

# Center-Pedal Area Products in Elliptic Billiards

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## Abstract

We prove constancy of the product of the signed area of an elliptic billiard orbit and the signed area of its pedal polygon at the common center, for a fixed nondegenerate confocal elliptic caustic and least period odd or divisible by four. Primitive star trajectories are included. A reflection-coordinate telescope expresses the centered pedal area as a fixed multiple of a Jacobi trace. After complexifying the billiard parameter, the original area and this trace have complementary simple-pole sets; odd pairing and reversed-edge pairing remove every pole of their product. This establishes invariant  $k_{203,b}$  under an explicitly stated genuine-period convention. Exact six-periodic controls show why an unrestricted repeated-list interpretation would be false. No absolute priority claim is made.

## 1 Statement and conventions

Fix  $a > b > 0$  and  $0 < \lambda < b^2$ . The boundary and caustic are

$$E : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad C : \frac{x^2}{a^2 - \lambda} + \frac{y^2}{b^2 - \lambda} = 1. \quad (1)$$

Let  $P = (P_i)_{i \in \mathbb{Z}/N\mathbb{Z}}$  be a billiard orbit on  $E$  tangent to  $C$ . Throughout,  $N \geq 3$  is the *least* positive period. Every admitted coprime winding number is allowed, including stars. For an ordered polygon define

$$\mathcal{A}(V) = \frac{1}{2} \sum_i \det(V_i, V_{i+1}).$$

Write  $A = \mathcal{A}(P)$ . Let  $Q_i$  be the perpendicular foot from the center  $O = (0, 0)$  onto the original sideline  $P_i P_{i+1}$ , and put  $A_O = \mathcal{A}(Q)$ . All areas are signed; they are not sums of absolute areas of bounded faces.

**Theorem 1.** *If  $N$  is odd or divisible by four, the product  $AA_O$  is constant as the initial vertex varies in the fixed-caustic family (1).*

This is the center-pedal assertion  $k_{203,b}$  in [1, 2, Table 3], under the genuine-period convention. The source does not explicitly define minimal period; we state this qualification rather than attribute it verbatim to that source. An arbitrary repeated list is different: Section 6 gives an exact excluded six-periodic example whose double traversal has twelve entries but still least period six. Hyperbolic or degenerate caustics are outside the present two-ellipses setting.

Set

$$\alpha = a^2, \quad \beta = b^2, \quad d = \alpha - \beta, \quad D = \alpha\beta, \quad s = \alpha + \beta.$$

Orient the orbit with the caustic on the left of each directed edge. Reversal changes both area signs, leaving their product unchanged.

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## 2 A reflection telescope and the center-pedal area

At  $P = (a \cos \theta, b \sin \theta)$  put

$$W = \beta \cos^2 \theta + \alpha \sin^2 \theta, \quad n = \frac{(b \cos \theta, a \sin \theta)}{\sqrt{W}}, \quad t = \frac{(-a \sin \theta, b \cos \theta)}{\sqrt{W}}.$$

Here  $n$  is the outward unit normal,  $t$  is the positive unit tangent, and  $\det(n, t) = 1$ . Reflection and confocal tangency give the incoming and outgoing unit velocities

$$v_- = ht + jn, \quad v_+ = ht - jn, \quad j = \sqrt{\frac{\lambda}{W}}, \quad h = \sqrt{\frac{W - \lambda}{W}}. \quad (2)$$

For completeness, a chord between parameters  $m - \delta, m + \delta$  has direction  $(-a \sin m, b \cos m)/\sqrt{U}$ , where  $U = \alpha \sin^2 m + \beta \cos^2 m$ . Tangency gives  $\sin^2 \delta = \lambda U/D$ , and the squared normal component is  $D \sin^2 \delta / (UW) = \lambda/W$ . Since the caustic contains  $O$ , choose  $0 < \delta < \pi/2$ . The endpoint tangent-component numerators are  $U \cos \delta \pm d \sin m \cos m \sin \delta$ . Their sum is positive and their product is  $U(DU - \lambda V)/D > 0$ , where  $V = \alpha^2 \sin^2 m + \beta^2 \cos^2 m$ ; indeed  $DU - \beta V = \beta^2 d \cos^2 m \geq 0$  and  $\lambda < \beta$ . Thus both tangent components are positive, fixing the signs in (2).

