

Pedal Areas of Periodic Elliptic Billiard Trajectories with Respect to an Arbitrary Point

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Abstract

Reznik, Garcia and Koiller listed many quantities that are experimentally constant along the families of N -periodic billiard trajectories in an ellipse whose caustic is a confocal ellipse. We treat three of their rows on the signed areas of pedal polygons with respect to an arbitrary point M . Let A be the signed area of the trajectory, A' that of its outer polygon, and A_M, A'_M the signed areas of their pedal polygons with respect to M . The rows state that A/A_M is constant for $N \equiv 0 \pmod{4}$ (row $k_{203,a}$), A/A_M for $N \equiv 2 \pmod{4}$ (row k_{204}) and A'/A'_M for $N \equiv 0 \pmod{4}$ (row k_{304}). We prove them for trajectories of least period N , every winding number and every point M , real or complex (the two ratios wherever they are defined). For $k_{203,a}$ we use a Jacobi parametrisation of the family, in which the product is an elliptic function of the phase. Its possible poles lie at the poles of the vertices and at the points where a side normal is isotropic; these two sets are disjoint exactly when N is odd or divisible by 4, an order-reversing symmetry makes each factor vanish at the poles of the other, and Liouville's theorem concludes. For $N \equiv 2 \pmod{4}$ we prove $A_M = \kappa(M)A$, and for the outer polygon with $N \equiv 0 \pmod{4}$ we prove $A'_M = \kappa'(M)A'$, where κ and κ' are quadratic in M with positive constant terms. So the ratios are constant wherever $\kappa(M) \neq 0$, resp. $\kappa'(M) \neq 0$; the exceptional set can be a real circle in star families (an exact example with $N = 8$ for the outer polygon). For even N the part of A_M that is linear in M vanishes. The rows are due to Reznik, Garcia and Koiller, and the parametrisation is Stachel's. The case $M = O$ of $k_{203,a}$ is in an earlier note of the author, and the dependence of A'_M on M for every even N was proved by Kriebel; for k_{304} only the constancy of A'/A'_O for $N \equiv 0 \pmod{4}$ is new. This is an unrefereed note.

1 Introduction

1.1 The three rows

Let $\mathcal{E}$ be the ellipse $x^2/a^2 + y^2/b^2 = 1$ with $a > b > 0$ and centre O , and fix λ with $0 < \lambda < b^2$. The ellipse

$$\mathcal{C} : \frac{x^2}{a^2 - \lambda} + \frac{y^2}{b^2 - \lambda} = 1$$

is confocal with $\mathcal{E}$ and lies inside it. A *billiard trajectory* in $\mathcal{E}$ is a sequence $(P_l)_{l \in \mathbb{Z}}$ of points of $\mathcal{E}$ with $P_{l+1} \neq P_l$ such that at every P_l the chords $P_{l-1}P_l$ and P_lP_{l+1} are mirror images of each other in the normal of $\mathcal{E}$ at P_l (the law of reflection); it has the *caustic* $\mathcal{C}$ if all its chords are tangent to $\mathcal{C}$. It is an N -periodic if $P_{l+N} = P_l$ for all l and $N \geq 3$ is the least such number. If one trajectory with caustic $\mathcal{C}$ is N -periodic, then all are, with the same winding number τ around O (Poncelet's porism; see Lemma 2.2); they form a one-parameter family, which we call the family.

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For a closed polygon $W = (W_1, \dots, W_N)$, indices taken modulo N , the *signed area* is $S(W) = \frac{1}{2} \sum_i W_i \times W_{i+1}$, where $x \times y = x_1 y_2 - x_2 y_1$. For an N -periodic P we write $A = S(P)$. For a point M let Q_i be the foot of the perpendicular from M to the line $P_i P_{i+1}$; the *pedal polygon* of P with respect to M is $(Q_1, \dots, Q_N)$, in the order of the sides, and A_M is its signed area. The *outer polygon* P' has the vertices V_i , the intersection points of the tangents to $\mathcal{E}$ at P_i and P_{i+1} ; its sides lie on these tangents, and we write $A' = S(P')$. The pedal polygon of P' with respect to M consists of the feet Q'_i of the perpendiculars from M to the tangents at P_i , in this order, and A'_M is its signed area. These are the definitions of Reznik, Garcia and Koiller [RGK21], who work with signed areas throughout.

In [RGK21] the authors list several dozen quantities that they found experimentally to be constant along such families. We treat three rows of their tables of pedal polygons, on products and ratios of signed areas for an arbitrary point M (Table 1); the same tables have further rows for an arbitrary point, among them $k_{303,a}$ (Section 1.3). The three rows are marked as not proven, both in the arXiv version [RGK21, v11], whose row codes the corpus [ula26] uses, and in Tables 3 and 4 of the journal version. The corresponding records of the UnsolvedMath collection [ula26] ask to prove that, for every M , the quantity is constant as P varies through the family.

record	row	quantity	N	M
AMR-050-0011	$k_{203,a}$	$A A_M$	$\equiv 0 \pmod{4}$	all
AMR-050-0013	k_{204}	A/A_M	$\equiv 2 \pmod{4}$	all
AMR-050-0016	k_{304}	A'/A'_M	$\equiv 0 \pmod{4}$	all

Table 1: The three rows of [RGK21, v11] treated here. The neighbouring rows $k_{203,b}$ ($A A_M$ for $M = O$ and $N \not\equiv 2 \pmod{4}$) and $k_{303,a}$, $k_{303,b}$ ($A' A'_M$) are discussed in Section 1.3.

1.2 Results

For points $M = (M_1, M_2)$ we allow complex coordinates; the feet are then defined by the same formulas (Section 3), and we write $M \cdot M = M_1^2 + M_2^2$, which is $|M|^2$ for real M .

Theorem A (row $k_{203,a}$). *Let $a > b > 0$ and $0 < \lambda < b^2$, and suppose that the trajectories with caustic $\mathcal{C}$ are N -periodic with $N \equiv 0 \pmod{4}$, of any winding number. Then there are constants $C_0 > 0$ and $C_2 \in \mathbb{R}$, depending only on a , b and λ , such that*

$$A A_M = C_0 + C_2 M \cdot M$$

for every trajectory of the family and every $M \in \mathbb{C}^2$. In particular $A A_M$ is constant along the family for every point M . For $N = 4$ one has $C_2 = 0$ and $C_0 = 8a^4 b^4 / (a^2 + b^2)^2$.

Theorem B (row k_{204}). *Let a , b , λ be as in Theorem A, but with $N \equiv 2 \pmod{4}$. Then there are constants $\kappa_0 > 0$ and $\kappa_2 \in \mathbb{R}$, depending only on a , b and λ , such that*

$$A_M = \kappa(M) A, \quad \kappa(M) := \kappa_0 + \kappa_2 M \cdot M,$$

for every trajectory of the family and every $M \in \mathbb{C}^2$. Here $A \neq 0$ for every trajectory. Consequently, for every M with $\kappa(M) \neq 0$ the ratio $A/A_M = 1/\kappa(M)$ is defined and constant along the family, and for every M with $\kappa(M) = 0$ the pedal area A_M vanishes for every trajectory of the family. The real points with $\kappa(M) = 0$ form the empty set if $\kappa_2 \geq 0$ and the circle $|M|^2 = -\kappa_0/\kappa_2$ about O if $\kappa_2 < 0$.

Theorem C (row k_{304}). *Let a, b, λ be as in Theorem A, with $N \equiv 0 \pmod{4}$. Then there are constants $\kappa'_0 > 0$ and $\kappa'_2 \in \mathbb{R}$, depending only on a, b and λ , such that*

$$A'_M = \kappa'(M) A', \quad \kappa'(M) := \kappa'_0 + \kappa'_2 M \cdot M,$$

for every trajectory of the family and every $M \in \mathbb{C}^2$. Here $A' \neq 0$. Consequently, for every M with $\kappa'(M) \neq 0$ the ratio $A'/A'_M = 1/\kappa'(M)$ is defined and constant along the family, and for every M with $\kappa'(M) = 0$ the area A'_M vanishes for every trajectory.

The exceptional sets are not empty in general. By a theorem of Kriebel [Kri26c, Theorem 1, (2)], valid for every even N ,

$$\frac{\kappa'_2}{\kappa'_0} = \frac{\delta}{2F}, \quad \delta := 1 - \lambda \left(\frac{1}{a^2} + \frac{1}{b^2} \right), \quad F := \frac{a^2 + b^2}{2} \delta + 2\lambda > 0, \quad (1)$$

so that in Theorem C the real exceptional set is empty if $\delta \geq 0$, which is the case for all families of winding number 1 [Kri26c, Theorem 1], and is the circle $|M|^2 = -2F/\delta$ if $\delta < 0$. For the star family with $N = 8$, $\tau = 3$, $a^2 = 32/27$, $b = 1$ and $\lambda = 32/35$ we find exactly $\kappa'(M) = (85 - 27 M \cdot M)/600$: the outer pedal area vanishes for every trajectory of this family when $|M|^2 = 85/27$ (Section 6). For Theorem B we found numerically $\kappa_2 < 0$, hence a real exceptional circle, for $N = 10$, $\tau = 3$, and $\kappa_2 > 0$ in all families of winding number 1 that we tested; that $\kappa_2 > 0$ for winding number 1 is not proved here.

On the way we obtain the following fact (Proposition 3.2): for even N , $A_{-M} = A_M$ and $A'_{-M} = A'_M$, so the parts of A_M and A'_M that are linear in M vanish. It is a direct consequence of the central symmetry of even periodic trajectories.

