

Half-Angle Sines, Steiner Pedal Areas and Evolute Areas of N -Periodics in the Elliptic Billiard

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Abstract

Reznik, Garcia and Koiller listed many quantities that they found experimentally to be constant along the families of N -periodic billiard trajectories in an ellipse with a confocal elliptic caustic. We treat four rows of their tables (records AMR-050-0002, -0028, -0039 and -0032 of the UnsolvedMath collection) for trajectories of least period N and every winding number, with signed areas; the source's families are simple, and our statements include them. Let A and A' be the areas of the trajectory and of its outer polygon and θ_i the angles of the trajectory. Row k_{108} states that $(A'/A)/\prod_i \sin(\theta_i/2)$ is constant for $N \equiv 2 \pmod{4}$. A. Kriebel showed in a Zenodo deposit, with an exact six-periodic example, that this is false; we prove that the product $(A'/A)\prod_i \sin(\theta_i/2)$ is constant on every family with $N \equiv 2 \pmod{4}$, $N \geq 6$, and that the printed quotient is constant on none. Row k_{502} states that $A'/A'_{K'}$ is constant for odd N , where $A'_{K'}$ is the area of the pedal polygon of the outer polygon with respect to its Steiner curvature centroid; we prove that it equals $c^2\sigma/(2(\pi_g - \pi_f))$, with three further constants σ , π_f , π_g of the family, wherever it is defined, while its definedness on every family is not proved. Row k_{701} states that A/A_{ev} is constant for $N > 4$, A_{ev} being the area of the evolute polygon; we prove $A_{ev} = \nu A$ for every $N \geq 5$ with an explicit constant ν , so the ratio is constant if $\nu \neq 0$, which is observed numerically but not proved. For $N = 6$, $A_{ev} = -c^4 A/(4a^2 b^2)$ for every centrally symmetric hexagon inscribed in the ellipse with semi-axes $a > b$, $c^2 = a^2 - b^2$, and for $N = 4$ the ratio is not constant. Row k_{603} , on focal antipedal distances, was proved by Kriebel in another deposit; we only record a confirmation. The proofs use Stachel's parametrisation by Jacobi elliptic functions, the zeros and poles of the quantities on the period torus, and Liouville's theorem. No printed or deposited proof of the product identity, of row k_{502} or of row k_{701} was found in the sources searched. This is an unrefereed note.

1 Introduction

1.1 The rows

Let $\mathcal{E}$ be the ellipse $x^2/a^2 + y^2/b^2 = 1$ with $a > b > 0$, centre O and foci $f_1 = (-c, 0)$, $f_2 = (c, 0)$, $c = \sqrt{a^2 - b^2}$. A billiard trajectory in $\mathcal{E}$ whose chords are all tangent to a confocal ellipse $\mathcal{C}$, its *caustic*, closes after N reflections for one starting point if and only if it does so for every starting point, and the closed trajectories with a given caustic form a one-parameter family (Poncelet's porism). Reznik, Garcia and Koiller [RGK21] list quantities that they found experimentally to be constant along such families. We treat the following four rows of their tables [RGK21, v11, Tables 2, 6, 7 and 8; the same codes and entries are in Tables 2, 6, 7 and 8 of the journal version]:

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code	invariant	value	which N	proven
k_{108}	$k_{103}/k_{105} = (A'/A)/\prod_i \sin(\theta_i/2)$	?	$\equiv 2 \pmod{4}$	?
k_{502}	A'/A'_k	?	odd	?
k_{603}	$\sum_i q_{1,i}^* / \sum_i q_{2,i}^*$	1	all	?
k_{701}	A/A_{ev}	?	> 4	?

Here $P = (P_1, \dots, P_N)$ is the trajectory, A its signed area and θ_i its angle at P_i ; A' is the signed area of the *outer polygon* P' , whose vertices are the intersections of the tangents to $\mathcal{E}$ at consecutive vertices of P (the caption of Figure 1 of [RGK21, v11] describes it as tangent to the billiard ellipse at the vertices of the trajectory); A'_k is the signed area of the pedal polygon of P' with respect to the Steiner curvature centroid of P' , which we denote by $\mathcal{K}'$ (the source writes K' , a letter we need for a quarter period); $q_{j,i}^*$ is the distance from f_j to the i -th vertex of the antipedal polygon of P with respect to f_j ; and A_{ev} is the signed area of the evolute polygon of P , whose vertices are the intersections of the perpendicular bisectors of consecutive sides. The precise definitions are recalled in Section 2. The source writes the foci as $(\pm c, 0)$ without saying which one is f_1 ; the four rows are symmetric under $f_1 \leftrightarrow f_2$. The rows are the records AMR-050-0002 (k_{108}), AMR-050-0028 (k_{502}), AMR-050-0032 (k_{603}) and AMR-050-0039 (k_{701}) of the UnsolvedMath collection [ula26], each of which asks to prove that the quantity is constant along the family for the N indicated.

The source does not discuss winding numbers; its experimental method starts from vertices spread evenly over the ellipse [RGK21, v11, Section 4], that is, from simple (convex) trajectories. We consider trajectories of least period N and every winding number τ coprime to N ; all our statements include the source's case $\tau = 1$. Areas are signed, as in the source.

1.2 Results

Write $S = \prod_{i=1}^N \sin(\theta_i/2)$. Each family is parametrised by a real phase u (Lemma 2.1).

Theorem 1 (row k_{108}). *Let $N \equiv 2 \pmod{4}$, $N \geq 6$, and let P range over a family of billiard trajectories in $\mathcal{E}$ of least period N with a confocal elliptic caustic, of any winding number. Then A and A' do not vanish, and*

- (a) $(A'/A)S$ is constant on the family;
- (b) $(A'/A)/S$ is not constant on the family: it is a non-constant real-analytic function of the phase.

For $a = 1$, $b = 4/5$ the 6-periodic family contains members with $(A'/A)/S = 6908733/2151296$ and $29760696/9282325$, and $(A'/A)S = 91/162$ on this family (Proposition 3.2).

So row k_{108} is false as printed for every family with $N \equiv 2 \pmod{4}$, and the product, not the quotient, is invariant. For $N \equiv 0 \pmod{4}$ our computations show the opposite behaviour (Remark 3.3); the parity conditions of rows k_{107} and k_{108} appear to be interchanged in the source.

For a point M we write A'_M for the signed area of the pedal polygon of P' with respect to M ; it is a polynomial of degree at most two in the coordinates of M (Lemma 4.1), and we evaluate this polynomial also at the points $g_1 = (0, ic)$ and $g_2 = (0, -ic)$, the imaginary foci of $\mathcal{E}$. Let θ'_i be the angles of P' and $S' = \sum_i \sin 2\theta'_i$.

Theorem 2 (row k_{502}). *Let $N \geq 3$ be odd and let P range over a family as in Theorem 1. Then*

$$\sigma = A'S', \quad \pi_f = A'_{f_1}A'_{f_2}, \quad \pi_g = A'_{g_1}A'_{g_2}$$

are constant on the family, and at every member at which $\mathcal{K}'$ is defined and $A'_{\mathcal{K}'} \neq 0$

$$\frac{A'}{A'_{\mathcal{K}'}} = \frac{c^2 \sigma}{2(\pi_g - \pi_f)}.$$

The centroid K' is defined at every member if $\sigma \neq 0$ and at no member if $\sigma = 0$; if $\sigma \neq 0$, then $A'_{K'} \neq 0$ at every member if $\pi_g \neq \pi_f$ and at no member if $\pi_g = \pi_f$. For $N = 3$ the ratio equals 4.

