness, evenness, parallel-residue, and reversal identities. We have therefore exhausted and removed every possible pole of $A'B_M$. The product is an entire function with periods $4K$ and $4iK'$. It is bounded on a fundamental parallelogram, so Liouville's theorem makes it constant. Restricting to real phases proves Theorem 1. $\square$

All real tangent intersections are finite because $c_0 > 0$; all real feet are finite because $n \cdot n > 0$. No division by a polygon area has been used, so a zero signed pedal area is not an exception. Orientation reversal changes both area signs and leaves the product unchanged.

6 The center pedal and all periods not divisible by four

We now prove Theorem 2. First suppose N is the least period and $4 \nmid N$. For $M = O$ we have $R_M = 0$, so

$$B_O(u) = \mathcal{A}((U(u + jh))_j)$$

directly, without using central pairing. This is the crucial change that allows odd periods. The parametrization, tangent-intersection formula and pole descriptions above do not otherwise need N even.

The possible pole sets are still those of (11). At Jacobi poles U is regular with value zero. The roots of W are still simple and exactly those of (12). Adjacent original vertices could both be at Jacobi poles only if $N \mid 2\tau$, impossible for primitive $N \geq 3$; hence A' has at most simple poles. For B_O , evenness of the normal at $Z \pm \Delta$ makes the two residues parallel. The possible extra adjacent coincidence $\Delta = K/2$ would force primitive $N = 4$, which is excluded. Thus its poles are also at most simple.

Disjointness of the two pole sets again reduces to

$$4\tau j = N(2\ell + 1).$$

For odd N the right side is odd; for $N \equiv 2 \pmod{4}$ it is two modulo four. Both contradict the left side. Thus every step in the pole-order and finiteness argument is valid in both claimed period classes, including all coprime winding numbers.

At $u = z_p - \Delta$, oddness of U about z_p and $Nh = 4\tau K$ give $U_{1-j} = -U_j$. Reversal and global negation therefore imply $B_O = -B_O = 0$. For odd N , the self-paired index has argument $z_p + Nh/2 = z_p + 2\tau K$, a removable Jacobi-pole value of U equal to zero. No exceptional undefined vertex is being ignored. At $u = Z - \Delta$, all $P(Z + jh)$ are finite, and their evenness under $j \mapsto -j$ again gives $A(Z) = 0$ and then $A'(u) = 0$. All possible simple poles of $A'B_O$ are removed. The same entire elliptic-function argument proves constancy for primitive $4 \nmid N$.

Finally let L be any closed traversal length not divisible by four, and let N be its least period. Since $N \mid L$, necessarily $4 \nmid N$. For $r = L/N$, the ordered original, outer and foot polygons are each repeated r times. Their signed determinant areas are multiplied by r , so the product is multiplied by r^2 . The primitive result proves constancy for this repeated traversal as well. This completes the proof of Theorem 2. $\square$

Proposition 5. *The centered product is not constant on every four-periodic family. For any $a > b > 0$ and $\lambda = a^2b^2/(a^2 + b^2)$, two members of the same family have products $8a^2b^2$ and $2(a^2 + b^2)^2$, differing by $2(a^2 - b^2)^2 > 0$.*

Proof. The axis diamond $(a, 0), (0, b), (-a, 0), (0, -b)$ is a billiard trajectory; its slanted sidelines satisfy $(a^2 - \lambda)/a^2 + (b^2 - \lambda)/b^2 = 1$, the confocal tangency equation. The outer tangent polygon is the rectangle with vertices $(\pm a, \pm b)$, of area $4ab$. Its center feet are the original diamond vertices, of area $2ab$.