The hypotheses enter the proof of Theorem A at two places only: the condition $N \equiv 0 \pmod{4}$ in Lemma 4.4, which says that the poles of A and of A_M lie in disjoint sets, and the evenness of N in Lemma 4.5, through an involution of the sides. Least period and winding number coprime to N are used throughout. The conclusions fail where the hypotheses fail (Section 6): for $N \equiv 2 \pmod{4}$ the product $A A_M$ is not constant, for odd N it is constant for $M = O$ [Fer26a] but not for the other points tested, and a trajectory of least period 3, listed four times as a closed walk of length 12, does not have constant $A A_M$.

1.3 What is new and what is not

The three rows, and the experimental evidence for them, are due to Reznik, Garcia and Koiller [RGK21]. The parametrisation of the family by Jacobi elliptic functions that we use (Section 2) is classical; it is Stachel's canonical parametrisation [Sta22, Theorem 2], and we include the verifications we need. That confocal conics are caustics of the elliptic billiard is classical (see [Tab05]); Lemma 2.2 checks the reflection law directly in the parametrisation.

The case $M = O$ of Theorem A is the row $k_{203,b}$, proved for N odd or divisible by 4 in an earlier note of the author [Fer26a]. For the outer polygon and $N \equiv 2 \pmod{4}$ the product $A' A'_M$ is constant for every M , and for $N \not\equiv 0 \pmod{4}$ the product $A' A'_O$ is constant (rows $k_{303,a}$, $k_{303,b}$, [Fer26b]); that note also observes that the outer polygon is an affine image of the trajectory shifted by half a step (our Lemma 5.2(a)). Kriebel [Kri26c], in a note on the area centroid of the outer pedal polygon (row k_{307}), proves for every even N that $A'_M = A'_O (2F + \delta |M|^2)/(2F)$ with $A'_O > 0$, and that $\delta \geq 0$ for winding number 1. This is the dependence of A'_M on M in Theorem C, together with the exceptional circle; so in Theorem C only the constancy of A'/A'_O for $N \equiv 0 \pmod{4}$ is new. Our proof of Theorem C does not use [Kri26c]; we use it only for (1). Kriebel also proved that the pedal areas of the trajectory and of the outer polygon with respect to a focus are proportional (row

k_{606} , [Kri26a]), and the odd-period row $k_{403,a}$ on pedal and antipedal polygons with respect to O [Kri26b]. The method, which complexifies the phase, locates the possible poles and removes them by symmetries, followed by Liouville's theorem or by a comparison of residues, is the one of these notes [Fer26a, Fer26b, Kri26a, Kri26b], and we do not claim it as new. Of the other rows of the same table, the source credits $k_{202,a}$ and $k_{202,b}$ (products of distances) to Bialy and Tabachnikov [BT22], and k_{201} and k_{205} to private communications of A. Akopyan.

To our knowledge the following is new: Theorem A for an arbitrary point M , Theorem B, and, in Theorem C, the constancy of A'/A'_O for $N \equiv 0 \pmod{4}$. The searches on which this is based are described in the paragraph “Scope and priority” at the end of Section 6; we make no priority claim.

1.4 Conventions

Least period. The source does not define “ N -periodic” formally; in its experiments each N has its own caustic, which is the reading of least period. The reading “closed walk of length N ” changes the answer: if a trajectory of least period n is listed r times, then A and A_M (and A' , A'_M) are multiplied by r , so the products and ratios are constant exactly when they are for n . For $n = 3$, $r = 4$ the product AA_M is not constant (Section 6).

Orientation. Reversing the order of the vertices changes the signs of A , A_M , A' and A'_M and does not affect the statements. We use the counterclockwise order, in which A , A' , A_O and A'_O are positive (Lemma 3.4).

The caustic is a confocal ellipse, as in the source (“confocal ellipse pair”). Hyperbolic caustics are not treated.

2 The parametrisation

Throughout, $\alpha = a^2$, $\beta = b^2$, $0 < \lambda < \beta$, and

$$k^2 = \frac{\alpha - \beta}{\alpha - \lambda} \in (0, 1), \quad k'^2 = 1 - k^2 = \frac{\beta - \lambda}{\alpha - \lambda}, \quad K = K(k), \quad K' = K(k').$$

The functions sn , cn , dn have modulus k . For vectors $x, y \in \mathbb{C}^2$ we write $x \cdot y = x_1 y_1 + x_2 y_2$ (bilinear, also for complex vectors) and $x \times y = x_1 y_2 - x_2 y_1$. We use the following standard facts [DLM26].

(J1) sn , cn , dn are meromorphic on $\mathbb{C}$. Their poles are the points of $\Pi := iK' + 2K\mathbb{Z} + 2iK'\mathbb{Z}$, all simple; dn has simple zeros exactly at the points of $\Pi' := K + iK' + 2K\mathbb{Z} + 2iK'\mathbb{Z}$; and every point of $\Lambda := 4K\mathbb{Z} + 4iK'\mathbb{Z}$ is a period of all three functions ([DLM26, §22.4(i), Tables 22.4.1 and 22.4.2]).

(J2) For all w : $\operatorname{sn}(w + 2K) = -\operatorname{sn} w$, $\operatorname{cn}(w + 2K) = -\operatorname{cn} w$, $\operatorname{sn}(w + 2iK') = \operatorname{sn} w$, $\operatorname{cn}(w + 2iK') = -\operatorname{cn} w$, $\operatorname{dn}(w + 2iK') = -\operatorname{dn} w$,

$$\begin{aligned} \operatorname{sn}(w + iK') &= \frac{1}{k \operatorname{sn} w}, & \operatorname{cn}(w + iK') &= \frac{-i \operatorname{dn} w}{k \operatorname{sn} w}, \\ \operatorname{sn}(w + K + iK') &= \frac{\operatorname{dn} w}{k \operatorname{cn} w}, & \operatorname{cn}(w + K + iK') &= \frac{-ik'}{k \operatorname{cn} w} \end{aligned}$$

([DLM26, Table 22.4.3]).

(J3) $\operatorname{sn}^2 + \operatorname{cn}^2 = 1$ and $\operatorname{dn}^2 + k^2 \operatorname{sn}^2 = 1$ ([DLM26, §22.6(i)]), and the addition theorems

$$\operatorname{sn}(t \pm \Delta) = \frac{\operatorname{sn} t \operatorname{cn} \Delta \operatorname{dn} \Delta \pm \operatorname{sn} \Delta \operatorname{cn} t \operatorname{dn} t}{D}, \quad \operatorname{cn}(t \pm \Delta) = \frac{\operatorname{cn} t \operatorname{cn} \Delta \mp \operatorname{sn} t \operatorname{sn} \Delta \operatorname{dn} t \operatorname{dn} \Delta}{D},$$

$$D = 1 - k^2 \operatorname{sn}^2 t \operatorname{sn}^2 \Delta \text{ ([DLM26, §22.8(i)]).}$$

(J4) For real u , $\operatorname{sn} u = \sin \operatorname{am} u$ and $\operatorname{cn} u = \cos \operatorname{am} u$, where am is real analytic, $\operatorname{am}' = \operatorname{dn} > 0$ on $\mathbb{R}$, and $\operatorname{am}(u + 2K) = \operatorname{am} u + \pi$ ([DLM26, §22.16(i)]).

Let $\Delta \in (0, K)$ be the number with $\operatorname{sn}^2 \Delta = \lambda/\beta$, and put $s_0 = \operatorname{sn} \Delta$, $c_0 = \operatorname{cn} \Delta$, $d_0 = \operatorname{dn} \Delta$, all positive, and $h := 2\Delta$. Then

$$c_0^2 = \frac{\beta - \lambda}{\beta}, \quad d_0^2 = 1 - \frac{k^2 \lambda}{\beta} = \frac{\alpha(\beta - \lambda)}{\beta(\alpha - \lambda)}, \quad \frac{d_0^2}{\alpha} - \frac{1}{\beta} = -\frac{k^2}{\beta}. \quad (2)$$

Define, for $u, t, v \in \mathbb{C}$,

$$P(u) := (-a \operatorname{sn} u, b \operatorname{cn} u), \quad m(t) := \left(-\frac{d_0 \operatorname{sn} t}{a c_0}, \frac{\operatorname{cn} t}{b c_0} \right), \quad n(v) := \left(-\frac{\operatorname{sn} v}{a}, \frac{\operatorname{cn} v}{b} \right),$$

and $V(t) := (-ad_0 \operatorname{sn} t/c_0, b \operatorname{cn} t/c_0)$. For real u the point $P(u)$ lies on $\mathcal{E}$, and the tangent of $\mathcal{E}$ at $P(u)$ is the line $n(u) \cdot X = 1$. The ellipse $\mathcal{C}$ is touched by the line $px + qy = 1$ if and only if $(\alpha - \lambda)p^2 + (\beta - \lambda)q^2 = 1$.

Lemma 2.1 (chords). *For all $t \in \mathbb{C}$,*

$$m(t) \cdot P(t - \Delta) = m(t) \cdot P(t + \Delta) = 1, \quad n(t - \Delta) \cdot V(t) = n(t + \Delta) \cdot V(t) = 1.$$

For real t the line $L(t) := \{X : m(t) \cdot X = 1\}$ passes through the two distinct points $P(t \pm \Delta)$ of $\mathcal{E}$ and is tangent to $\mathcal{C}$, and $V(t)$ is the intersection point of the tangents of $\mathcal{E}$ at $P(t - \Delta)$ and $P(t + \Delta)$.