Thus row k_{502} holds on every family on which the ratio is defined at one member. We do not prove that $\sigma \neq 0$ and $\pi_g \neq \pi_f$ on every family; in the families computed both hold (Remark 4.4).

Theorem 3 (row k_{701}). *Let $N \geq 5$ and let P range over a family as in Theorem 1, with caustic parameter λ . There is a number $\kappa = \kappa(a, b, \lambda)$, given in closed form in Proposition 5.2, such that*

$$A_{ev} = \nu A, \quad \nu = -\frac{c^4}{16a^2b^2} \kappa,$$

at every member. Moreover $A > 0$ at every member traversed counterclockwise, so $A/A_{ev} = 1/\nu$ at every member if $\nu \neq 0$, and $A_{ev} = 0$ at every member if $\nu = 0$. For $N = 6$, $\kappa = 4$; in fact $A_{ev} = -c^4 A/(4a^2b^2)$ for every centrally symmetric hexagon inscribed in $\mathcal{E}$. As $N \rightarrow \infty$ with τ fixed, $\kappa \rightarrow 6$.

For $N = 4$ the ratio is not constant: A_{ev}/A equals $-c^4/(4a^2b^2)$ at the member with vertices on the axes and 0 at the rectangular member (Proposition 5.5); for $N = 3$, $A_{ev} = 0$. Whether $\nu \neq 0$ for every family with $N \geq 5$ is not proved; in all families computed $\kappa > 0$ for $\tau/N < 1/3$ and $\kappa < 0$ for $\tau/N > 1/3$ (Remark 5.6).

Row k_{603} was proved by Kriebel [Kri26d, Theorem 1] for every closed trajectory; in Section 6 we record a short confirmation.

The proofs use Stachel's parametrisation of the family by Jacobi elliptic functions [Sta22, Theorem 2], in which every quantity above is an elliptic function of the phase; we locate its zeros and poles on the period torus and apply Liouville's theorem. The same method is used for other rows in the author's notes [Fer26f, Fer26a, Fer26c] and in [Kri26c].

1.3 Credit and limits

The rows, their conventions and the constructions are due to Reznik, Garcia and Koiller [RGK21]. The parametrisation is Stachel's [Sta22, Theorem 2], and so are the reflection property, the angle formula and the central symmetry of even trajectories that we use [Sta21, Lemmas 2.1, 2.2 and Corollary 4.2(i)]; we include short proofs of the parametrisation, the angle formula and the symmetry. The falsity of row k_{108} as printed was shown first by Kriebel [Kri26a, Proposition 1], with two exact members of one six-periodic family on $x^2/4 + y^2 = 1$; he takes the coordinates of the member with a vertex at $(2, 0)$, and its caustic, from Garcia and Reznik [GR22, Appendix B.5.1] and verifies the other member himself. We claim no novelty for the disproof. Row k_{603} was proved by Kriebel [Kri26d]. The constancy of π_f in Theorem 2 is row $k_{605,a}$ of the source, proved first by Kriebel [Kri26c] and independently in [Fer26c]; we include a short proof because the proof for π_g is the same. The author's notes [Fer26g, Fer26e] treat the rows k_{501} and k_{503} (trajectory and inner polygon), not k_{502} , and [Fer26h] proves the rows k_{702} and k_{703} (outer and inner polygons), not k_{701} . Three ingredients of our proofs are in these notes: the quadratic form of the pedal area with Steiner's centroid as its stationary point (Lemma 4.1(b), (d)), a classical fact, is derived for real polygons in [Fer26g, equation (8)] and [Fer26e, Lemma 2]; the constancy of σ in Theorem 2 is the analogue for P' of [Fer26g, Lemma 3], which gives the constancy of A times a multiple of $\sum_i \sin 2\theta_i$ for odd N ; and the circumcentre formula behind Lemma 5.1 is used on the caustic in [Fer26h, Lemma 3]. New, to our knowledge, are the product identity of Theorem 1(a) with the non-constancy of the printed quotient on every family, Theorem 2 for every odd N on its domain, and Theorem 3 for every $N \geq 5$; no printed or deposited proof of them was found in the sources searched ("Scope and priority" in Section 7), and a search that finds nothing is not a proof of novelty. Not proved: $\nu \neq 0$, $\sigma \neq 0$ and $\pi_g \neq \pi_f$. Hyperbolic caustics are not treated.

2 Setting

2.1 Definitions

For vectors $u, v \in \mathbb{C}^2$ we write $u \cdot v = u_1 v_1 + u_2 v_2$ (bilinear, also for complex vectors), $u \times v = u_1 v_2 - u_2 v_1$, $Ju = (-u_2, u_1)$ and $Ru = (u_1, -u_2)$. The signed area of a polygon $W = (W_1, \dots, W_N)$ is $[W] = \frac{1}{2} \sum_{i=1}^N W_i \times W_{i+1}$, indices modulo N , as in [RGK21].

The billiard. For $0 < \lambda < b^2$ let $\mathcal{C} = \mathcal{C}_\lambda$ be the ellipse $x^2/a''^2 + y^2/b''^2 = 1$ with $a''^2 = a^2 - \lambda$, $b''^2 = b^2 - \lambda$. It is confocal with $\mathcal{E}$, lies inside $\mathcal{E}$, and O, f_1, f_2 lie inside $\mathcal{C}$ (as $c < a''$). A line $ux + vy = 1$ is tangent to $\mathcal{C}$ if and only if $a''^2 u^2 + b''^2 v^2 = 1$, and then it touches $\mathcal{C}$ at $(a''^2 u, b''^2 v)$. The two tangents from a point $X \in \mathcal{E}$ to $\mathcal{C}$ are symmetric with respect to the normal of $\mathcal{E}$ at X [Sta21, Lemma 2.1]. A *billiard trajectory with caustic $\mathcal{C}$* is a sequence $(P_k)_{k \in \mathbb{Z}}$ of points of $\mathcal{E}$ such that every line $P_k P_{k+1}$ is tangent to $\mathcal{C}$ and, for every k , the lines $P_{k-1} P_k$ and $P_k P_{k+1}$ are the two different tangents from P_k to $\mathcal{C}$; by the symmetry just quoted this is the reflection law. It is *N -periodic* if $P_{k+N} = P_k$ for all k and $N \geq 3$ is the least such number; we view it as the polygon $P = (P_1, \dots, P_N)$, indices modulo N .

Derived polygons. Let P be N -periodic. The angle $\theta_i \in (0, \pi)$ of P at P_i is the angle between $P_{i-1} - P_i$ and $P_{i+1} - P_i$; with this reading of the source's angles, its row k_{101} , $\sum_i \cos \theta_i = JL - N$ [AST22, Theorem 1], holds (for the regular N -gon in a circle both sides are $-N \cos(2\pi/N)$), while turning angles would change its sign. Let T_X be the tangent of $\mathcal{E}$ at X . The *outer polygon* P' has the vertices $P'_i = T_{P_i} \cap T_{P_{i+1}}$; its side lines are the tangents T_{P_i} , and θ'_i is its angle at P'_i , defined as for P . For a polygon W and a point M , the *pedal polygon* of W with respect to M has as vertices the feet of the perpendiculars from M to the side lines of W , in the order of the sides; for $W = P'$ its signed area is A'_M . The *Steiner curvature centroid* of W is

$$\mathcal{K}(W) = \frac{\sum_i \sin(2\theta_i) W_i}{\sum_i \sin(2\theta_i)},$$

θ_i the angles of W [RGK21, v11, equation (4)] (there the vertices are denoted R_i); we write $\mathcal{K}' = \mathcal{K}(P')$. The *antipedal polygon* of P with respect to M has the vertices $Q_i^* = \ell_i \cap \ell_{i+1}$, where ℓ_i is the line through P_i perpendicular to $P_i - M$ (in the source: the intersections of rays shot from the P_i along $(P_i - M)^\perp$). The *evolute polygon* of P has as i -th vertex the intersection of the perpendicular bisectors of $P_{i-1} P_i$ and $P_i P_{i+1}$, as in [RGK21], where the construction is attributed to [AFI⁺17]; this is the centre E_i of the circle through P_{i-1}, P_i, P_{i+1} (three distinct points of $\mathcal{E}$ are never collinear). We write $A = [P]$, $A' = [P']$ and $A_{ev} = [(E_1, \dots, E_N)]$. A cyclic shift of the labels does not change any of these quantities.