**Lemma 2.** For  $S_r = \sum_i \sqrt{W_i - \lambda}/W_i^r$ , one has

$$2DS_2 = sS_1. \quad (3)$$

*Proof.* Multiplying velocity coordinates in (2) gives

$$(v_-)_x(v_-)_y - (v_+)_x(v_+)_y = \frac{2\sqrt{\lambda}\sqrt{W - \lambda}(2D - sW)}{dW^2}.$$

The left side telescopes around the orbit, since every outgoing velocity is the next incoming velocity. This proves the identity.  $\square$

**Lemma 3.** The center-pedal area is

$$A_O = \frac{\sqrt{\lambda}[sD - \lambda(\alpha^2 + \beta^2)]}{2D} S_1. \quad (4)$$

This identity requires no parity hypothesis.

*Proof.* Let  $q_-, q_+$  be the center's feet on the incoming and outgoing sidelines at  $P$ , and write

$$P_n = P \cdot n = \frac{ab}{\sqrt{W}}, \quad P_t = P \cdot t = -\frac{d \sin \theta \cos \theta}{\sqrt{W}}.$$

The unit normals of those two lines give their actual projections:

$$q_- = (hP_n - jP_t)(hn - jt), \quad q_+ = (hP_n + jP_t)(hn + jt).$$

Hence their ordered half determinant is

$$\frac{1}{2} \det(q_-, q_+) = hj(h^2 P_n^2 - j^2 P_t^2).$$

Using  $d^2 \sin^2 \theta \cos^2 \theta = (W - \beta)(\alpha - W)$ , the bracket simplifies to

$$\frac{D(W - \lambda) - \lambda(W - \beta)(\alpha - W)}{W^2} = \lambda + \frac{D - s\lambda}{W}.$$

Summing once over the vertices and applying (3),

$$A_O = \sqrt{\lambda}[\lambda S_1 + (D - s\lambda)S_2] = \frac{\sqrt{\lambda}[sD - \lambda(\alpha^2 + \beta^2)]}{2D} S_1.$$

These are feet on original sidelines, not on the boundary tangents. The coefficient is positive, since its bracketed numerator exceeds  $\beta^2(\alpha - \beta) > 0$ .  $\square$

### 3 Complexification of the billiard parameter

Stachel's parametrization [3, Theorem 2, pp. 11–12] is

$$P(u) = (-a \operatorname{sn} u, b \operatorname{cn} u), \quad P_i(u) = P(u + i\eta), \quad \eta = 2\Delta = \frac{4\tau K}{N}, \quad \gcd(\tau, N) = 1, \quad (5)$$

where  $k^2 = d/(\alpha - \lambda)$ ,  $K = K(k)$ , and  $K' = K(\sqrt{1 - k^2})$ . The modulus is suppressed in the Jacobi functions. Choose  $0 < \tau < N/2$ , so  $0 < \Delta < K$ . The caustic condition is

$$\operatorname{sn}^2 \Delta = \frac{\lambda}{\beta}, \quad \operatorname{cn}^2 \Delta = \frac{\beta - \lambda}{\beta}, \quad \operatorname{dn}^2 \Delta = \frac{\alpha(\beta - \lambda)}{\beta(\alpha - \lambda)}. \quad (6)$$

In particular the point  $P$  has real period  $4K$ , not  $2K$ .

Define the meromorphic functions

$$\begin{aligned} W(u) &= \alpha - d \operatorname{sn}^2 u, & f(u) &= \frac{\operatorname{dn} u}{W(u)}, & S(u) &= \sum_{i=0}^{N-1} f(u + i\eta), \\ A(u) &= \frac{1}{2} \sum_{i=0}^{N-1} \det(P(u + i\eta), P(u + (i+1)\eta)). \end{aligned} \quad (7)$$

They have common periods  $4K, 4iK'$  and their cyclic traces  $A, S$  are  $\eta$ -periodic. On the real axis,  $W - \lambda = (\alpha - \lambda) \operatorname{dn}^2 u$  and  $\operatorname{dn} u > 0$ , so

$$S_1 = \sqrt{\alpha - \lambda} S(u). \quad (8)$$

By Lemma 3, it suffices to prove that  $A(u)S(u)$  is constant. We do so by removing all its possible complex poles.