Proof. Write $s = \operatorname{sn} t$, $c = \operatorname{cn} t$, $e = \operatorname{dn} t$. Both identities say $d_0 s \operatorname{sn}(t \pm \Delta) + c \operatorname{cn}(t \pm \Delta) = c_0$, and by (J3)

$$d_0 s (s c_0 d_0 \pm s_0 c e) + c (c c_0 \mp s s_0 e d_0) = c_0 (s^2 d_0^2 + c^2) = c_0 (1 - k^2 s^2 s_0^2) = c_0 D.$$

For real t , $P(t + \Delta) = P(t - \Delta)$ would force $2\Delta \in 4K\mathbb{Z}$ by (J4), which is false; the tangency condition reads $(\alpha - \lambda)d_0^2 s^2 / (\alpha c_0^2) + (\beta - \lambda)c^2 / (\beta c_0^2) = 1$, which holds by (2) because $(\alpha - \lambda)d_0^2 / \alpha = (\beta - \lambda) / \beta = c_0^2$. The tangents at $P(t \pm \Delta)$ are not parallel, as $P(t + \Delta) \neq -P(t - \Delta)$ (this would force $2\Delta \in 2K + 4K\mathbb{Z}$). $\square$

Lemma 2.2 (trajectories). (a) *For real u, v one has $P(u) = P(v)$ if and only if $u - v \in 4K\mathbb{Z}$. For real u , the lines $L(u - \Delta)$ and $L(u + \Delta)$ are the two tangents from $P(u)$ to $\mathcal{C}$, and they are mirror images of each other in the normal of $\mathcal{E}$ at $P(u)$.*

(b) *For real u the points $P_l := P(u + lh)$, $l \in \mathbb{Z}$, form a billiard trajectory with caustic $\mathcal{C}$, which runs counterclockwise around O ; the chord $P_l P_{l+1}$ lies on $L(u + lh + \Delta)$.*

(c) *This trajectory is N -periodic if and only if $Nh = 4\tau K$ for an integer τ coprime to N . Then $0 < \tau < N/2$, and τ is the winding number of the trajectory around O .*

(d) *Every billiard trajectory with caustic $\mathcal{C}$ is, after a reversal of its order if necessary, of the form in (b).*

Proof. (a) The first statement follows from (J4). By Lemma 2.1, $L(u \pm \Delta)$ pass through $P(u)$ and are tangent to $\mathcal{C}$. They are distinct, since otherwise one line would contain the three distinct points $P(u - 2\Delta)$, $P(u)$, $P(u + 2\Delta)$ of $\mathcal{E}$ ($0 < 4\Delta < 4K$). As $P(u)$ lies outside $\mathcal{C}$, these are the two tangents from $P(u)$. For the mirror symmetry put $T = (a \operatorname{cn} u, b \operatorname{sn} u)$, a tangent vector of $\mathcal{E}$ at $P(u)$ orthogonal to $n = n(u)$. Two lines through $P(u)$ with normal vectors m_1, m_2 are mirror images in the normal line of $\mathcal{E}$ if and only if $(m_1 \cdot T)(m_2 \cdot n) + (m_2 \cdot T)(m_1 \cdot n) = 0$ (write m_j in the orthogonal basis

T , n ; the reflection changes the sign of the T -component). With $s = \operatorname{sn} u$, $c = \operatorname{cn} u$, $e = \operatorname{dn} u$ and $D = 1 - k^2 s^2 s_0^2 > 0$, the addition theorems and (2) give

$$c_0 D m(u \pm \Delta) \cdot T = s_0 (X \mp Y), \quad c_0 D m(u \pm \Delta) \cdot n = U \pm W,$$

where $X = k^2 s_0 c_0 s c$, $Y = d_0 e$, $U = c_0 e^2 / \beta$ and $W = s_0 d_0 s c e (\alpha^{-1} - \beta^{-1})$; here we used $d_0^2 s^2 / \alpha + c^2 / \beta = e^2 / \beta$, which is (2) together with (J3). Hence the left-hand side of the criterion, for $m_{1,2} = m(u \pm \Delta)$, equals $2s_0(XU + YW)/(c_0 D)^2$, and

$$XU + YW = s_0 s c e^2 \left(\frac{k^2 c_0^2}{\beta} - \frac{d_0^2 (\alpha - \beta)}{\alpha \beta} \right) = s_0 s c e^2 \frac{(\alpha - \beta)(\beta - \lambda)}{\beta^2 (\alpha - \lambda)} (1 - 1) = 0.$$

(b) By Lemma 2.1 with $t = u + lh + \Delta$, the chord $P_l P_{l+1}$ lies on $L(u + lh + \Delta)$ and is tangent to $\mathcal{C}$, and $P_{l+1} \neq P_l$ because $0 < h < 4K$. At P_l the two chords lie on $L(u + lh \mp \Delta)$, which are mirror images in the normal of $\mathcal{E}$ by (a); this is the reflection law. By (J4) the eccentric angle $\operatorname{am}(u + lh) + \pi/2$ of P_l increases with l , by less than π in each step since $0 < h < 2K$.

(c) By (a), $P_{l+N} = P_l$ for all l if and only if $Nh \in 4K\mathbb{Z}$; write $Nh = 4\tau K$ with $\tau > 0$. If $g = \gcd(\tau, N) > 1$ then N/g is also a period; conversely, if $\gcd(\tau, N) = 1$ and N' is a period, then $N'h = 4K\tau N'/N \in 4K\mathbb{Z}$ forces $N \mid N'$. Moreover $\tau/N = h/(4K) = \Delta/(2K) \in (0, \frac{1}{2})$, and by (J4) the eccentric angle increases by $\operatorname{am}(u + 4\tau K) - \operatorname{am} u = 2\pi\tau$ along one period.

(d) Let (P_l) be a billiard trajectory with caustic $\mathcal{C}$ and choose a real u with $P_0 = P(u)$. The chord $P_0 P_1$ is one of the two tangents from P_0 to $\mathcal{C}$, so by (a) $P_1 = P(u + h)$ or $P_1 = P(u - h)$; after a reversal of the order we may assume $P_1 = P(u + h)$. If $P_{l-1} = P(u + (l-1)h)$ and $P_l = P(u + lh)$, the reflection law and (a) show that the chord $P_l P_{l+1}$ lies on $L(u + lh + \Delta)$, whose second point on $\mathcal{E}$ is $P(u + (l+1)h)$. By induction, and in the same way for negative l , $P_l = P(u + lh)$ for all l . $\square$

Parts (b) and (c) are Stachel's description of the family [Sta22, Theorem 2]: the canonical parametrisation $(-a \operatorname{sn} \tilde{u}, b \operatorname{cn} \tilde{u})$ with the modulus k (the numerical eccentricity of $\mathcal{C}$), vertex parameters $\tilde{u}_0 + 2l\Delta\tilde{u}$ with $b_{\mathcal{C}} = b \operatorname{cn} \Delta\tilde{u}$, and $\Delta\tilde{u} = 2\tau K/N$ for N -periodics. From now on N and τ are as in (c); by (d), the family of N -periodics is $\{(P(u + lh))_l : u \in \mathbb{R}\}$, and a quantity is constant along the family if and only if it is constant as a function of the real phase u . Since P , m , n have the period $4K$ and $Nh = 4\tau K$, the vectors $P(u + lh)$, $m(u + lh + \Delta)$ and $n(u + lh)$ depend only on l modulo N .

Lemma 2.3 (normals). *For all $t, v \in \mathbb{C}$,*

$$m(t) \cdot m(t) = \frac{\operatorname{dn}^2 t}{\beta c_0^2}, \quad n(v) \cdot n(v) = \frac{\alpha - (\alpha - \beta) \operatorname{sn}^2 v}{\alpha \beta}.$$

Hence $m \cdot m$ has double zeros exactly at the points of Π' and double poles exactly at the points of Π , while $n \cdot n$ has simple zeros exactly at the points of $\Pi' \pm \Delta$ and double poles at the points of Π . For real arguments both are positive.

Proof. By (2), $c_0^2 m \cdot m = d_0^2 \operatorname{sn}^2 t / \alpha + \operatorname{cn}^2 t / \beta = (1 - k^2 \operatorname{sn}^2 t) / \beta$; the zeros and poles then come from (J1). The formula for $n \cdot n$ is immediate. By (J2), $\operatorname{sn}^2(K + iK' \pm \Delta) = d_0^2 / (k^2 c_0^2) = \alpha / (\alpha - \beta)$ by (2), so $n \cdot n$ vanishes at $\Pi' \pm \Delta$. The function sn^2 is elliptic with the periods $2K$, $2iK'$ and has one double pole in a period parallelogram, so it takes each value exactly twice there, counted with multiplicity; the two classes $K + iK' \pm \Delta$ modulo $2K\mathbb{Z} + 2iK'\mathbb{Z}$ are different because $0 < 2\Delta < 2K$, so these zeros are all the zeros and are simple. $\square$

In particular the side normals $m(t)$ are *isotropic*, $m \cdot m = 0$, exactly at the zeros of dn , and the tangent normals $n(v)$ exactly at $\Pi' \pm \Delta$.

Lemma 2.4 (parities). *Let ϕ be one of P , m , n , and let $R := \text{diag}(1, -1)$.*

- (a) *For $z \in \Pi$ and all w : $\phi(z + w) = -\phi(z - w)$.*
- (b) *For $Z \in \Pi'$ and all w : $\phi(Z + w) = \phi(Z - w)$.*
- (c) *$\phi(w + 2K) = -\phi(w)$ and $\phi(w + 2iK') = R\phi(w)$.*

Moreover ϕ is holomorphic outside Π and has simple poles at the points of Π .