2.2 The parametrisation

Let $m = c/a'' \in (0, 1)$, $m' = \sqrt{1 - m^2} = b''/a''$, let $\text{sn}, \text{cn}, \text{dn}$ be the Jacobi elliptic functions of modulus m , $K = K(m)$, $K' = K(m')$ and $\text{cs} = \text{cn}/\text{sn}$ [DLM26, Chapter 22]. We use the following standard facts [DLM26, §§22.2, 22.4, 22.6, 22.8, 22.13, 22.16]. The poles of $\text{sn}, \text{cn}, \text{dn}$ are simple and are exactly the points of $iK' + \Lambda_0$, $\Lambda_0 = 2K\mathbb{Z} + 2iK'\mathbb{Z}$; sn is odd, cn and dn are even; $\text{sn}^2 + \text{cn}^2 = 1$,

$\operatorname{dn}^2 + m^2 \operatorname{sn}^2 = 1$, $\operatorname{sn}' = \operatorname{cn} \operatorname{dn}$, $\operatorname{cn}' = -\operatorname{sn} \operatorname{dn}$, $\operatorname{dn}' = -m^2 \operatorname{sn} \operatorname{cn}$; and

$$\begin{aligned} \operatorname{sn}(x + 2K) &= -\operatorname{sn} x, & \operatorname{cn}(x + 2K) &= -\operatorname{cn} x, & \operatorname{dn}(x + 2K) &= \operatorname{dn} x, \\ \operatorname{sn}(x + 2iK') &= \operatorname{sn} x, & \operatorname{cn}(x + 2iK') &= -\operatorname{cn} x, & \operatorname{dn}(x + 2iK') &= -\operatorname{dn} x, \\ \operatorname{sn}(x + iK') &= \frac{1}{m \operatorname{sn} x}, & \operatorname{cn}(x + iK') &= \frac{-i \operatorname{dn} x}{m \operatorname{sn} x}, & \operatorname{dn}(x + iK') &= \frac{-i \operatorname{cn} x}{\operatorname{sn} x}, \\ \operatorname{sn}(x + K + iK') &= \frac{\operatorname{dn} x}{m \operatorname{cn} x}, & \operatorname{cn}(x + K + iK') &= \frac{-im'}{m \operatorname{cn} x}, & \operatorname{dn}(x + K + iK') &= \frac{im' \operatorname{sn} x}{\operatorname{cn} x}. \end{aligned} \quad (1)$$

For real x , $\operatorname{sn} x = \sin \operatorname{am} x$, $\operatorname{cn} x = \cos \operatorname{am} x$ with am increasing and $\operatorname{am}(x + 2K) = \operatorname{am} x + \pi$, and $m' \leq \operatorname{dn} x \leq 1$, with $\operatorname{dn} x = 1$ only for $x \in 2K\mathbb{Z}$. Finally, the addition theorems [DLM26, (22.8.1)–(22.8.3)] give

$$\operatorname{cn} x \operatorname{cn} y + \operatorname{sn} x \operatorname{sn} y \operatorname{dn}(x - y) = \operatorname{cn}(x - y). \quad (2)$$

Indeed, with $D' = 1 - m^2 \operatorname{sn}^2 x \operatorname{sn}^2 y$ they give $\operatorname{cn}(x - y) = (\operatorname{cn} x \operatorname{cn} y + \operatorname{sn} x \operatorname{sn} y \operatorname{dn} x \operatorname{dn} y)/D'$ and $\operatorname{dn}(x - y) = (\operatorname{dn} x \operatorname{dn} y + m^2 \operatorname{sn} x \operatorname{sn} y \operatorname{cn} x \operatorname{cn} y)/D'$, and inserting the second expression turns the left side of (2) into the first.

Let $\Delta \in (0, K)$ be defined by $\operatorname{cn} \Delta = b''/b$. Then $\operatorname{sn}^2 \Delta = \lambda/b^2$, and $\operatorname{dn}^2 \Delta = 1 - m^2 \operatorname{sn}^2 \Delta = (a''^2 b^2 - c^2 \lambda)/(a''^2 b^2) = a^2 b''^2/(a''^2 b^2)$, because $a''^2 b^2 - c^2 \lambda = a^2 b''^2$; thus

$$\operatorname{cn} \Delta = \frac{b''}{b}, \quad \operatorname{dn} \Delta = \frac{ab''}{a''b}, \quad \frac{\operatorname{dn} \Delta}{\operatorname{cn} \Delta} = \frac{a}{a''}, \quad \operatorname{sn}^2 \Delta = \frac{\lambda}{b^2}. \quad (3)$$

Put

$$\begin{aligned} P(u) &= (-a \operatorname{sn} u, b \operatorname{cn} u), & V(u) &= \left(-\frac{a^2}{a''} \operatorname{sn} u, \frac{b^2}{b''} \operatorname{cn} u \right), \\ n(u) &= \left(-\frac{\operatorname{sn} u}{a}, \frac{\operatorname{cn} u}{b} \right), & \ell(u) &= \left(-\frac{\operatorname{sn} u}{a''}, \frac{\operatorname{cn} u}{b''} \right). \end{aligned}$$

For real u , $P(u) = (a \cos \varphi, b \sin \varphi)$ with eccentric anomaly $\varphi = \operatorname{am} u + \pi/2$, so P is a bijection of $\mathbb{R}/4K\mathbb{Z}$ onto $\mathcal{E}$ running counterclockwise, and $T_{P(u)}$ is the line $n(u) \cdot X = 1$.

Lemma 2.1 (Stachel's parametrisation). *(i) For every u , $\ell(u) \cdot P(u \pm \Delta) = 1$ and $n(u \pm \Delta) \cdot V(u) = 1$. For real u the line through $P(u - \Delta)$ and $P(u + \Delta)$ is tangent to $\mathcal{C}$, and the tangents of $\mathcal{E}$ at these two points meet at $V(u)$.*

(ii) The billiard trajectories with caustic $\mathcal{C}$ are the sequences $P_k = P(u_0 + 2\varepsilon k \Delta)$, $k \in \mathbb{Z}$, with $u_0 \in \mathbb{R}$, $\varepsilon \in \{1, -1\}$.

(iii) Such a trajectory is N -periodic if and only if $\Delta = 2K\tau/N$ with integers $0 < \tau < N/2$, $\gcd(\tau, N) = 1$. For $\varepsilon = 1$ the eccentric anomaly increases from vertex to vertex by an amount in $(0, \pi)$, and by $2\pi\tau$ over N steps: the vertices follow each other counterclockwise and go τ times around $\mathcal{E}$.