### 4 Pole classes and their separation

The following Jacobi identities are standard; see [4, Tables 22.4.1 and 22.4.3]. Their complete pole lattice is

$$z_p = 2mK + (2n+1)iK', \quad m, n \in \mathbb{Z}.$$

At the base pole,

$$\operatorname{sn}(iK' + w) = \frac{1}{k \operatorname{sn} w}, \quad \operatorname{cn}(iK' + w) = -\frac{i \operatorname{dn} w}{k \operatorname{sn} w}, \quad \operatorname{dn}(iK' + w) = -\frac{i \operatorname{cn} w}{\operatorname{sn} w}. \quad (9)$$

Thus all three are odd about every  $z_p$ ; translations only change some signs. There  $W$  has a double pole and  $\operatorname{dn}$  a simple pole, so  $f$  extends analytically with value zero and is odd about  $z_p$ .

Put  $Z = z_p + K$ . At the representative  $Z = K + iK'$ ,

$$\operatorname{sn}(Z + w) = \frac{\operatorname{dn} w}{k \operatorname{cn} w}, \quad \operatorname{cn}(Z + w) = -\frac{ik'}{k \operatorname{cn} w}, \quad k' = \sqrt{1 - k^2}.$$

Consequently  $P(Z + w) = P(Z - w)$ , also at every translated  $Z$ .

**Lemma 4.** *The poles of  $f$  are simple and occur exactly at*

$$Z \pm \Delta \pmod{2K\mathbb{Z} + 2iK'\mathbb{Z}}. \quad (10)$$

*Their orbit shifts are disjoint from all shifted vertex poles if and only if  $N \not\equiv 2 \pmod{4}$ .*

*Proof.* Equations (6) and the quarter-period identity give

$$\operatorname{sn}^2(K \pm \Delta) = \frac{\operatorname{cn}^2 \Delta}{\operatorname{dn}^2 \Delta} = \frac{\alpha - \lambda}{\alpha}.$$

The imaginary shift in (9) then gives  $\operatorname{sn}^2(Z \pm \Delta) = \alpha/d$ , hence  $W = 0$ . These two roots are distinct because  $0 < \Delta < K$  and exhaustive because  $\operatorname{sn}^2$  has one double pole and order two on the indicated lattice. At either root,

$$\operatorname{sn}^2 u = \frac{\alpha}{d}, \quad \operatorname{cn}^2 u = -\frac{\beta}{d}, \quad \operatorname{dn}^2 u = -\frac{\lambda}{\alpha - \lambda}.$$

All are nonzero, so  $W' = -2d \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u \neq 0$ . The apparent  $f$ -singularities at Jacobi poles are already removed.

A collision of shifted pole classes is equivalent to

$$j\eta \equiv K \pm \Delta \pmod{2K}, \quad 2\tau(2j \pm 1) = N(2\ell + 1) \quad (11)$$

for integers  $j, \ell$  (changing the sign of  $j$  has no effect). For odd  $N$  the two sides have different parity. If  $4 \mid N$ , coprimality makes  $\tau$  odd: the left side has exactly one factor of two, while the right has at least two. Conversely, if  $N = 2r$  with  $r$  odd, then  $\tau$  is odd; take  $2j \pm 1 = r$  and  $2\ell + 1 = \tau$ . This produces a collision. All translated complex pole classes are covered by the same congruence.  $\square$

## 5 Cancellation and proof of the theorem

Assume henceforth that  $N$  is odd or divisible by four. The only possible poles of  $A$  are shifted Jacobi vertex poles. Two lifted vertices can pole together only if their index difference times  $\eta$  is in  $2K\mathbb{Z}$ . For even  $N$  there can indeed be two antipodal lifted vertices separated by  $N/2$  indices. They are never adjacent: that would force  $N \mid 2\tau$ , impossible for primitive  $N \geq 3$ . Every edge determinant therefore contains at most one simple-pole factor. Thus  $A$  has at most a simple pole.

