

Focal Distance Products of Outer Elliptic Billiard Polygons

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Abstract

For a billiard with a fixed nondegenerate confocal elliptic caustic, form the outer polygon by intersecting consecutive tangents to the boundary ellipse at its orbit vertices. If the billiard has least period divisible by four, the product of distances from either original focus to the outer vertices is independent of phase. We give an explicit positive formula valid for every admissible coprime star winding and for repeated traversals of such primitive cycles. The proof factors the squared focal distance, uses a Jacobi quarter-period product, and cancels the divisor of a cyclic elliptic-function denominator with multiplicities. An exact repeated triangle shows that a listed length divisible by four alone does not suffice. The result addresses the genuine-period interpretation of the source-listed invariant k_{115} ; no unqualified resolution of an ambiguous repetition convention or absolute-priority claim is made.

1 The outer polygon and the theorem

Let

$$E : \frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1, \quad E_c : \frac{X^2}{A^2} + \frac{Y^2}{B^2} = 1, \quad a > b > 0, \quad A > B > 0,$$

where $a^2 - A^2 = b^2 - B^2 = \lambda > 0$. The fixed inner ellipse E_c is the caustic. Write

$$c^2 = a^2 - b^2 = A^2 - B^2, \quad k = c/A, \quad m = k^2, \quad k' = B/A.$$

All Jacobi functions have modulus $k \in (0, 1)$, and $K = K(k)$ and $K' = K(k')$ are complete quarter periods. For a closed reflected billiard on E , its *outer vertices* are the intersections of boundary tangents at consecutive original vertices. This is not an antipedal, a focal polar, the caustic contact polygon, or a conjugate billiard on E .

Reznik, Garcia and Koiller list the focal-distance product as invariant k_{115} in Table 2 of [2] and in the final article [1, p. 345]. Their row specifies a period divisible by four. We distinguish least period from the length of a list that may repeat a shorter orbit. The distinction is essential in Section 5.

Theorem 1. *Suppose a billiard on E tangent to E_c has least period $n = 4r$. Choose its orientation so that its canonical Jacobi step is*

$$h = 2\delta = \frac{4\tau K}{n}, \quad \gcd(\tau, n) = 1, \quad 0 < \tau < n/2.$$

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Put $x = \text{sn}^2 \delta$, $q = \text{cn}^2 \delta = 1 - x$ and $V = 1 - mx^2$. For either focus $F_\varepsilon = (\varepsilon c, 0)$, $\varepsilon \in \{1, -1\}$, the product over the n outer vertices is

$$\prod_{j=0}^{n-1} |R_j - F_\varepsilon| = \left(\frac{A^2 V}{q^2} \right)^{n/2} (k')^{n/2} Q^{1/4}, \quad Q = \prod_{j=0}^{n-1} (1 - m \text{sn}^2(jh) \text{sn}^2 h) > 0. \quad (1)$$

In particular it is independent of phase. All the outer vertices are finite. The assertion includes every admissible coprime star winding; for d traversals of a primitive cycle the product is the d th power of (1).

The theorem compares phases for the same caustic and winding, not different caustics. It makes no assertion about hyperbolic or degenerate caustics. The elementary circular case is treated separately below.

2 The canonical parameter and tangent intersections

The classical confocal billiard parameter, used in [3, Theorem 4.3], can be recovered directly from addition formulas. Parametrize the caustic by

$$T(u) = (-A \text{sn } u, B \text{cn } u).$$

The intersection of its tangents at $u = v - \delta$ and $u = v + \delta$ is

$$P(v) = \left(-\frac{A \text{dn } \delta}{\text{cn } \delta} \text{sn } v, \frac{B}{\text{cn } \delta} \text{cn } v \right) = (-a \text{sn } v, b \text{cn } v). \quad (2)$$

Indeed there is a unique $\delta \in (0, K)$ with $\text{cn } \delta = B/b$, and confocality gives $a = A \text{dn } \delta / \text{cn } \delta$. The parameter v traverses E once modulo $4K$. The caustic tangent at $T(v + \delta)$ is shared by $P(v)$ and $P(v + 2\delta)$, so the forward billiard map is translation by $h = 2\delta$. Closure with least period n is exactly the stated coprime condition. Since $n = 4r$, the integer τ is odd: $n/4$ steps add τK and $n/2$ steps add $2\tau K$.