Proof. (i) By (3), $\ell(u) \cdot P(v) = n(v) \cdot V(u) = (\operatorname{dn} \Delta \operatorname{sn} u \operatorname{sn} v + \operatorname{cn} u \operatorname{cn} v)/\operatorname{cn} \Delta$, and for $v = u \pm \Delta$ this is 1 by (2). For real u , $a''^2 \ell_1(u)^2 + b''^2 \ell_2(u)^2 = \operatorname{sn}^2 u + \operatorname{cn}^2 u = 1$, so the line $\ell(u) \cdot X = 1$ is tangent to $\mathcal{C}$; it contains the points $P(u \pm \Delta)$, which are distinct since $0 < 2\Delta < 4K$.

(ii) Let $P_0 = P(u_0)$. By (i) the lines from P_0 to $P(u_0 \pm 2\Delta)$ are tangent to $\mathcal{C}$ at $(a''^2 \ell_1, b''^2 \ell_2)(u_0 \pm \Delta) = (-a'' \operatorname{sn}, b'' \operatorname{cn})(u_0 \pm \Delta)$, two distinct points, so they are the two tangents from P_0 to $\mathcal{C}$, and $P_1 = P(u_0 + 2\varepsilon \Delta)$ for some $\varepsilon = \pm 1$. If $P_{k-1} = P(v - 2\varepsilon \Delta)$ and $P_k = P(v)$, the line $P_k P_{k+1}$ is the other tangent from P_k , so $P_{k+1} = P(v + 2\varepsilon \Delta)$. Induction in both directions gives the claim, and every such sequence is a trajectory with caustic $\mathcal{C}$.

(iii) As P is injective on $\mathbb{R}/4K\mathbb{Z}$, $P_k = P_0$ if and only if $2k\Delta \in 4K\mathbb{Z}$. So the least period is N if and only if $\Delta/(2K) = \tau/N$ in lowest terms, and $0 < \tau/N < 1/2$ since $0 < \Delta < K$. The anomaly $\text{am } u + \pi/2$ increases by $\text{am}(u + 2\Delta) - \text{am } u \in (0, \pi)$ at each step, as $0 < 2\Delta < 2K$, and by $\text{am}(u + 4K\tau) - \text{am } u = 2\pi\tau$ over N steps. $\square$

Part (i), the vertex parameters $u_0 + 2k\Delta$ and the periodicity condition are Stachel's [Sta22, Theorem 2 and its proof], with modulus m and $b'' = b \text{cn } \Delta$ as here. From now on the caustic $\mathcal{C}$ is fixed and its trajectories are N -periodic, so $\Delta = 2K\tau/N$ as in (iii), and we call

$$P_j(u) = P(u + 2j\Delta), \quad j \in \mathbb{Z}/N\mathbb{Z}, \quad u \in \mathbb{R},$$

the *family* of $\mathcal{C}$, with winding number τ ; all its members are traversed counterclockwise ($\varepsilon = -1$ gives the same polygons with reversed labels, which changes the sign of every signed area and leaves every quantity of the four rows unchanged). By (i) the outer vertices are $P'_j = V_j(u) = V(u + (2j + 1)\Delta)$; they lie on the ellipse with semi-axes a^2/a'' , b^2/b'' , with eccentric anomaly $\text{am}(u + (2j + 1)\Delta) + \pi/2$.

Lemma 2.2. *At every member of a family, $P_j \times P_{j+1} > 0$, $P'_j \times P'_{j+1} > 0$, $(P_j - P_{j-1}) \times (P_{j+1} - P_j) > 0$ and $(P'_j - P'_{j-1}) \times (P'_{j+1} - P'_j) > 0$ for all j . In particular $A > 0$ and $A' > 0$, and every turning angle of P and of P' lies in $(0, \pi)$.*

Proof. For $X(\varphi) = (\alpha \cos \varphi, \beta \sin \varphi)$ with $\alpha, \beta > 0$ one has $X(\varphi_0) \times X(\varphi_1) = \alpha\beta \sin(\varphi_1 - \varphi_0)$, and with $\delta_1 = \varphi_1 - \varphi_0$, $\delta_2 = \varphi_2 - \varphi_1$

$$(X(\varphi_1) - X(\varphi_0)) \times (X(\varphi_2) - X(\varphi_1)) = \alpha\beta (\sin \delta_1 + \sin \delta_2 - \sin(\delta_1 + \delta_2)) = 4\alpha\beta \sin \frac{\delta_1}{2} \sin \frac{\delta_2}{2} \sin \frac{\delta_1 + \delta_2}{2}.$$

Both are positive if $\delta_1, \delta_2 \in (0, \pi)$, which holds for the anomalies of consecutive vertices of P and of P' by Lemma 2.1(iii). $\square$

2.3 Complex phases

Every quantity of the four rows is, as a function of the phase u , a rational function of sn , cn , dn evaluated at u plus real constants (this is seen in each section). We use the same formulas for complex u ; they define meromorphic functions on $\mathbb{C}$, and an identity between such functions that holds for real u holds on $\mathbb{C}$. A quantity is therefore constant on a family if and only if the corresponding meromorphic function is constant. We use two standard facts on an elliptic function f with period lattice L : if f has no poles it is constant; otherwise it has as many zeros as poles in $\mathbb{C}/L$, counted with multiplicity, and the sum of its residues there is zero. We write $\text{ord}_z f$ for the order of f at z (negative at poles).

By (1), for $F \in \{P, V, n, \ell\}$,

$$F(u + 2K) = -F(u), \quad F(u + 2iK') = RF(u), \quad (4)$$

and P, V, n, ℓ have simple poles exactly at the points of $iK' + \Lambda_0$. Near the points iK' and $K + iK'$,

$$\begin{aligned} P(iK' + x) &= \frac{-1}{m \text{sn } x} (a, ib \text{dn } x), & P(K + iK' + x) &= \frac{-1}{m \text{cn } x} (a \text{dn } x, ibm'), \\ V(iK' + x) &= \frac{-1}{m \text{sn } x} \left(\frac{a^2}{a''}, \frac{ib^2}{b''} \text{dn } x \right), & V(K + iK' + x) &= \frac{-1}{m \text{cn } x} \left(\frac{a^2}{a''} \text{dn } x, \frac{ib^2 m'}{b''} \right), \\ n(iK' + x) &= \frac{-1}{m \text{sn } x} \left(\frac{1}{a}, \frac{i \text{dn } x}{b} \right), & n(K + iK' + x) &= \frac{-1}{m \text{cn } x} \left(\frac{\text{dn } x}{a}, \frac{im'}{b} \right). \end{aligned} \quad (5)$$

So P, V, n are odd functions of x at iK' and even functions of x at $K + iK'$, and their residues at iK' are

$$R_P = -\frac{1}{m} (a, ib), \quad R_V = -\frac{1}{m} \left(\frac{a^2}{a''}, \frac{ib^2}{b''} \right), \quad R_n = -\frac{1}{m} \left(\frac{1}{a}, \frac{i}{b} \right); \quad (6)$$

$R_n \cdot R_n = -c^2/(a^2 b^2 m^2) \neq 0$. Finally $n(v) \cdot n(v) = (a^2 - c^2 \operatorname{sn}^2 v)/(a^2 b^2)$ vanishes exactly when $\operatorname{sn}^2 v = a^2/c^2$, that is, by (1) and $\operatorname{sn}(K - \Delta) = \operatorname{cn} \Delta / \operatorname{dn} \Delta = a''/a$, exactly at the points of

$$\pm(K - \Delta) + iK' + \Lambda_0, \quad (7)$$

and the zeros are simple: sn^2 has order two on $\mathbb{C}/\Lambda_0$ and these are two distinct points of $\mathbb{C}/\Lambda_0$. We call them the *isotropic phases*: there the tangent $n(v) \cdot X = 1$ is an isotropic line. At an isotropic phase $\operatorname{cn}^2 v = -b^2/c^2$.