Proof. Each ϕ has the form $(c_1 \text{sn } w, c_2 \text{cn } w)$ with nonzero constants. By (J2), $\text{sn}(iK' + w)$ and $\text{cn}(iK' + w)$ are odd in w , and $\text{sn}(K + iK' + w)$ and $\text{cn}(K + iK' + w)$ are even in w . A translation of z by $2pK + 2qiK'$ multiplies sn by $(-1)^p$ and cn by $(-1)^{p+q}$, independently of w , which gives (a) and (b) for all points of Π and Π' ; (c) is (J2) again, and the last statement is (J1). $\square$

Lemma 2.5 (orientation). *For real v, w ,*

$$P(v) \times P(w) = ab \sigma, \quad m(v) \times m(w) = \frac{d_0}{ab c_0^2} \sigma, \quad n(v) \times n(w) = \frac{\sigma}{ab}, \quad \sigma = \sin(\text{am } w - \text{am } v).$$

In particular all three are positive if $0 < w - v < 2K$.

Proof. Each of the three cross products equals a positive constant times $\text{sn } w \text{cn } v - \text{sn } v \text{cn } w = \sin(\text{am } w - \text{am } v)$ by (J4). If $0 < w - v < 2K$, then $0 < \text{am } w - \text{am } v < \pi$ because am is increasing and $\text{am}(v + 2K) = \text{am } v + \pi$. $\square$

3 Pedal areas as functions of the phase

From now on the trajectories with caustic $\mathcal{C}$ are N -periodics with winding number τ , so $Nh = 4\tau K$ and $\text{gcd}(\tau, N) = 1$ (Lemma 2.2). For a complex phase u put

$$u_l := u + lh, \quad t_i := u_i + \Delta, \quad P_l := P(u_l), \quad m_i := m(t_i), \quad n_l := n(u_l) \quad (l, i \in \mathbb{Z}/N).$$

For $M \in \mathbb{C}^2$ and $x \in \mathbb{C}^2$ with $x \cdot x \neq 0$ let

$$\Phi_M(x) := M + \frac{1 - x \cdot M}{x \cdot x} x. \quad (3)$$

For real x and M , $\Phi_M(x)$ is the foot of the perpendicular from M to the line $x \cdot X = 1$: it lies on the line, and $\Phi_M(x) - M$ is parallel to x . We define, as functions of $u \in \mathbb{C}$,

$$A := \frac{1}{2} \sum_l P_l \times P_{l+1}, \quad A' := \frac{1}{2} \sum_l V_l \times V_{l+1}, \quad V_l := V(u_l + \Delta),$$

$$A_M := \frac{1}{2} \sum_i Q_i \times Q_{i+1}, \quad Q_i := \Phi_M(m_i), \quad A'_M := \frac{1}{2} \sum_l Q'_l \times Q'_{l+1}, \quad Q'_l := \Phi_M(n_l).$$

For a real phase u and a real point M these are the quantities of the introduction for the trajectory $(P(u + lh))_l$: the side $P_i P_{i+1}$ lies on $L(t_i)$ by Lemma 2.2(b), the side of P' through P_l lies on the tangent $n_l \cdot X = 1$, and V_l is the intersection of the tangents at P_l and P_{l+1} by Lemma 2.1. For complex M we take these formulas as the definition. All four functions are meromorphic in u , since the denominators $m_i \cdot m_i$ and $n_l \cdot n_l$ are not identically zero (Lemma 2.3). They are invariant under $u \mapsto u + h$, which only shifts the indices, and under translations by Λ , by (J1). Since $V(t) = \Gamma P(t)$ with $\Gamma = \text{diag}(d_0/c_0, 1/c_0)$,

$$A'(u) = \det \Gamma \cdot A(u + \Delta) = \frac{d_0}{c_0^2} A(u + \Delta). \quad (4)$$

Lemma 3.1 (dependence on M). *Let $x_1, \dots, x_N \in \mathbb{C}^2$ with $x_i \cdot x_i \neq 0$, indices modulo N , and put $w_i := (x_i \times x_{i+1}) / ((x_i \cdot x_i)(x_{i+1} \cdot x_{i+1}))$. Then for every $M \in \mathbb{C}^2$*

$$\frac{1}{2} \sum_i \Phi_M(x_i) \times \Phi_M(x_{i+1}) = \frac{1}{2} \sum_i w_i - \frac{1}{2} \sum_i w_i (x_i + x_{i+1}) \cdot M + \left(\frac{1}{4} \sum_i w_i (x_i \cdot x_{i+1}) \right) M \cdot M.$$

Consequently $A_M = A_O + c_1 \cdot M + c_2 M \cdot M$ and $A'_M = A'_O + c'_1 \cdot M + c'_2 M \cdot M$, where A_O, c_1, c_2 (with $x_i = m_i$) and A'_O, c'_1, c'_2 (with $x_i = n_i$) are meromorphic functions of u given by these formulas.

Proof. The signed area of a closed polygon does not change under translation, since $\sum_i (M \times Q_{i+1} + Q_i \times M) = \sum_i (M \times Q_{i+1} - M \times Q_i) = 0$. Translating by $-M$ and using $\Phi_M(x) - M = (1 - x \cdot M)x / (x \cdot x)$, the left-hand side becomes $\frac{1}{2} \sum_i (1 - x_i \cdot M)(1 - x_{i+1} \cdot M) w_i$. The constant and the linear terms are as stated. For the quadratic term $\frac{1}{2} \sum_i w_i (x_i \cdot M)(x_{i+1} \cdot M)$ write $y^\pm := y_1 \pm iy_2$ for $y \in \mathbb{C}^2$, so that $y \cdot z = (y^+ z^- + y^- z^+) / 2$, $y \times z = (y^- z^+ - y^+ z^-) / (2i)$ and $y \cdot y = y^+ y^-$; note $x_i^\pm \neq 0$. Then

$$(x_i \cdot M)(x_{i+1} \cdot M) = \frac{1}{4} (x_i^+ x_{i+1}^+ (M^-)^2 + x_i^- x_{i+1}^- (M^+)^2 + (x_i^+ x_{i+1}^- + x_i^- x_{i+1}^+) M^+ M^-),$$

and the coefficients of $(M^-)^2$ and $(M^+)^2$ in the quadratic term telescope to zero:

$$\frac{1}{8} \sum_i w_i x_i^+ x_{i+1}^+ = \frac{1}{16i} \sum_i \left(\frac{x_{i+1}^+}{x_{i+1}^-} - \frac{x_i^+}{x_i^-} \right) = 0, \quad \frac{1}{8} \sum_i w_i x_i^- x_{i+1}^- = \frac{1}{16i} \sum_i \left(\frac{x_i^-}{x_i^+} - \frac{x_{i+1}^-}{x_{i+1}^+} \right) = 0.$$

The remaining term is $\frac{1}{8} \sum_i w_i (x_i^+ x_{i+1}^- + x_i^- x_{i+1}^+) M^+ M^- = \frac{1}{4} \sum_i w_i (x_i \cdot x_{i+1}) M \cdot M$. For the last statement apply this with $x_i = m_i$ (resp. $x_i = n_i$) at all phases where no $m_i \cdot m_i$ (resp. $n_i \cdot n_i$) vanishes; both sides are meromorphic in u . $\square$

For real vectors the lemma contains the classical observation, going back to Steiner, that the pedal area of a fixed polygon is constant on circles about a fixed centre.

Proposition 3.2 (even N). *Let N be even. Then $P_{l+N/2} = -P_l$, $m_{i+N/2} = -m_i$ and $n_{l+N/2} = -n_l$, and $A_{-M} = A_M$, $A'_{-M} = A'_M$ for all $M \in \mathbb{C}^2$ and all phases. Hence $c_1 = c'_1 = 0$, that is,*

$$A_M = A_O + c_2 M \cdot M, \quad A'_M = A'_O + c'_2 M \cdot M.$$

Proof. As N is even and $\gcd(\tau, N) = 1$, τ is odd, so $\frac{N}{2}h = 2\tau K \in 2K + 4K\mathbb{Z}$; the first three identities follow from Lemma 2.4(c) and the period $4K$. From (3), $\Phi_M(-x) = -\Phi_{-M}(x)$, so $Q_{i+N/2}(M) = -Q_i(-M)$, and shifting the index of summation by $N/2$ gives $2A_M = \sum_i Q_{i+N/2}(M) \times Q_{i+N/2+1}(M) = \sum_i Q_i(-M) \times Q_{i+1}(-M) = 2A_{-M}$. The same argument applies to A'_M . By Lemma 3.1, $A_M - A_{-M} = 2c_1 \cdot M$, so $c_1 = 0$; likewise $c'_1 = 0$. $\square$

Lemma 3.3 (shifts). *For every N , $A(u + 2K) = A(u)$ and $A(u + 2iK') = -A(u)$. If N is even, then each of the functions A_O, c_2, A', A'_O, c'_2 is also invariant under $u \mapsto u + 2K$ and changes sign under $u \mapsto u + 2iK'$. In particular all these functions, and A , are invariant under the group $G' := h\mathbb{Z} + 2K\mathbb{Z} + 4iK'\mathbb{Z}$.*

Proof. By Lemma 2.4(c), $u \mapsto u + 2K$ replaces P_l, m_i, n_l by their negatives, and $u \mapsto u + 2iK'$ replaces them by RP_l, Rm_i, Rn_l . The area A is quadratic in the P_l , and $Rx \times Ry = -x \times y$. For the pedal areas, $\Phi_M(-x) = -\Phi_{-M}(x)$ gives $A_M(u + 2K) = A_{-M}(u) = A_M(u)$ by Proposition 3.2, and $\Phi_M(Rx) = R\Phi_{RM}(x)$ (because $Rx \cdot M = x \cdot RM$ and $R^2 = 1$) gives $A_M(u + 2iK') = -A_{RM}(u)$. As $RM \cdot RM = M \cdot M$, comparing the coefficients in $A_M = A_O + c_2 M \cdot M$ gives the claims for A_O and c_2 ; the outer pedal area is treated in the same way, and A' by (4). Invariance under h was noted above. $\square$

Lemma 3.4 (positivity). *For every real phase u : $A > 0$, $A' > 0$, $A_O > 0$ and $A'_O > 0$.*

Proof. Since $0 < h < 2K$, Lemma 2.5 gives $P_l \times P_{l+1} > 0$, $m_i \times m_{i+1} > 0$ and $n_l \times n_{l+1} > 0$, and $m_i \cdot m_i > 0$, $n_l \cdot n_l > 0$ by Lemma 2.3. Hence $A > 0$, and $A_O = \frac{1}{2} \sum w_i > 0$, $A'_O > 0$ by Lemma 3.1 with $M = O$; finally $A' > 0$ by (4). $\square$

For A'_O this positivity is also [Kri26c, (5)]. For $N = 4$ the quadratic coefficients vanish: then $h = K$, so $m_{i+2} = -m_i$ and $n_{l+2} = -n_l$, whence $w_{i+1} = w_i$ and $c_2 = \frac{1}{2}w_0(m_0 \cdot m_1 + m_1 \cdot m_2) = 0$, and likewise $c'_2 = 0$. Thus for $N = 4$ neither pedal area depends on M .