The tangents to the *boundary* at $P(v \pm \delta)$ have equations

$$-\text{sn}(v \pm \delta) \frac{X}{a} + \text{cn}(v \pm \delta) \frac{Y}{b} = 1.$$

Their intersection is

$$R(v) = \left(-\frac{a \text{dn } \delta}{\text{cn } \delta} \text{sn } v, \frac{b}{\text{cn } \delta} \text{cn } v \right). \quad (3)$$

To check this, insert (3) and use the Jacobi addition formulas [4, Section 22.8]. Mixed terms cancel and the remaining numerator is

$$\text{dn}^2 \delta \text{sn}^2 v + \text{cn}^2 v = 1 - m \text{sn}^2 \delta \text{sn}^2 v,$$

which is the addition denominator. Because $0 < h < 2K$, the original endpoints are neither equal nor antipodal, so their tangents intersect in a unique finite point. Up to a harmless shift of initial parameter, the outer polygon consists of $R(v + jh)$.

3 Factorization of the squared focal distance

Set

$$D = 1 - mx, \quad U = 1 - 2mx + mx^2, \quad J = 1 - 2x + mx^2.$$

Doubling δ gives

$$U + J = 2Dq, \quad \operatorname{dn} h = U/V, \quad \operatorname{cn} h = J/V. \quad (4)$$

Since $B^2 = A^2(1-m)$, formula (3) also reads $R = (-AD \operatorname{sn} v/q, B \operatorname{cn} v/q)$. Direct expansion, followed by (4), gives

$$\begin{aligned} |R(v) - F_\varepsilon|^2 &= \frac{A^2}{q^2} (U + mJ \operatorname{sn}^2 v + 2\varepsilon k Dq \operatorname{sn} v) \\ &= C(1 + \varepsilon k \operatorname{sn} v)(\operatorname{dn} h + \varepsilon k \operatorname{cn} h \operatorname{sn} v), \quad C = \frac{A^2 V}{q^2} > 0. \end{aligned} \quad (5)$$

Both factors are positive on the real axis. The first is at least $1 - k > 0$. For the second, $\operatorname{dn} h > 0$ and

$$\operatorname{dn}^2 h - m \operatorname{cn}^2 h = 1 - m > 0,$$

so $\operatorname{dn} h > k|\operatorname{cn} h|$. Thus no outer vertex is a focus, and all square roots used below are positive Euclidean lengths.

4 A quarter-period product and a cyclic divisor

For $v_j = v + jh$, write

$$H_\varepsilon(v) = \prod_{j=0}^{n-1} (1 + \varepsilon k \operatorname{sn} v_j), \quad G_\varepsilon(v) = \prod_{j=0}^{n-1} (\operatorname{dn} h + \varepsilon k \operatorname{cn} h \operatorname{sn} v_j).$$

Pairing j with $j + n/2$ changes $\operatorname{sn} v_j$ to its negative, hence

$$H_\varepsilon(v) = \prod_{j=0}^{n/2-1} \operatorname{dn}^2 v_j.$$

Now pair this half-grid by $j \mapsto j + n/4$. The function dn has real period $2K$, and $\operatorname{dn}(u+K) = k'/\operatorname{dn} u$. Since τ is odd, each pair contributes $(k')^2$. There are $n/4$ pairs, giving

$$H_\varepsilon(v) = (k')^{n/2}. \quad (6)$$

This quarter-shift method is classical; compare [3, Lemma 5.3]. It is the step that requires the genuine period condition rather than a divisible-by-four repeated-list length.

Define $D_h(v) = 1 - m \operatorname{sn}^2 v \operatorname{sn}^2 h$. The addition formula gives

$$(1 + \varepsilon k \operatorname{sn}(v+h))(1 + \varepsilon k \operatorname{sn}(v-h)) = \frac{(\operatorname{dn} h + \varepsilon k \operatorname{cn} h \operatorname{sn} v)^2}{D_h(v)}. \quad (7)$$

For completeness, the sum of the two shifted sn values is $2 \operatorname{sn} v \operatorname{cn} h \operatorname{dn} h / D_h(v)$, and their product is $(\operatorname{sn}^2 v - \operatorname{sn}^2 h) / D_h(v)$. Substitution proves (7) without a branch choice. Multiplying it around the closed grid gives

$$G_\varepsilon(v)^2 = H_\varepsilon(v)^2 Q(v), \quad Q(v) = \prod_{j=0}^{n-1} D_h(v + jh). \quad (8)$$

Lemma 2. *The function $Q(v)$ is a positive constant on the real axis.*

Proof. Work on the complex torus with lattice $\Lambda = 2K\mathbb{Z} + 2iK'\mathbb{Z}$. The elliptic function sn^2 has degree two with its sole double pole at $p = iK'$. Its standard shift identity is $\text{sn}(iK' + z) = 1/(k \text{sn } z)$; see [4, Section 22.4]. Consequently the zeros of D_h are $p + h$ and $p - h$ modulo Λ . Since $0 < h < 2K$, $\text{sn } h \neq 0$. These zeros are simple unless $h = K$, when the classes coincide and $D_h = \text{dn}^2$ has a double zero. Thus the divisor, with multiplicities, is

$$\text{div } D_h = [p + h] + [p - h] - 2[p].$$

Because $nh = 4\tau K \in \Lambda$, shifting the index by either one or minus one permutes the finite cyclic multiset. Therefore

$$\text{div } Q = \sum_{j=0}^{n-1} ([p + h - jh] + [p - h - jh] - 2[p - jh]) = 0.$$