3 The half-angle product: row k_{108}

3.1 The half angles

Lemma 3.1. *Let $P(v)$, v real, be a vertex of a billiard trajectory with caustic $\mathcal{C}$, and θ its angle there. Then*

$$\sin^2 \frac{\theta}{2} = g(v), \quad g(v) = \frac{a''^2 \operatorname{dn}^2 v}{a^2 - c^2 \operatorname{sn}^2 v}.$$

Proof. The two chords at $P = P(v)$ lie on the inner side of T_P and are symmetric with respect to the normal, so each makes the same angle $\beta \in (0, \pi/2)$ with T_P , and $\theta = \pi - 2\beta$, $\sin(\theta/2) = \cos \beta$. We use two classical facts: the tangent T_P makes equal angles α with the focal radii Pf_1 , Pf_2 ; and the product of the distances of the two foci from a tangent of an ellipse with minor semi-axis β_0 is β_0^2 . Let $d_j = |P - f_j|$; from $P = (-a \operatorname{sn} v, b \operatorname{cn} v)$ one gets $d_1 = a - c \operatorname{sn} v$, $d_2 = a + c \operatorname{sn} v$. The distances of the foci from T_P are $d_j \sin \alpha$, so $d_1 d_2 \sin^2 \alpha = b^2$. Measure angles at P from one direction of T_P towards the inner side: the focal radii have the directions α and $\pi - \alpha$, and a chord the direction β (or $\pi - \beta$, which is symmetric). Both foci lie inside $\mathcal{C}$, hence on the same side of the chord, so $\beta < \alpha < \pi - \beta$ (the other side would need $\beta > \pi/2$), and the distances of the foci from the chord are $d_1 \sin(\alpha - \beta)$ and $d_2 \sin(\alpha + \beta)$ (up to the order). The chord is tangent to $\mathcal{C}$, so $b''^2 = d_1 d_2 \sin(\alpha - \beta) \sin(\alpha + \beta) = d_1 d_2 (\sin^2 \alpha - \sin^2 \beta) = b^2 - d_1 d_2 \sin^2 \beta$. Hence $\sin^2 \beta = \lambda/(d_1 d_2) = \lambda/(a^2 - c^2 \operatorname{sn}^2 v)$ and $\cos^2 \beta = (a''^2 - c^2 \operatorname{sn}^2 v)/(a^2 - c^2 \operatorname{sn}^2 v) = g(v)$, since $a''^2 - c^2 \operatorname{sn}^2 v = a''^2 \operatorname{dn}^2 v$. $\square$

The formula $\sin^2 \beta = \lambda/(d_1 d_2)$ is Stachel's [Sta21, Lemma 2.2], in other notation. The function g is elliptic with period lattice Λ_0 . Its denominator has simple zeros exactly at the isotropic phases (7), where its numerator equals $a''^2 - c^2 a^2/c^2 = -\lambda \neq 0$; at iK' it is finite, since $g(iK' + x) = -a''^2 \operatorname{cn}^2 x/(a^2 \operatorname{sn}^2 x - a''^2)$ by (1); and dn^2 has a double zero at $K + iK'$, where the denominator equals $a^2 - a''^2 = \lambda$. So on $\mathbb{C}/\Lambda_0$

$$\operatorname{div} g = 2[K + iK'] - [K - \Delta + iK'] - [-K + \Delta + iK']. \quad (8)$$

3.2 Proof of Theorem 1

Let $N = 2N'$ with $N' \geq 3$ odd. Then τ is odd and coprime to N' , and $\Delta = K\tau/N'$. Since $2N'\Delta = 2K\tau$, (4) gives the central symmetry $P_{j+N'} = -P_j$ and $V_{j+N'} = -V_j$ [Sta21, Corollary 4.2(i)], [Fer26b]; hence $\theta_{j+N'} = \theta_j$, and by Lemma 3.1 for real u

$$S(u) = \prod_{j=0}^{N-1} \sin \frac{\theta_j}{2} = \prod_{j=0}^{N'-1} g(u + 2j\Delta) > 0, \quad (9)$$

an elliptic function of u without square roots. Let

$$L = 2\Delta\mathbb{Z} + 2K\mathbb{Z} + 2iK'\mathbb{Z} = \frac{2K}{N'}\mathbb{Z} + 2iK'\mathbb{Z}, \quad L_A = \frac{2K}{N'}\mathbb{Z} + 4iK'\mathbb{Z},$$

where $2\Delta\mathbb{Z} + 2K\mathbb{Z} = (2K/N')\mathbb{Z}$ because $\gcd(\tau, N') = 1$. Note that $K \notin L$, while $\Delta - K = K(\tau - N')/N' \in L$ since $\tau - N'$ is even. Put $X = iK'$ and $Y = K + iK'$.

The function S has the period L : the factors of (9) have the periods $2K$, $2iK'$, and $u \mapsto u + 2\Delta$ permutes them, because $g(u + 2N'\Delta) = g(u + 2K\tau) = g(u)$. The functions $A = \frac{1}{2} \sum_j P_j \times P_{j+1}$ and $A' = \frac{1}{2} \sum_j V_j \times V_{j+1}$ are invariant under $u \mapsto u + 2\Delta$ (relabelling) and $u \mapsto u + 2K$ (by (4), $P_j \mapsto -P_j$), and change sign under $u \mapsto u + 2iK'$ (by (4), $P_j \mapsto RP_j$ and $\det R = -1$). So A, A' have the period lattice L_A , and A'/A , $(A'/A)S$ and $(A'/A)/S$ have the period lattice L .

The function S . The points $u + 2j\Delta$, $0 \leq j < N'$, are the N' preimages of $u \in \mathbb{C}/L$ in $\mathbb{C}/\Lambda_0$, so by (9) the order of S at u is the sum of the orders of g at these preimages. The two poles of g lie over X , as $\pm(K - \Delta) \in L$, and are distinct preimages; the double zero $K + iK'$ lies over Y . So $\text{ord}_X S = -2$, $\text{ord}_Y S = 2$, and S has no other zeros or poles on $\mathbb{C}/L$.