Put $s = a^2 + b^2$. The second trajectory is the axis-parallel rectangle with cyclic vertices

$$\frac{(a^2, b^2)}{\sqrt{s}}, \quad \frac{(-a^2, b^2)}{\sqrt{s}}, \quad \frac{(-a^2, -b^2)}{\sqrt{s}}, \quad \frac{(a^2, -b^2)}{\sqrt{s}}.$$

Its vertices lie on E , and its normals have coordinate magnitudes $1/\sqrt{s}$, so the horizontal and vertical sides obey the reflection law. The identities $a^2 - \lambda = a^4/s$ and $b^2 - \lambda = b^4/s$ show that its sides are tangent to the same caustic. Its boundary tangent lines have equations $\pm x \pm y = \sqrt{s}$, giving an outer diamond of area $2s$. Their center feet are the corners $(\pm\sqrt{s}/2, \pm\sqrt{s}/2)$, of area s . The two area products and their stated difference follow. $\square$

For $a = 5, b = 3$, these products are 1800 and 2312, with the same $\lambda = 225/34$. This refutes an all-period extension, not Theorem 2. Exact checks also give product 243/64 in five three-periodic instances with $a^2 = 1, b^2 = 3/8, \lambda = 9/25$, and verify the r^2 scaling on repeated lengths 6, 9, 15. These checks supplement the analytic proof; they do not establish general constancy by sampling. No exact five-periodic control is claimed.

7 Dependence on the pedal point and the six-periodic constant

There is a useful algebraic description of the arbitrary-point dependence. For any cyclic collection of lines $n_i \cdot X = 1$ with nonzero real normals, put $s_i = n_i \cdot n_i$ and $D_i = \det(n_i, n_{i+1})$. Translating the foot polygon by $-M$ gives the exact identity

$$B_M = \frac{1}{2} \sum_i \frac{D_i(1 - n_i \cdot M)(1 - n_{i+1} \cdot M)}{s_i s_{i+1}}. \quad (13)$$

For $n = (x, y), q = (z, w), D = xw - yz, s = x^2 + y^2, t = z^2 + w^2$,

$$D(xz - yw) = szw - txy, \quad D(xw + yz) = sw^2 - ty^2.$$

After division by st , both expressions telescope when summed cyclically. Thus the homogeneous quadratic part of (13) is radial. For even primitive N , central pairing eliminates the linear part, yielding

$$B_M = B_0 + C|M|^2, \quad C = \frac{1}{4} \sum_i \frac{D_i(n_i \cdot n_{i+1})}{s_i s_{i+1}} = \frac{1}{8} \sum_i \sin(2(\phi_{i+1} - \phi_i)), \quad (14)$$

where ϕ_i is a real normal angle. It follows that the invariant in Theorem 1 has the form $C_0 + C_2|M|^2$ for the fixed conic pair.

Proposition 6. *For the six-periodic family with $\lambda = a^2 b^2 / (a + b)^2$,*

$$A' B_M = ab(a + 2b)(2a + b) \left(1 + \frac{|M|^2}{(a + b)^2} \right). \quad (15)$$

Proof. By Theorem 1, it suffices to evaluate one member. Put $r = \sqrt{b(2a + b)}$ and $s = a + b$. Take

$$P_1 = (a, 0), \quad P_2 = (a^2/s, br/s), \quad P_3 = (-a^2/s, br/s), \quad P_{i+3} = -P_i.$$

Since $a^2 + r^2 = s^2$, all six distinct vertices lie on E . At P_1 the horizontal normal reflects direction (a, r) to $(-a, r)$. At P_2 the normal $(1/s, r/(bs))$ has squared length $2/(bs)$ and dot product 1 with $(-a, r)$, so reflection sends that direction to $(-s, 0)$. Symmetry verifies the other reflections. The horizontal edge is tangent to the caustic because $b^2 - \lambda = b^2 r^2 / s^2$; the slant-edge tangency identity is $\lambda(r^2 + a^2) = a^2 b^2$. Its contact parameter from P_1 to P_2 is $b/s \in (0, 1)$, so the contacts lie on the segments. Thus this is a genuine six-periodic member.