4 Proof of Theorem A

Let

$$G := h\mathbb{Z} + 2K\mathbb{Z} + 2iK'\mathbb{Z}, \quad \mathcal{P}_A := \Pi - h\mathbb{Z} = iK' + G, \quad \mathcal{P}_{\text{iso}} := \Pi' - \Delta - h\mathbb{Z} = K + iK' - \Delta + G.$$

$\mathcal{P}_A$ is the set of phases u at which some vertex parameter u_l is a pole of the Jacobi functions, and $\mathcal{P}_{\text{iso}}$ the set of phases at which some side parameter t_i lies in Π' , so that the normal m_i is isotropic (Lemma 2.3). Both are cosets of the group G , and both lie on the lines $\text{Im } u \in K' + 2K'\mathbb{Z}$; in particular no real phase lies in $\mathcal{P}_A \cup \mathcal{P}_{\text{iso}}$, and all quantities are finite on the real family.

Lemma 4.1 (poles of A). *A is holomorphic outside $\mathcal{P}_A$. For $u \in \mathcal{P}_A$ the set $J(u) := \{l \in \mathbb{Z}/N : u_l \in \Pi\}$ consists of one index if N is odd and of two indices $j, j + N/2$ if N is even; it never contains two adjacent indices. At every point of $\mathcal{P}_A$, A has at most a simple pole.*

Proof. A is a polynomial in the coordinates of the P_l , which are holomorphic unless $u_l \in \Pi$ (Lemma 2.4). If $u_l, u_{l'} \in \Pi$, then $(l - l')h \in \Pi - \Pi = 2K\mathbb{Z} + 2iK'\mathbb{Z}$, and as h is real, $(l - l')h \in 2K\mathbb{Z}$, that is $N \mid 2\tau(l - l')$, that is $N \mid 2(l - l')$ since $\gcd(\tau, N) = 1$. So $l' = l$, or N is even and $l' = l + N/2$; in the latter case $u_{l+N/2} = u_l + 2\tau K \in \Pi$ indeed. Adjacent indices would need $N \mid 2$. Hence every term $P_l \times P_{l+1}$ has at most one factor with a pole, and that pole is simple. $\square$

Lemma 4.2 (feet). *Let $u_1 \in \mathbb{C}$ and $i \in \mathbb{Z}/N$.*

- (a) *If $t_i(u_1) \in \Pi$, then Q_i has a removable singularity at u_1 ; we use its holomorphic extension.*
- (b) *If $t_i(u_1) \in \Pi'$, then Q_i has at most a pole of order 2 at u_1 .*
- (c) *Otherwise Q_i is holomorphic at u_1 .*

Consequently A_M is holomorphic outside $\mathcal{P}_{\text{iso}}$. For $u \in \mathcal{P}_{\text{iso}}$ the set $I(u) := \{i \in \mathbb{Z}/N : t_i \in \Pi'\}$ consists of one index if N is odd and of two indices $i, i + N/2$ if N is even, never adjacent.

Proof. (a) Near u_1 write $w = u - u_1$. By Lemma 2.4, $m_i = v/w + O(1)$ with $v \neq 0$, and $m_i \cdot m_i = (v \cdot v)/w^2 + O(1/w)$ with $v \cdot v \neq 0$, because $m \cdot m$ has a double pole at Π (Lemma 2.3). Then $(1 - m_i \cdot M)m_i = -(v \cdot M)v/w^2 + O(1/w)$, so Q_i is bounded near u_1 . (b) m_i is holomorphic at u_1 and $m_i \cdot m_i$ has a zero of order exactly 2 there. (c) m_i is holomorphic and $m_i \cdot m_i \neq 0$. The statement on $I(u)$ follows as in Lemma 4.1, since $\Pi' - \Pi' = 2K\mathbb{Z} + 2iK'\mathbb{Z}$ as well. $\square$

Lemma 4.3 (simple poles of A_M). *At every point $u_1 \in \mathcal{P}_{\text{iso}}$, A_M has at most a simple pole.*

Proof. Put $\tilde{Q}_j = Q_j - M$, so that $2A_M = \sum_j \tilde{Q}_j \times \tilde{Q}_{j+1}$ (translation invariance). By Lemma 4.2 no term contains two singular factors, and for $i \in I(u_1)$ the two terms containing $\tilde{Q}_i$ add up to $\tilde{Q}_i \times (Q_{i+1} - Q_{i-1})$, where $Q_{i\pm 1}$ are holomorphic at u_1 . Let $Z := t_i(u_1) \in \Pi'$. Then $t_{i\pm 1}(u_1 + w) = Z \pm h + w$, and by Lemma 2.4(b), $m(Z - h + w) = m(Z + h - w)$, that is, $m_{i-1}(u_1 + w) = m_{i+1}(u_1 - w)$. Hence $Q_{i-1}(u_1 + w) = Q_{i+1}(u_1 - w)$ near $w = 0$ (off the removable singularities of Lemma 4.2(a), and then by continuity everywhere), so $Q_{i+1} - Q_{i-1}$ vanishes at u_1 . A function with a pole of order at most 2 times a function with a zero has at most a simple pole. $\square$

For $N = 4$ one has $h = K$ and $Z \pm h \in \Pi$: the neighbours of an isotropic side are at poles of m , and Lemma 4.2(a) is needed. For $N \equiv 0 \pmod{4}$, $N \geq 8$, this does not happen, since $h \in K + 2K\mathbb{Z}$ would mean $4\tau = N(2p + 1)$, so that $N/4$ would divide τ .

Lemma 4.4 (the residue class modulo 4). *If N is odd or $N \equiv 0 \pmod{4}$, then $\mathcal{P}_A \cap \mathcal{P}_{\text{iso}} = \emptyset$. If $N \equiv 2 \pmod{4}$, then $\mathcal{P}_A = \mathcal{P}_{\text{iso}}$.*

Proof. Two cosets of G are equal or disjoint, and $\mathcal{P}_A = \mathcal{P}_{\text{iso}}$ if and only if $K - \Delta \in G$. As $K - \Delta$ is real and the real elements of G form $h\mathbb{Z} + 2K\mathbb{Z}$, this means $rK = s\Delta$ with odd integers r, s , and with $\Delta = 2\tau K/N$ it means $rN = 2\tau s$. If N is odd, the left side is odd and the right side even. If $N \equiv 0 \pmod{4}$, then τ is odd, so the right side is $\equiv 2 \pmod{4}$ and the left side $\equiv 0 \pmod{4}$. If $N = 2N'$ with N' odd, then τ is odd and $r = \tau$, $s = N'$ is a solution. $\square$

Lemma 4.5 (zeros of A_M). *Let $N \equiv 0 \pmod{4}$, $u_0 \in \mathcal{P}_A$ and $M \in \mathbb{C}^2$. Then A_M is holomorphic at u_0 and $A_M(u_0) = 0$.*

Proof. By Lemma 4.4, $u_0 \notin \mathcal{P}_{\text{iso}}$, so A_M is holomorphic at u_0 by Lemma 4.2. Let $j \in J(u_0)$, $z := u_0 + jh \in \Pi$, and $w_k := (k - j + \frac{1}{2})h$ for $k \in \mathbb{Z}$, so that $t_k(u_0) = z + w_k$. No $t_k(u_0)$ lies in Π : otherwise $w_k \in 2K\mathbb{Z}$, that is $N \mid \tau(2(k - j) + 1)$, which is impossible for even N . Hence all $m_k(u_0)$ are finite with $m_k \cdot m_k \neq 0$, and by the proof of Lemma 3.1,

$$2A_M(u_0) = \sum_k \Psi(m_k, m_{k+1}), \quad \Psi(x, y) := (1 - x \cdot M)(1 - y \cdot M) \frac{x \times y}{(x \cdot x)(y \cdot y)},$$

with $\Psi(y, x) = -\Psi(x, y)$. Let $\rho(k) := 2j - 1 - k + N/2$. Then $w_{\rho(k)} = -w_k + \frac{N}{2}h = -w_k + 2\tau K$, and since τ is odd, Lemma 2.4(c),(a) give

$$m_{\rho(k)} = m(z - w_k + 2\tau K) = -m(z - w_k) = m(z + w_k) = m_k.$$

As ρ is a bijection of $\mathbb{Z}/N$ with $\rho(k + 1) = \rho(k) - 1$,

$$\sum_k \Psi(m_k, m_{k+1}) = \sum_k \Psi(m_{\rho(k)}, m_{\rho(k)-1}) = \sum_k \Psi(m_k, m_{k-1}) = -\sum_k \Psi(m_{k-1}, m_k),$$

so $A_M(u_0) = -A_M(u_0) = 0$. $\square$

Lemma 4.6 (zeros of A). *Let N be odd or $N \equiv 0 \pmod{4}$, and $u_1 \in \mathcal{P}_{\text{iso}}$. Then A is holomorphic at u_1 and $A(u_1) = 0$.*