The function A . A pole of A requires a vertex $P_j(u)$ at a pole of P , that is, $u + 2j\Delta \in iK' + \Lambda_0$, so $u \in X + L$, which modulo L_A is $\{X, X + 2iK'\}$. At $u = X$ exactly P_0 and $P_{N'} = -P_0$ have poles, since $2j\Delta \in 2K\mathbb{Z}$ if and only if $N' \mid j$. Near X the singular part of $2A$ comes from $(P_{-1} - P_1) \times P_0 + (P_{N'-1} - P_{N'+1}) \times P_{N'} = 2(P_{-1} - P_1) \times P_0$ (central symmetry). As $P(iK' + x)$ is odd, $P_{-1}(X) = -P_1(X)$, and by (5), (6)

$$\text{Res}_X A = -2P_1(X) \times R_P = -\frac{2}{m^2 \text{sn } 2\Delta} (a, ib \text{dn } 2\Delta) \times (a, ib) = -\frac{2iab(1 - \text{dn } 2\Delta)}{m^2 \text{sn } 2\Delta} \neq 0,$$

because $0 < 2\Delta < 2K$ gives $\text{sn } 2\Delta > 0$ and $\text{dn } 2\Delta < 1$. As $A(u + 2iK') = -A(u)$, A has a simple pole at $X + 2iK'$ as well, so it has exactly two zeros on $\mathbb{C}/L_A$, counted with multiplicity. At $u = Y$ no vertex has a pole ($Y + 2j\Delta \in iK' + \Lambda_0$ would need $N' + 2j\tau \equiv 0 \pmod{2N'}$, but $N' + 2j\tau$ is odd). As $P(K + iK' + x)$ is even, $P_{-j}(Y) = P_j(Y)$; in $2A(Y) = \sum_j P_j \times P_{j+1}$ the terms j and $-1 - j$ are $P_j \times P_{j+1}$ and $P_{j+1} \times P_j$, and $j \mapsto -1 - j$ has no fixed point in $\mathbb{Z}/N\mathbb{Z}$ because N is even. So $A(Y) = 0 = A(Y + 2iK')$; these are the two zeros, and $\text{ord}_X A = -1$, $\text{ord}_Y A = 1$.

The function A' . A pole of A' requires $u + (2j + 1)\Delta \in iK' + \Lambda_0$, so $u \in Y + L$ (as $(2j + 1)\Delta \equiv (2j + 1)K \equiv K \pmod{L}$), which modulo L_A is $\{Y, Y + 2iK'\}$. At $u = Y$ the vertex V_j has a pole if and only if $K + (2j + 1)\Delta \in 2K\mathbb{Z}$, that is, $(2j + 1)\tau \equiv N' \pmod{2N'}$; since τ has an odd inverse modulo $2N'$, this means $2j + 1 \equiv N' \pmod{2N'}$, so $j \in \{j_0, j_0 + N'\}$ with $j_0 = (N' - 1)/2$. Now $(2j_0 + 1)\Delta = N'\Delta = K\tau$, so with $s = (\tau + 1)/2$ and (4), $V_{j_0}(Y + x) = V(iK' + 2sK + x) = (-1)^s V(iK' + x)$ and $V_{j_0 \pm 1}(Y) = (-1)^s V(iK' \pm 2\Delta)$. As for A , the singular part of $2A'$ is $2(V_{j_0-1} - V_{j_0+1}) \times V_{j_0}$, and since $V(iK' + x)$ is odd,

$$\text{Res}_Y A' = -2V(iK' + 2\Delta) \times R_V = -\frac{2}{m^2 \text{sn } 2\Delta} \left(\frac{a^2}{a''}, \frac{ib^2}{b''} \text{dn } 2\Delta \right) \times \left(\frac{a^2}{a''}, \frac{ib^2}{b''} \right) = -\frac{2ia^2b^2(1 - \text{dn } 2\Delta)}{a''b''m^2 \text{sn } 2\Delta} \neq 0.$$

So A' has exactly two poles on $\mathbb{C}/L_A$, both simple, at Y and $Y + 2iK'$. At $u = X$ no outer vertex has a pole ($X \notin Y + L$ since $K \notin L$). As $V(iK' + x)$ is odd, $V_{-1-j}(X) = -V_j(X) = V_{j+N'}(X)$, that is, $V_l(X) = V_{N'-1-l}(X)$ for all l . So the terms l and $N' - 2 - l$ of $2A'(X)$ are $V_l \times V_{l+1}$ and $V_{l+1} \times V_l$, and $l \mapsto N' - 2 - l$ has no fixed point in $\mathbb{Z}/N\mathbb{Z}$, because $N' - 2$ is odd and N is even. Hence $A'(X) = 0 = A'(X + 2iK')$, $\text{ord}_X A' = 1$, $\text{ord}_Y A' = -1$, and A' has no other zeros or poles on $\mathbb{C}/L_A$.

Conclusion. On $\mathbb{C}/L$, A'/A has a double zero at X , a double pole at Y , and no other zeros or poles. Therefore $(A'/A)S$ has no zeros and no poles on $\mathbb{C}/L$ and is constant, which proves (a); and $(A'/A)/S$ has a zero of order 4 at X and a pole of order 4 at Y , so it is not constant. The zeros and poles of A, A' found above have imaginary parts in $K' + 2K'\mathbb{Z}$, and $S > 0$ on $\mathbb{R}$ by (9); so $(A'/A)/S$ is real-analytic on $\mathbb{R}$, and it is not constant there, since it would otherwise be constant on $\mathbb{C}$. This proves (b). $\square$

3.3 An exact example

Proposition 3.2. *Let $a = 1$, $b = 4/5$, $\lambda = 16/81$, so that $c = 3/5$ and $\mathcal{C}$ is $x^2/(65/81) + y^2/(896/2025) = 1$. Put $h = 8\sqrt{14}/45$. The hexagons*

$$H_1 = ((1, 0), (\frac{5}{9}, h), (-\frac{5}{9}, h), (-1, 0), (-\frac{5}{9}, -h), (\frac{5}{9}, -h)),$$

$$H_2 = ((0, \frac{4}{5}), (-\frac{\sqrt{65}}{9}, \frac{16}{45}), (-\frac{\sqrt{65}}{9}, -\frac{16}{45}), (0, -\frac{4}{5}), (\frac{\sqrt{65}}{9}, -\frac{16}{45}), (\frac{\sqrt{65}}{9}, \frac{16}{45}))$$

are 6-periodic billiard trajectories with caustic $\mathcal{C}$, traversed counterclockwise, and so members of one family with $N = 6$, $\tau = 1$. They have the following exact values:

	A	A'	A'/A	$\sin^2(\theta_i/2), i = 1, 2, 3$	S	$(A'/A)/S$	$(A'/A)S$
H_1	$224\sqrt{14}/405$	$26\sqrt{14}/35$	$1053/784$	$56/81, 7/9, 7/9$	$2744/6561$	$6908733/2151296$	$91/162$
H_2	$104\sqrt{65}/405$	$112\sqrt{65}/325$	$1134/845$	$65/81, 13/18, 13/18$	$10985/26244$	$29760696/9282325$	$91/162$

Here $i = 1, 2, 3$ refers to the first three vertices as listed, and $\theta_{i+3} = \theta_i$. The two values of $(A'/A)/S$, 3.21142... and 3.20616..., differ by 105038782209/19969028643200.