Its outer vertices and center feet are, respectively,

$$\begin{aligned} V_1 &= (a, b^2/r), & V_2 &= (0, bs/r), & V_3 &= (-a, b^2/r), \\ Q_1 &= (a, 0), & Q_2 &= (b/2, r/2), & Q_3 &= (-b/2, r/2), \end{aligned}$$

with $V_{i+3} = -V_i$ and $Q_{i+3} = -Q_i$. Shoelace evaluation gives

$$A' = \frac{2ab(a + 2b)}{r}, \quad B_0 = \frac{r(2a + b)}{2}.$$

Writing $\phi = \arg(b+ir)$, the normal-angle increments are $\phi, \pi-2\phi, \phi$, repeated twice. Since $\sin 2\phi = r/s$ and $\sin^2 \phi = (2a+b)/(2s)$,

$$C = \frac{4 \sin 2\phi - 2 \sin 4\phi}{8} = \frac{r(2a+b)}{2s^2} = \frac{B_0}{s^2}.$$

Substitution in (14) proves (15). $\square$

For example, $a = 5, b = 3$ gives $\lambda = 225/64$ and $A'B_M = 2145(1 + |M|^2/64)$. The supplementary standard-library checker independently reconstructs six-periodic orbits for four semiaxis pairs and four initial phases, verifies their reflection laws and strict caustic contacts, and checks all coefficients of the quadratic area polynomial. The circle case among those controls is not needed for the theorem's $a > b$ domain. These sixteen orbit instances are finite checks, not the proof of all-family invariance.

8 Least period, reproducibility and limitations

The least-period qualification in Theorem 1, for arbitrary M , cannot be dropped in favor of arbitrary list length. The centered Theorem 2 has the separate repetition argument already proved above. Fix

$$a^2 = 1, \quad b^2 = \frac{3}{8}, \quad \lambda = \frac{9}{25}, \quad M = (1, 0).$$

The triangle

$$P = \left((1, 0), \left(-\frac{4}{5}, \frac{3\sqrt{6}}{20} \right), \left(-\frac{4}{5}, -\frac{3\sqrt{6}}{20} \right) \right)$$

and its central reflection $-P$ are genuine three-periodic billiards with this same caustic. At P , the incoming and outgoing directions at $(1, 0)$ reflect in the horizontal normal. At the upper-left vertex their representative directions $(-12, \sqrt{6})$ and $(0, -1)$ satisfy the reflection law with normal $(-4/5, 2\sqrt{6}/5)$; the remaining vertex follows by symmetry. The vertical side is tangent because $a^2 - \lambda = 16/25$, and the two slant sides satisfy the confocal tangency equation. These identities and strict segment contacts are also checked exactly in the supplement.

For P the outer area and pedal area at M are

$$A' = \frac{27\sqrt{6}}{16}, \quad B_M = \frac{81\sqrt{6}}{200},$$

while for $-P$ they are

$$A' = \frac{27\sqrt{6}}{16}, \quad B_M = \frac{21\sqrt{6}}{200}.$$

The respective products are $6561/1600$ and $1701/1600$. Repeating each list twice produces six entries and doubles both signed areas, giving products

$$\frac{6561}{400} \quad \text{and} \quad \frac{1701}{400}.$$

Their difference is $243/20 > 0$. This refutes the padded-list extension, not Theorem 1: these orbits still have least period three and do not have the even-orbit central pairing used in (8).

The supplement uses exact rational polynomial arithmetic and real quadratic fields. It verifies the two universal telescoping identities behind (14), elementary axis identities, the six-periodic controls, and the padded triangle countercontrol. The general theorem rests on the written analytic argument, not on those finite checks or on numerical exploration.

The source conjecture, confocal geometry, Jacobi parametrization and complex pole-cancellation method are credited prior mathematics. A bounded primary-literature search did not locate an earlier proof of this precise arbitrary-point outer-pedal product. This is not evidence of absolute priority, and the source’s historical question mark is not itself a statement about the current literature. The closest cited area-product and arbitrary-point projection results concern different quantities, as described in Section 1.

This is an unrefereed preprint. AI assistance was used in derivation, internal checking, coding and manuscript preparation. No independent human peer review or formal proof-assistant verification is claimed.

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