Proof. By Lemma 4.4, $u_1 \notin \mathcal{P}_A$, so all P_l are holomorphic at u_1 . Let $i \in I(u_1)$, $Z := t_i(u_1) \in \Pi'$ and $w'_l := (l - i - \frac{1}{2})h$, so that $u_l(u_1) = Z + w'_l$. The involution $\rho'(l) := 2i + 1 - l$ satisfies $w'_{\rho'(l)} = -w'_l$, so $P_{\rho'(l)} = P(Z - w'_l) = P(Z + w'_l) = P_l$ by Lemma 2.4(b), and $\rho'(l + 1) = \rho'(l) - 1$. As in the previous proof, $2A(u_1) = \sum_l P_{\rho'(l)} \times P_{\rho'(l)-1} = \sum_l P_l \times P_{l-1} = -2A(u_1)$. $\square$

Proof of Theorem A. Let $N \equiv 0 \pmod{4}$ and $M \in \mathbb{C}^2$. The function $f := AA_M$ is meromorphic and Λ -periodic, and by Lemmas 4.1 and 4.2 its poles lie in $\mathcal{P}_A \cup \mathcal{P}_{\text{iso}}$. At $u_0 \in \mathcal{P}_A$, A has at most a simple pole and A_M is holomorphic with a zero (Lemma 4.5), so f is holomorphic at u_0 . At $u_1 \in \mathcal{P}_{\text{iso}}$, A_M has at most a simple pole (Lemma 4.3) and A is holomorphic with a zero (Lemma 4.6), so f is holomorphic at u_1 . Thus f is entire and Λ -periodic, hence bounded, hence constant by Liouville's theorem.

By Proposition 3.2, $f = AA_O + (Ac_2)M \cdot M$. Taking $M = O$ and $M = (1, 0)$ shows that $C_0 := AA_O$ and $C_2 := Ac_2$ are constant; they are real, as all quantities are real for a real phase, and $C_0 > 0$ by Lemma 3.4. They depend only on a, b, λ , which determine the parametrisation. For $N = 4$, $c_2 = 0$ by the remark after Lemma 3.4, so $C_2 = 0$, and at the phase $u = 0$ the trajectory is the rhombus $(0, b), (-a, 0), (0, -b), (a, 0)$ (as $h = K$). There $A = 2ab$, and the feet of the perpendiculars from O on the four sides are $(-ab^2, a^2b), (-ab^2, -a^2b), (ab^2, -a^2b), (ab^2, a^2b)$, each divided by $a^2 + b^2$: the vertices of a positively oriented rectangle of area $4a^3b^3/(a^2 + b^2)^2$. Hence $C_0 = 8a^4b^4/(a^2 + b^2)^2$. $\square$

Remark 4.7 (where the hypotheses enter). The congruence $N \equiv 0 \pmod{4}$ is used only through Lemma 4.4, in Lemmas 4.5 and 4.6; the evenness of N and the oddness of τ are used, in addition, in Lemma 4.5, for the involution ρ and to see that no side parameter lies in Π at u_0 . Lemmas 4.1 to 4.3 hold for every N . If $N \equiv 2 \pmod{4}$, the two pole sets coincide and the argument breaks down; indeed AA_M is then not constant (Section 6), and Theorem B holds instead. For odd N the available reversal $k \mapsto 2j - 1 - k$ maps m_k to $-m_k$ rather than to m_k , and the proof of Lemma 4.5 does not apply; numerically AA_M is then constant for $M = O$, which is the row $k_{203,b}$ proved in [Fer26a], but not for the other points tested (Section 6).

5 Proofs of Theorems B and C

Theorems B and C are statements about ratios. Their proofs compare residues, and for this they need that the residues of A do not vanish. Note that $\mathcal{P}_A = iK' + G$ is the union of the two cosets $iK' + G'$ and $3iK' + G'$ of the subgroup $G' = h\mathbb{Z} + 2K\mathbb{Z} + 4iK'\mathbb{Z}$.

Lemma 5.1 (residues of A). *Let $\nu = 1$ if N is odd and $\nu = 2$ if N is even, and*

$$r_A := \nu \frac{iab(\operatorname{dn} h - 1)}{k^2 \operatorname{sn} h} \neq 0.$$

At every point of $iK' + G'$ the function A has a simple pole with residue r_A , and at every point of $3iK' + G'$ a simple pole with residue $-r_A$.

Proof. By Lemma 3.3, A is invariant under G' and changes sign under $u \mapsto u + 2iK'$, so it suffices to compute the residue at $u_0 = iK' - jh$, where $u_j(u_0) = iK'$. With $w = u - u_0$, (J2) gives $P_j = (-a/(k \operatorname{sn} w), -ib \operatorname{dn} w/(k \operatorname{sn} w)) = r/w + O(w)$ with $r = (-a, -ib)/k$. The two terms of $2A$ containing P_j add up to $P_j \times (P_{j+1} - P_{j-1})$, and $P_{j\pm 1}(u_0) = P(iK' \pm h)$ are finite because $h \notin 2K\mathbb{Z}$. By Lemma 2.4(a), $P(iK' - h) = -P(iK' + h)$, so the residue of these two terms is

$$2r \times P(iK' + h) = \frac{2}{k} \left((-a) \frac{-ib \operatorname{dn} h}{k \operatorname{sn} h} - (-ib) \frac{-a}{k \operatorname{sn} h} \right) = \frac{2iab(\operatorname{dn} h - 1)}{k^2 \operatorname{sn} h}.$$

For even N the antipodal vertex $j + N/2$ contributes the same amount, because $P_{j+N/2} = -P_j$ and $P_{j+N/2\pm 1} = -P_{j\pm 1}$ (Proposition 3.2); by Lemma 4.1 there are no other singular vertices. Dividing by 2 gives r_A . Finally $0 < h < 2K$ gives $\operatorname{sn} h \neq 0$ and $0 < \operatorname{dn} h < 1$, so $r_A \neq 0$. $\square$

Proof of Theorem B. Let $N \equiv 2 \pmod{4}$. By Proposition 3.2, $A_M = A_O + c_2 M \cdot M$ with $A_O = A_{(0,0)}$ and $c_2 = A_{(1,0)} - A_{(0,0)}$. By Lemmas 4.2 and 4.3, which hold for every N , the functions A_O and c_2 are holomorphic outside $\mathcal{P}_{\text{iso}}$ and have at most simple poles on $\mathcal{P}_{\text{iso}}$, and $\mathcal{P}_{\text{iso}} = \mathcal{P}_A$ by Lemma 4.4. By Lemma 3.3 the residues of A_O and of c_2 are constant on $iK' + G'$ and change sign on $3iK' + G'$, exactly as those of A do (Lemma 5.1). Hence

$$\kappa_0 := \frac{\text{Res } A_O}{\text{Res } A}, \quad \kappa_2 := \frac{\text{Res } c_2}{\text{Res } A}$$

are the same at every point of $\mathcal{P}_A$. The functions $g_0 := A_O - \kappa_0 A$ and $g_2 := c_2 - \kappa_2 A$ are meromorphic and Λ -periodic, with at most simple poles, all in $\mathcal{P}_A$, where their residues vanish. So they are entire and hence constant, and since they change sign under $u \mapsto u + 2iK'$ (Lemma 3.3), they vanish. Thus $A_O = \kappa_0 A$, $c_2 = \kappa_2 A$ and $A_M = (\kappa_0 + \kappa_2 M \cdot M) A$. For a real phase, $A > 0$ and $A_O > 0$ (Lemma 3.4), so $\kappa_0 = A_O/A > 0$, and κ_2 is real. The remaining statements follow from $A_M = \kappa(M)A$ and $A \neq 0$. $\square$

For the outer polygon we need the analogue of Lemmas 4.2 and 4.3. Recall that $\Pi' + \Delta = \Pi' - \Delta + h$, so $(\Pi' \pm \Delta) - h\mathbb{Z} = \mathcal{P}_{\text{iso}}$.

Lemma 5.2 (outer polygon). (a) $V_l = \Gamma P(u_l + \Delta)$ with $\Gamma = \text{diag}(d_0/c_0, 1/c_0)$, and $A'(u) = (d_0/c_0^2) A(u + \Delta)$. Hence A' has simple poles exactly at the points of $\mathcal{P}_A - \Delta$; its residues are $(d_0/c_0^2) r_A$ on $iK' - \Delta + G'$ and $-(d_0/c_0^2) r_A$ on $3iK' - \Delta + G'$.

(b) Let N be even. For every $M \in \mathbb{C}^2$, A'_M is holomorphic outside $\mathcal{P}_{\text{iso}}$ and has at most simple poles on $\mathcal{P}_{\text{iso}}$.