Proof. All twelve points satisfy $x^2 + 25y^2/16 = 1$; for instance $25/81 + (25/16)(896/2025) = 1$ and $65/81 + (25/16)(256/2025) = 1$. The side of H_1 from $(1, 0)$ to $(\frac{5}{9}, h)$ is the line $x + \frac{5}{2\sqrt{14}}y = 1$, and $\frac{65}{81} \cdot 1 + \frac{896}{2025} \cdot \frac{25}{56} = 1$; the side from $(\frac{5}{9}, h)$ to $(-\frac{5}{9}, h)$ is $y = h$, and $h^2 = 896/2025 = b'^2$. The side of H_2 from $(0, \frac{4}{5})$ to $(-\frac{\sqrt{65}}{9}, \frac{16}{45})$ is $-\frac{\sqrt{65}}{13}x + \frac{5}{4}y = 1$, and $\frac{65}{81} \cdot \frac{65}{169} + \frac{896}{2025} \cdot \frac{25}{16} = 1$; the next side is $x = -\sqrt{65}/9 = -a''$. The remaining sides follow by the symmetries $x \mapsto -x, y \mapsto -y$, which preserve $\mathcal{E}$ and $\mathcal{C}$. So every side is tangent to $\mathcal{C}$, consecutive sides are different tangents, and both hexagons are billiard trajectories with caustic $\mathcal{C}$; they have six distinct vertices, so they are 6-periodic, and they run counterclockwise. By Lemma 2.1(ii) both are members of the family of $\mathcal{C}$. The reflection law can also be seen directly: at a vertex, both sides make the same angle γ with the inward normal, where $\cos \gamma = 5/9$ at $(\pm 1, 0)$, $\sqrt{2}/3$ at $(\pm \frac{5}{9}, \pm h)$, $4/9$ at $(0, \pm \frac{4}{5})$ and $\sqrt{10}/6$ at $(\pm \frac{\sqrt{65}}{9}, \pm \frac{16}{45})$; for example, at $(-\frac{5}{9}, h)$ the inward unit normal is $(\frac{\sqrt{2}}{3}, -\frac{\sqrt{7}}{3})$ and the unit vectors of the two sides are $(1, 0)$ and $(-\frac{5}{9}, -\frac{2\sqrt{14}}{9})$. Since $\theta = 2\gamma$, $\sin^2(\theta/2) = 1 - \cos^2 \gamma$, which gives the column of half angles, in agreement with Lemma 3.1. The outer vertices of H_1 are $(\pm 1, \pm \frac{4\sqrt{14}}{35})$ and $(0, \pm \frac{9\sqrt{14}}{35})$, those of H_2 are $(\pm \frac{\sqrt{65}}{13}, \pm \frac{4}{5})$ and $(\pm \frac{9\sqrt{65}}{65}, 0)$; the areas follow from the shoelace formula, and the last two columns from the others. $\square$

By Theorem 1(a), $(A'/A)S = 91/162$ on the whole family of Proposition 3.2. Kriebel's members [Kri26a, Proposition 1], on $x^2/4 + y^2 = 1$ with $\lambda = 4/9$, give $(A'/A)/S = 11664/3125$ and $3645/1024$; he notes that $(A'/A)S = 5/9$ at both and that this proves neither an identity at every phase nor a replacement for all periods. The program of Section 7 recomputes both examples exactly.

Remark 3.3. For $N \equiv 0 \pmod{4}$ the proof above does not apply: then $K \in L$ and the points X and Y coincide on $\mathbb{C}/L$. On the families with $N = 4, 8$ (winding numbers 1, 3) and 12 (winding number 5) computed in Section 7, the quotient $(A'/A)/S$ is constant and the product $(A'/A)S$ is not. Row k_{107} of the source asserts the constancy of the product for $N \equiv 0 \pmod{4}$; its failure for $N = 4$ is proved in the deposits [Fer26d, Kri26b]. So the parity conditions of the rows k_{107} and k_{108} appear to be interchanged in the source. We do not prove the constancy of the quotient for $N \equiv 0 \pmod{4}$. The arXiv source of [GR22] (versions v1–v3) contains an appendix file, `app/320_app_invariant_numeric.tex`, that is not included in any compiled version or in the journal version; it tabulates numerical values of k_{107} for $N = 4, 8, 12$ and of k_{108} for $N = 6, 10$, each at one member, for five aspect ratios. For $N = 6$ and $a/b = 1.25$ its value 3.211 agrees with the member H_1 of Proposition 3.2, and the member H_2 gives 3.206.

4 The outer Steiner pedal ratio: row k_{502}

4.1 Pedal areas of a polygon given by its side lines

For $p, q \in \mathbb{C}^2$ with $p \cdot p \neq 0 \neq q \cdot q$ put

$$\rho(p, q) = -2 \frac{(p \cdot q)(p \times q)}{(p \cdot p)(q \cdot q)}.$$

It is homogeneous of degree 0 in p and in q , so $\rho(\pm p, \pm q) = \rho(p, q)$; moreover $\rho(q, p) = -\rho(p, q)$, $\rho(p, p) = 0$ and $\rho(Rp, Rq) = -\rho(p, q)$.

Lemma 4.1. *Let $n_1, \dots, n_N \in \mathbb{C}^2$ (indices modulo N) with $n_i \cdot n_i \neq 0$ and $n_i \times n_{i+1} \neq 0$, let ℓ_i be the line $n_i \cdot X = 1$, and $W_i = J(n_i - n_{i+1})/(n_i \times n_{i+1})$, the intersection of ℓ_i and ℓ_{i+1} . For $M \in \mathbb{C}^2$ put $Q_i = M + \frac{1-n_i \cdot M}{n_i \cdot n_i} n_i$ (the foot of M on ℓ_i when everything is real) and $A_M = \frac{1}{2} \sum_i Q_i \times Q_{i+1}$. Let $w_i = (n_i \times n_{i+1})/((n_i \cdot n_i)(n_{i+1} \cdot n_{i+1}))$, $\rho_i = \rho(n_i, n_{i+1})$ and $\Sigma = \sum_i \rho_i$.*

- (a) $A_M = \frac{1}{2} \sum_i (1 - n_i \cdot M)(1 - n_{i+1} \cdot M) w_i$; in particular A_M is a polynomial of degree at most two in M .
- (b) If $\Sigma \neq 0$, then $A_M = A_K - \frac{\Sigma}{8}(M - K) \cdot (M - K)$ for all M , where $K = \Sigma^{-1} \sum_i \rho_i W_i$.
- (c) If $\Sigma \neq 0$ and $c \neq 0$, then $A_K = 2(A_{g_1} A_{g_2} - A_{f_1} A_{f_2})/(c^2 \Sigma)$, where $f_{1,2} = (\mp c, 0)$ and $g_{1,2} = (0, \pm ic)$.
- (d) If the n_i are real and every turning angle of the polygon (W_i) lies in $(0, \pi)$, then $\rho_i = \sin 2\theta_i$, where θ_i is the angle of (W_i) at W_i . So $\Sigma = \sum_i \sin 2\theta_i$, and K is the Steiner curvature centroid of (W_i) .

Proof. (a) $Q_i = M + p_i n_i$ with $p_i = (1 - n_i \cdot M)/(n_i \cdot n_i)$. Since $\sum_i (M \times (Q_{i+1} - M) + (Q_i - M) \times M) = \sum_i M \times (Q_{i+1} - Q_i) = 0$, we get $\sum_i Q_i \times Q_{i+1} = \sum_i p_i p_{i+1} n_i \times n_{i+1}$, which is the claim.