After a cyclic shift, take a possible area pole at  $u = z_p$ . By Lemma 4,  $S$  is analytic there. Its terms cancel under  $j \mapsto N - j$ , by oddness of  $f$  about  $z_p$  and the period  $N\eta = 4\tau K$ . The self-paired  $j = 0$  term is zero. If  $N$  is even, the other self-paired term  $j = N/2$  is also zero, because  $z_p + 2\tau K$  is a Jacobi pole. Therefore  $S(z_p) = 0$ , removing every possible simple pole of  $A$  in the product  $AS$ .

Now take a possible trace pole at  $u_0 = Z - \Delta$ . Evenness of  $P$  about  $Z$  gives

$$P_i(u_0) = P(Z + (2i - 1)\Delta) = P(Z - (2i - 1)\Delta) = P_{1-i}(u_0).$$

All these vertices are finite by Lemma 4. The edge  $i \rightarrow i + 1$  is paired with its reverse  $-i \rightarrow 1 - i$ :

$$\det(P_i, P_{i+1}) = -\det(P_{-i}, P_{1-i}).$$

Hence  $A(u_0) = 0$ . Here  $A$  is analytic and  $S$  has at most a simple pole, even if several summands pole simultaneously; the product again has a removable singularity. The other root  $Z + \Delta$  is an  $\eta$ -translate and both traces are  $\eta$ -periodic. Translations by  $2K$  change the overall sign of  $P$ , while translations by  $2iK'$  change one coordinate sign. They preserve the zero and pairing arguments. Thus every trace-pole class, not just the base pair, has been treated.

We have exhausted and removed every possible pole of  $AS$ . An entire elliptic function is constant. Equations (4) and (8) now prove Theorem 1. No division by a signed polygon area was used.  $\square$

## 6 Period scope, exact checks and provenance

For the primitive four-periodic family,  $\lambda = D/s$  and

$$AA_O = \frac{8D^2}{s^2}.$$

For example, at  $a = 4, b = 3$ , the axial diamond and rectangular orbit have respective area pairs  $(24, 6912/625)$  and  $(576/25, 288/25)$ , both with product  $165888/625$ . The supplement contains a direct four-periodic derivation, a separate elementary three-periodic proof, and exact coefficient and geometric checks. These finite controls are not used to infer the general theorem.

Least period is essential. Fix  $a = 5, b = 3, \lambda = 225/64$ . Primitive six-periodic trajectories beginning at  $(5, 0)$  and  $(4, 9/5)$  have center-pedal products

$$\frac{72665775}{65536} \quad \text{and} \quad \frac{122650422111}{110166016}.$$

The second-minus-first difference is

$$\frac{7800849}{1721344} > 0.$$

The supplementary standard-library checker verifies exact closure, least period six, reflection, strict caustic contacts, and actual perpendicular-foot areas. Traversing each twice creates a twelve-entry list and multiplies each product by four, without changing its least period. This refutes a padded-list extension, not Theorem 1. Repeating an allowed primitive orbit does preserve constancy, scaling the product by the square of the repetition count.

The confocal billiard geometry, Jacobi parametrization and complex pole-cancellation method are established prior mathematics. Chávez-Caliz's related theorem [5, Theorem 6] concerns the original and outer tangent-polygon areas for even periods; it does not itself identify the centered pedal or handle the odd case here. Our argument supplies the explicit centered-pedal trace factor and applies the complementary pole cancellations to the stated invariant. A bounded primary-literature search did not identify this exact prior conclusion; this does not establish worldwide novelty or absolute priority.

This is an unrefereed preprint. AI assistance was used in derivation, internal checking, coding and manuscript preparation. No independent human peer review or formal proof-assistant verification is claimed. The written analytic argument, not the accompanying numerical scouting or finite controls, establishes the general result.

## References

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- [4] NIST Digital Library of Mathematical Functions, §22.4, Periods, Poles, and Zeros, Tables 22.4.1 and 22.4.3; version 1.2.8, checked 1 October 2026.
- [5] A. C. Chávez-Caliz, More about areas and centers of Poncelet polygons, *Arnold Math. J.* **7** (2021), 91–106. [doi:10.1007/s40598-020-00154-8](https://doi.org/10.1007/s40598-020-00154-8).