Proof. (a) is (4) with Lemmas 4.1 and 5.1. (b) By Lemma 2.3, $n \cdot n$ has simple zeros exactly at $\Pi' \pm \Delta$, and at the points of Π a double pole with the nonzero leading coefficient $(\alpha^{-1} - \beta^{-1})/k^2$. As in Lemma 4.2, the feet $Q'_l = \Phi_M(n_l)$ are therefore holomorphic except where $u_l \in \Pi' \pm \Delta$, where they have at most simple poles; so A'_M is holomorphic outside $\mathcal{P}_{\text{iso}}$. Let $u_1 \in \mathcal{P}_{\text{iso}}$. Shifting the index, we may assume $u_l(u_1) = Z - \Delta$ with $Z \in \Pi'$; then $u_{l+1}(u_1) = Z + \Delta$. As in Lemma 4.1, an index l' with $u_{l'}(u_1) \in \Pi' \pm \Delta$ satisfies $N \mid 2(l' - l)$ or $N \mid 2(l' - l - 1)$, so the singular indices are $l, l+1, l+N/2, l+1+N/2$. Put $\tilde{Q}'_j = Q'_j - M$, so that

$$\tilde{Q}'_j \times \tilde{Q}'_{j+1} = (1 - n_j \cdot M)(1 - n_{j+1} \cdot M) \frac{n_j \times n_{j+1}}{(n_j \cdot n_j)(n_{j+1} \cdot n_{j+1})}.$$

A term of $2A'_M = \sum_j \tilde{Q}'_j \times \tilde{Q}'_{j+1}$ with one singular factor has at most a simple pole. For the term with $j = l$ the denominator has a zero of order 2, but $n_l(u_1) = n(Z - \Delta) = n(Z + \Delta) = n_{l+1}(u_1)$ by Lemma 2.4(b), so the numerator vanishes at u_1 and the term has at most a simple pole; the same holds for $j = l + N/2$, where $n_j = -n_l$ and $n_{j+1} = -n_{l+1}$. If $N \geq 6$, these are all the terms with two singular factors. If $N = 4$, all four indices are singular, and the remaining terms $j = l+1, j = l+3$ also have a vanishing numerator: $h = K$ gives $n_{l+2} = -n_l$ and $n_{l+3} = -n_{l+1}$, so that $n_{l+1} \times n_{l+2} = n_{l+3} \times n_l = n_l \times n_{l+1}$, which vanishes at u_1 . $\square$

Proof of Theorem C. Let $N \equiv 0 \pmod{4}$. The cosets $\mathcal{P}_A - \Delta = iK' - \Delta + G$ and $\mathcal{P}_{\text{iso}} = K + iK' - \Delta + G$ coincide if and only if $K \in G$, that is, $K \in h\mathbb{Z} + 2K\mathbb{Z}$, that is, $N(2p+1) = 4\tau j$ for some integers j, p . This holds with $j = N/4$ and $2p+1 = \tau$, since τ is odd. So the poles of A' and the possible poles of A'_M lie in the same set $\mathcal{P}_A - \Delta$. By Proposition 3.2, $A'_M = A'_O + c'_2 M \cdot M$ with $A'_O = A'_{(0,0)}$ and $c'_2 = A'_{(1,0)} - A'_{(0,0)}$, which by Lemma 5.2(b) have at most simple poles, all in $\mathcal{P}_A - \Delta$. By Lemma 3.3, A', A'_O and c'_2 are invariant under G' and change sign under $u \mapsto u + 2iK'$, and the residues of A'

do not vanish (Lemma 5.2(a)). Exactly as in the proof of Theorem B, $\kappa'_0 := \text{Res } A'_O / \text{Res } A'$ and $\kappa'_2 := \text{Res } c'_2 / \text{Res } A'$ do not depend on the pole, the functions $A'_O - \kappa'_0 A'$ and $c'_2 - \kappa'_2 A'$ are entire, Λ -periodic and odd under $u \mapsto u + 2iK'$, hence zero, and $A'_M = (\kappa'_0 + \kappa'_2 M \cdot M)A'$. By Lemma 3.4, $\kappa'_0 = A'_O/A' > 0$ and $A' \neq 0$ for a real phase. $\square$

For $N \equiv 2 \pmod{4}$ the two cosets in this proof are disjoint, and the outer product $A'A'_M$ is constant for every M instead [Fer26b]. For odd N they are disjoint as well; then $A'A'_O$ is constant [Fer26b], but $A'A'_M$ is not constant for the other points tested (Section 6).

Remark 5.3 (comparison with Kriebel's theorem). For every even N and every real M , [Kri26c, Theorem 1, (2)] states, in our notation and with the same counterclockwise orientation, that $A'_M = A'_O (2F + \delta |M|^2)/(2F)$ with δ, F as in (1), and that $F > 0$. Both sides are polynomials in M , so the identity holds for $M \in \mathbb{C}^2$ with $M \cdot M$ in place of $|M|^2$. Together with Theorem C this gives $\kappa'(M) = \kappa'_0 (2F + \delta M \cdot M)/(2F)$, that is (1), and identifies the exceptional set of Theorem C as $\{2F + \delta M \cdot M = 0\}$. Our proof determines κ'_2 only as a quotient of residues; the dependence on M is Kriebel's. For $N = 4$ one has $\delta = 0$ [Kri26c] and $c'_2 = 0$ (Section 3), in accordance with each other; then $A'_M = A'/2$, as the outer polygon is a rectangle [Kri26c].

6 Computations and reproducibility

All theorems and lemmas of this note are proved without computation. The computations test the statements on trajectories that are constructed without the parametrisation of Section 2, give exact values in examples, and check that the statements fail where their hypotheses fail. The reproducibility package has five parts.

The program of this note (`check_note.py`, about ten seconds). (i) Trajectories are built by the reflection law alone: from a point of $\mathcal{E}$ the first chord is the tangent to $\mathcal{C}$ in the counterclockwise direction, and every further chord is the mirror image of the previous one in the normal of $\mathcal{E}$. The caustic of a family (N, τ) is found by bisection on the total advance of the eccentric angle after N reflections; the arithmetic has 50 digits. For 24 families, with $(a, b) \in \{(1.3, 1), (2, 1), (3.5, 1), (2.6, 1.7)\}$ and $(N, \tau) \in \{(4, 1), (8, 1), (8, 3), (12, 1), (12, 5), (16, 3)\}$, eight members each and the five points $O, (0.8, 0.3), (-2.5, 1.7), (12.3, -7.9)$ and $(0.4 + 1.1i, -0.6 + 0.2i)$, the closure and the tangency of every chord hold to $5 \cdot 10^{-42}$, and the relative spread of $A A_M$ over the members is at most $4 \cdot 10^{-42}$ (at most $3 \cdot 10^{-44}$ except in the star family $a = 3.5, (12, 5)$, where $b^2 - \lambda < 2 \cdot 10^{-8}$), while A_M/A varies. For $N = 4$, A_M does not depend on M . For $a = 2, b = 1, N = 8$ and $M = (0.8, 0.3)$ the program prints $A A_M = 33.4063768502875027135279705 \dots$, which agrees with the values printed by the programs of the first written version and of both verification runs. Controls, for $a = 2, b = 1$: for $(N, \tau) = (6, 1), (10, 1), (10, 3), (14, 3)$ the relative spread of $A A_M$ is at least $8 \cdot 10^{-6}$ for every M , while A_M/A is constant to $5 \cdot 10^{-48}$; for $(3, 1), (5, 1), (5, 2), (7, 2)$ the product $A A_O$ is constant to $6 \cdot 10^{-46}$, while $A A_M$ varies by at least $7 \cdot 10^{-3}$ for the four points $M \neq O$; the 3-periodic listed four times and the 6-periodic listed twice give spreads of at least 0.02, and the 4-periodic listed twice gives a constant product. (ii) Theorem B on 12 families ($a \in \{1.3, 2, 3.5\}, b = 1, (N, \tau) \in \{(6, 1), (10, 1), (10, 3), (14, 3)\}$): A_M/A is constant to $9 \cdot 10^{-48}$, and $\kappa_0 + \kappa_2 M \cdot M$, with κ_0, κ_2 computed from O and $(0.8, 0.3)$, reproduces A_M/A at all five points to $3 \cdot 10^{-46}$; $\kappa_0 > 0$ throughout, and $\kappa_2 < 0$ exactly in the three families with $(N, \tau) = (10, 3)$, for which A_M vanishes (to $6 \cdot 10^{-48}$, relative to A_O) at all members for points on the circle $|M|^2 = -\kappa_0/\kappa_2$ ($3.0378 \dots, 5.8229 \dots, 15.4158 \dots$). Theorem C on the 24 families of (i): A'_M/A' is constant to $2 \cdot 10^{-42}$, $\kappa'_0 > 0$, the ratio κ'_2/κ'_0 agrees with $\delta/(2F)$ of (1) to $7 \cdot 10^{-44}$, and $A'A'_M$ is not constant for $N \geq 8$. (iii) An exact certificate for $N = 4$ with symbolic a, b , phase and M (sympy). In the coordinates $(x/a, y/b)$, in which $\mathcal{E}$ is the unit circle, the 4-periodic through (X, Y) has the next vertex $(-Y, k'X)/r$ with $k' = b^2/a^2$ and $r^2 = Y^2 + k'^2 X^2$, and the vertices are

completed by central symmetry. The program verifies, modulo $X^2 + Y^2 = 1$ and $r^2 = Y^2 + k'^2 X^2$, that the second vertex lies on $\mathcal{E}$, that the four chords are tangent to the caustic $\lambda = a^2 b^2 / (a^2 + b^2)$, the reflection law at the four vertices, $A A_M = 8a^4 b^4 / (a^2 + b^2)^2$ and $A_M = A_O$; it also rejects three false identities. As a 4-periodic is determined by one vertex, this certifies the case $N = 4$ of Theorem A independently of Section 2. (iv) Exact examples in quadratic number fields. If the first vertex has rational coordinates in the coordinates $(x/a, y/b)$ and a^2, b^2, λ are rational, the tangent from it to $\mathcal{C}$ needs one square root, and the reflections and the second intersections with $\mathcal{E}$ are rational operations; so the whole trajectory lies in a field $\mathbb{Q}(\sqrt{D})$. The program computes such trajectories exactly for the four families of Table 2, from three or four starting points each, verifies exactly the closure after N steps, the least period, the tangency of every chord and the winding number, and computes A, A' and the pedal areas as exact polynomials in M . The identities of Table 2 hold exactly at all these members. In the star family the outer pedal area vanishes for $|M|^2 = 85/27$, and Kriebel's constants $\delta = -24/35, F = 68/63$ give $\delta/(2F) = -27/85 = \kappa'_2/\kappa'_0$ and $-2F/\delta = 85/27$. In the same family $A A_M = -512(27 M \cdot M - 13)/4725$, so by Theorem A and $A > 0$ the pedal area A_M itself vanishes at every member when $|M|^2 = 13/27$. A 50-digit computation along the star family confirms $A'_M = 0$ on the circle $|M|^2 = 85/27$ at ten members.