This cancels every zero and pole, including coincidences and repeated points modulo Λ ; translation invariance of a divisor alone would not suffice. The function Q is consequently holomorphic on a compact torus and constant. On the real axis each factor satisfies $D_h(v) \geq 1 - m > 0$, so the constant is positive and equals $Q(0)$. $\square$

By positivity, (8) gives $G_\varepsilon = H_\varepsilon \sqrt{Q}$. Multiplying (5) and taking the positive square root yields

$$\prod_{j=0}^{n-1} |R(v_j) - F_\varepsilon| = C^{n/2} H_\varepsilon Q^{1/4}.$$

Equation (6) proves Theorem 1. For d traversals, each distance appears d times; reversal and cyclic relabeling only permute factors. For concentric circles, all outer vertices of a regular star have radius $a/\cos \delta$, so a list of length N has product $(a/\cos \delta)^N$ directly. No limiting argument on a degenerating torus is needed.

5 Why listed length alone is insufficient

Consider $a^2 = 21$, $b^2 = 16$, and the positive focus $F = (\sqrt{5}, 0)$. The three vertices

$$(\sqrt{21}, 0), \quad (-3\sqrt{21}/5, 16/5), \quad (-3\sqrt{21}/5, -16/5)$$

form a primitive reflected triangle with side lengths $8, 32/5, 8$. Its caustic has squared axes $189/25, 64/25$, so $\lambda = 336/25$. At the axial vertex reflection reverses only the first velocity component. At the upper vertex the incoming and outgoing unit vectors are

$$u_{\text{in}} = (-\sqrt{21}/5, 2/5), \quad u_{\text{out}} = (0, -1).$$

Reflection in the normal $(-3/(5\sqrt{21}), 1/5)$ maps one to the other. Symmetry proves the third reflection.

The upper slanted sideline has normal $(2, \sqrt{21})$ and offset $2\sqrt{21}$. Its squared support on the caustic is

$$4\frac{189}{25} + 21\frac{64}{25} = 84 = (2\sqrt{21})^2.$$

The vertical sideline $X = -3\sqrt{21}/5$ is tangent at the caustic's left vertex. Thus these are actual reflected orbits in one fixed confocal family, not arbitrary inscribed triangles. Negation shifts the Jacobi phase by $2K$ in the same family.

The outer vertices are

$$(\sqrt{21}, 8), \quad (-5\sqrt{21}/3, 0), \quad (\sqrt{21}, -8).$$

Their squared distances to F have product $413600 + 5600\sqrt{105}$. For the negated triangle it is $413600 - 5600\sqrt{105}$. Both values are positive and unequal. Four traversals give listed length $N = 12$ but least period three; the corresponding products of *unsquared* distances are

$$(413600 + 5600\sqrt{105})^2 \quad \text{and} \quad (413600 - 5600\sqrt{105})^2,$$

which remain unequal. This refutes the unrestricted repeated-list extension, not Theorem 1. Since the source row does not explicitly settle the repetition convention, we do not claim an unqualified resolution of the whole source record.

6 Attribution and reproducible checks

The canonical parameter and quarter-period method are established background [3], and the Jacobi identities and divisor data are standard [4]. None is claimed as new. For $n = 4$, the focal product $4a^2b^2$ is already implicit in the author's retained four-periodic outer-rectangle calculation for the related invariant $k_{806,b}$ (AMR-050-0048). The present argument treats every admitted least period divisible by four. A bounded review of relevant sources did not identify a direct all-period proof of k_{115} , but does not establish absolute novelty or priority.

The accompanying standard-library Python checker verifies nine exact polynomial identities, a primitive eight-periodic geometric fixture with $a^2 = 40$, $b^2 = 15$, $\lambda = 24/7$ and focal product $2000000/3$, and the repeated-triangle product arithmetic. The triangle's reflection and tangency proof is given above. A separate optional 85-digit numerical calculation checks direct tangent intersections, reflections and products in 39 families at four phases per family, with admitted periods 4, 8, 12, 16, 20. These checks are regression evidence, not a substitute for the analytic proof.

AI assistance was used in developing the argument, checking algebra, reviewing sources and preparing this manuscript. The author is responsible for its claims and presentation. This is an unrefereed preprint; no independent human review or formal proof-assistant verification is claimed.

References

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