N	τ	a^2	b^2	λ	exact identities, for all $M \in \mathbb{C}^2$
4	1	16/9	1	16/25	$A A_M = 2048/625, \quad A'_M = A'/2$
6	1	4	1	4/9	$A_M = (7 + M \cdot M) A/9$
8	1	8/3	1	8/35	$A A_M = 256(13 + 3 M \cdot M)/175, \quad A'_M = 9(5 + M \cdot M) A'/50$
8	3	32/27	1	32/35	$A A_M = -512(27 M \cdot M - 13)/4725, \quad A'_M = (85 - 27 M \cdot M) A'/600$

Table 2: Exact values in four families, computed by `check_note.py` at three or four members each in quadratic number fields; the first verification run obtained the same values with symbolic phase, and the third at two members each, exactly in $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7})$. For $N = 4$, $2048/625 = 8a^4 b^4 / (a^2 + b^2)^2$.

The programs with which the results were first obtained (original/). `check_k203a.py` (60 digits, Jacobi form): the identities of Section 2 at random complex points, a geometric cross-check of three orbits, Theorem A on 18 families with $N = 4, 8, 12, 16$ and stars, five points M , the controls, the behaviour of A and A_M near the poles (simple poles, simple zeros, and the product equal to its value on the real axis to 25 digits), and the involutions of Lemmas 4.5 and 4.6. `exact_N4.py`: the case $N = 4$ in the Jacobi form with sympy. `check_siblings.py`: $c_1 = 0$ for even N , Theorems B and C, and the residue formula of Lemma 5.1 to 15 digits.

The programs of the verification runs, which are described in the next paragraph, with their recorded outputs: in the folder `verification_run_A/` those of the first run, in `verification_run_B/` those of the second, and in `independent_run_2/` those of the third run, which examined the text of this note. Re-running all programs of the package reproduces the recorded outputs, apart from timings, with two exceptions described in its README: `vb_geom.py` of the second run, as included, reproduces 22 of the 37 families of its recorded part A and does not construct the other 15 (its bisection interval was changed after that run), and the last two lines of the recorded output of `reflect_check.py` of the first run, the reference value above at two phases, were printed by a call that is not part of the program.

Verification. The results were first obtained, with the proofs of Theorems A, B and C given here and with the programs in `original/`. Two independent verification runs, both AI-assisted, followed; each wrote its own programs, the second before reading those of the first write-up. The first run compared the rows and the definitions with the tex source of [RGK21, v11] and with Tables 3 and 4

of the journal version; re-derived every lemma and the three theorems; checked the shift formulas of the Jacobi functions numerically; constructed trajectories by elementary geometry without Jacobi functions (68 families, N up to 32, a/b from 1.01 to 30, stars, a complex point M); obtained exact certificates, symbolic in the phase, for the four families of Table 2 by identities in a polynomial ring; and found Kriebel's note [Kri26c]. The second run constructed trajectories by the reflection law (more than forty families with N up to 40, and 10 further families, with N up to 48 and caustics within 10^{-26} of the focal segment, in the Jacobi form); tested every lemma at random complex phases, the orders of the poles and zeros, the residue formula and the quotients of residues in Theorems B and C (to 25 digits); showed that its programs detect wrong inputs, among them Poncelet polygons for a caustic that is not confocal; gave an exact certificate of the case $N = 4$, symbolic in a , b , the phase and M , with a parametrisation of its own; re-ran the original programs, whose outputs were reproduced byte for byte; and checked the bibliography. Neither run found a gap in the proofs of the three theorems. They required ten corrections, all of which were made: the citation of [Kri26c], with the novelty of Theorem C restricted accordingly; the explicit domains in Theorems B and C, including the real exceptional circles; the sign factor in Lemma 5.1, which is ± 1 ; one DOI for [Fer26a]; the attribution of the rows k_{201} and k_{205} to private communications; the removal of a remark on odd N whose argument was incomplete; and four minor ones (the explicit formula for c_1 , the notation $M \cdot M$ for complex M , the record of the check of the journal tables, and the reference for the caustic property, now checked directly in Lemma 2.2). When this note was written, the positivity of Lemma 3.4, the formula for c_2 in Lemma 3.1, the value of C_0 for $N = 4$ from the rhombus, the direct verification of the reflection law and the description of the pole sets as cosets in Lemma 4.4 were added. The text of this note was then examined by a third independent verification run, also AI-assisted. It read the sources again (the tex source of [RGK21, v11], Tables 3 and 4 of the journal version, Theorem 2 of [Sta22], [Kri26c] in full and the statement of [Fer26a]); re-derived every lemma and the three theorems, in particular these additions; and wrote its own programs before re-running those of the package: exact symbolic identities (Lemmas 2.1 to 2.3, the identities in the proof of Lemma 2.2(a), Lemma 3.1 for generic complex vectors, the rhombus with symbolic a , b and M , the residue of Lemma 5.1, and the coset and index arithmetic of Sections 4 and 5 for all $N \leq 400$); trajectories built by the reflection law for four further shapes and twenty pairs (N, τ) each, compared with the formulas of Section 3, with Theorems A to C, Lemma 3.4 and Proposition 3.2 at every member, the constants of this note, Kriebel's formula (1) for every even N , the controls (among them, for odd N , that $A'A'_M$ is not constant for $M \neq O$), and, at complex phases, the residues and zeros of Sections 4 and 5; and an exact computation of two members of each family of Table 2. It then re-ran all programs of the package, repeated the literature searches and checked the references. It found no error in the mathematics and required nine corrections, all of which were made: the description of the rows treated, in the abstract and in Section 1; a sentence on odd N after the proof of Theorem C; the definition of the law of reflection, which now excludes a reversal of the chord; the attribution of $k_{202,a}$ and $k_{202,b}$, and the removal of three references that had no role in the note; the description of two programs of the package; the record of the runs and of the searches; two numbers in this section; the notation in Lemmas 2.4, 4.5 and 4.6 and in (4); and the release documents. The program `check_note.py` tests the final statements. No proof depends on a computation.

Scope and priority. The note proves the rows $k_{203,a}$, k_{204} and k_{304} of [RGK21] for every ellipse, every confocal elliptic caustic whose trajectories have least period N in the respective class, every winding number and every point M , real or complex. For the two ratio rows the ratio is constant wherever it is defined, that is off the set $\kappa(M) = 0$, resp. $\kappa'(M) = 0$, on which the pedal area vanishes along the whole family; for k_{304} this set has no real points when the winding number is 1 [Kri26c], and for k_{204} this was only tested. Hyperbolic caustics are not treated, and for closed walks that

repeat a trajectory of smaller least period the statements are false in general. The rows and the observation that they hold are due to Reznik, Garcia and Koiller [RGK21]; the parametrisation is Stachel's [Sta22]; the caustic property of confocal conics is classical [Tab05]; the case $M = O$ of $k_{203,a}$ is in [Fer26a], the outer analogue for $N \equiv 2 \pmod{4}$ in [Fer26b], and the dependence of A'_M on M for every even N , with the exceptional circle, is Kriebel's [Kri26c]. To our knowledge new are Theorem A for an arbitrary point M , Theorem B, and in Theorem C the constancy of A'/A'_O for $N \equiv 0 \pmod{4}$. No printed or deposited proof of $k_{203,a}$ for arbitrary M or of k_{204} was found in the sources searched: the source in its arXiv and journal versions; the arXiv listings found by queries on pedal polygons together with billiards or Poncelet polygons, on invariants of the elliptic billiard, and on the works of the proposers and of Stachel; the works citing [RGK21] in OpenAlex (19) and Semantic Scholar (31); zbMATH Open and Crossref; the unrefereed 2026 deposits on Zenodo that a search for elliptic billiards returns (the latest search on 15 October 2026), which treat other rows, except [Kri26c] as described; and five web searches. A search that finds nothing is not a proof of novelty, and we make no priority claim.

What was read: [RGK21] in the tex source of arXiv v11 (preliminaries and the tables of pedal polygons), and Tables 3 and 4 of the journal version in a text extraction of the journal's file; Theorem 2 of [Sta22] in arXiv v2; [Kri26c] in full, in the deposited version 1.0; the abstracts of [Fer26a, Fer26b, Kri26a, Kri26b] and the statement of [Fer26a]; and §§22.4, 22.6, 22.8 and 22.16 of [DLM26]. The text of [BT22] and the book [Tab05] were not read for this note; their bibliographic data were checked with Crossref, those of the Zenodo deposits with DataCite. The third verification run read these sources again, in the same versions. The searches were made on 15 October 2026 by the first write-up and by the verification runs, with the public interfaces of the databases named above; the Zenodo search was repeated when this note was written and again by the third run.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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