(b) By (a), $A_M = A_0 - \frac{1}{2} \sum_i w_i (n_i + n_{i+1}) \cdot M + \frac{1}{2} \sum_i w_i (n_i \cdot M)(n_{i+1} \cdot M)$. We claim $\sum_i w_i (n_i \cdot M)(n_{i+1} \cdot M) = -\frac{\Sigma}{4} M \cdot M$. Both sides are rational functions of the coordinates of the n_i and of M , so it suffices to prove this for real $n_i = r_i(\cos t_i, \sin t_i)$ and $M = \mu(\cos \phi, \sin \phi)$. Then $\rho_i = -\sin 2(t_{i+1} - t_i)$ and

$$\begin{aligned} w_i (n_i \cdot M)(n_{i+1} \cdot M) &= \mu^2 \sin(t_{i+1} - t_i) \cos(t_i - \phi) \cos(t_{i+1} - \phi) \\ &= \frac{\mu^2}{4} (\sin(2t_{i+1} - 2\phi) - \sin(2t_i - 2\phi)) + \frac{\mu^2}{4} \sin 2(t_{i+1} - t_i); \end{aligned}$$

the first part telescopes, which proves the claim. So $A_M = A_K - \frac{\Sigma}{8}(M - K) \cdot (M - K)$ with $K = -\frac{2}{\Sigma} \sum_i w_i (n_i + n_{i+1})$, and it remains to show $\sum_i \rho_i W_i = -2 \sum_i w_i (n_i + n_{i+1})$. For $p, q \in \mathbb{C}^2$,

$$(p \cdot q)Jp - (p \times q)p = (p \cdot p)Jq, \quad (p \cdot q)Jq + (p \times q)q = (q \cdot q)Jp;$$

both sides of the first identity are quadratic in p and linear in q , and they agree for $p = (1, 0), (0, 1), (1, 1)$; similarly for the second. Subtracting and dividing by $(p \cdot p)(q \cdot q)$ gives, for $p = n_i, q = n_{i+1}$,

$$\rho_i W_i + 2w_i (n_i + n_{i+1}) = -2 \frac{(p \cdot q)J(p - q) - (p \times q)(p + q)}{(p \cdot p)(q \cdot q)} = -2J \left(\frac{n_{i+1}}{n_{i+1} \cdot n_{i+1}} - \frac{n_i}{n_i \cdot n_i} \right),$$

which telescopes to 0.

(c) Put $U = A_{\mathcal{K}} - \frac{\Sigma}{8}(c^2 + \mathcal{K} \cdot \mathcal{K})$, $U' = A_{\mathcal{K}} - \frac{\Sigma}{8}(\mathcal{K} \cdot \mathcal{K} - c^2)$ and $V_k = \frac{\Sigma}{4}c\mathcal{K}_k$. By (b), $A_{f_{1,2}} = U \mp V_1$ and $A_{g_{1,2}} = U' \pm iV_2$, so $A_{g_1}A_{g_2} - A_{f_1}A_{f_2} = (U' - U)(U' + U) + V_1^2 + V_2^2 = \frac{\Sigma c^2}{4}(2A_{\mathcal{K}} - \frac{\Sigma}{4}\mathcal{K} \cdot \mathcal{K}) + \frac{\Sigma^2 c^2}{16}\mathcal{K} \cdot \mathcal{K} = \frac{\Sigma c^2}{2}A_{\mathcal{K}}$.

(d) The side of (W_i) from W_{i-1} to W_i lies on ℓ_i , so its direction is $\pm Jn_i$. Hence the turning angle α_i at W_i differs from $t_{i+1} - t_i$ by a multiple of π , and $\rho_i = -\sin 2\alpha_i$. If $\alpha_i \in (0, \pi)$, then $\theta_i = \pi - \alpha_i$ and $\rho_i = \sin 2\theta_i$. $\square$

Part (b) contains Steiner's theorem that the pedal area is constant on the circles centred at the curvature centroid, as recalled in [RGK21, v11, Section 2]; part (d) identifies the stationary point with the weighted vertex average used by the source.

4.2 Feet of the four foci

The side lines of P' are the tangents $n(u_j) \cdot X = 1$, $u_j = u + 2j\Delta$. For $M \in \mathbb{C}^2$ let

$$Q_M(v) = M + \frac{1 - n(v) \cdot M}{n(v) \cdot n(v)} n(v), \quad A'_M(u) = \frac{1}{2} \sum_{j \in \mathbb{Z}/N\mathbb{Z}} Q_M(u_j) \times Q_M(u_{j+1}).$$

For real u and real M this is the signed area of the pedal polygon of P' with respect to M , and for all M it is the polynomial of Lemma 4.1(a) with $n_j = n(u_j)$.

Lemma 4.2. (i) $Q_M(v + 2K) = -Q_{-M}(v)$ and $Q_M(v + 2iK') = RQ_{RM}(v)$; hence $A'_M(u + 2K) = A'_{-M}(u)$ and $A'_M(u + 2iK') = -A'_{RM}(u)$.

(ii) Q_M is holomorphic at the points of $iK' + \Lambda_0$ and at every phase that is not isotropic.

(iii) Let v_0 be an isotropic phase. If $n(v_0) \cdot M = 1$, then Q_M is holomorphic at v_0 ; otherwise Q_M has a simple pole at v_0 whose residue is a multiple of $n(v_0)$.

(iv) At an isotropic phase v_0 , $n(v_0) \cdot M = 1$ holds for exactly one of $M = f_1$, $M = f_2$, and for exactly one of $M = g_1$, $M = g_2$.

Proof. (i) follows from (4), since R is orthogonal for the bilinear dot product, and from $\det R = -1$. (ii) At iK' , $n(iK' + x) = R_n/x + O(x)$ with $R_n \cdot R_n \neq 0$ by (5), (6), so $Q_M \rightarrow M - (R_n \cdot M)R_n/(R_n \cdot R_n)$; elsewhere n is holomorphic and Q_M can only be singular where $n \cdot n = 0$. (iii) The function $n \cdot n$ has a simple zero at v_0 , and $1 - n \cdot M$ is holomorphic there. (iv) $n(v) \cdot f_{1,2} = \pm \frac{c}{a} \sin v$ and $n(v) \cdot g_{1,2} = \pm i \frac{c}{b} \cos v$, and at v_0 we have $\frac{c^2}{a^2} \sin^2 v_0 = 1$ and $\frac{c^2}{b^2} \cos^2 v_0 = -1$. $\square$

4.3 Three invariants for odd N

Let N be odd and $s^* = (N - 1)/2$. Now

$$L = 2\Delta\mathbb{Z} + 2K\mathbb{Z} + 2iK'\mathbb{Z} = \frac{2K}{N}\mathbb{Z} + 2iK'\mathbb{Z},$$

since $\gcd(2\tau, N) = 1$; $\Delta = 2K\tau/N \in L$ and $K \notin L$. Put again $X = iK'$ and $Y = K + iK'$, and $n_j = n(u_j)$, $V_j = V(u + (2j + 1)\Delta)$, $S' = \sum_j \rho(n_j, n_{j+1})$.

Where the singularities are. (a) n_j has a pole if and only if $u + 2j\Delta \in iK' + \Lambda_0$, so only for $u \in X + L$; at $u = X$ only n_0 has a pole, as $2j\Delta = 4jK\tau/N \in 2K\mathbb{Z}$ forces $N \mid j$. (b) V_j has a pole only for $u \in X + L$; at $u = X$ only V_{s^*} , as $(2j + 1)\Delta \in 2K\mathbb{Z}$ forces $N \mid 2j + 1$. (c) By (7), n_j is isotropic only for $u \in Y + L$, since $\Delta \in L$ and $-K \equiv K$; at $u = Y$ exactly n_{s^*} and n_{s^*+1} are isotropic, as $K + 2j\Delta \equiv K - \Delta \pmod{2K}$ forces $N \mid 2j + 1$ and $K + 2j\Delta \equiv -(K - \Delta) \pmod{2K}$.

forces $N \mid 2j - 1$. In particular nothing is singular at X except n_0 and V_{s^*} , and at Y nothing except the isotropy of n_{s^*}, n_{s^*+1} .

Symmetries at X and Y . By (5), $n(iK' + x)$ and $V(iK' + x)$ are odd and $n(K + iK' + x)$ and $V(K + iK' + x)$ are even in x . Hence, where the values are finite,

$$n_{-j}(X) = -n_j(X), \quad V_{-1-j}(X) = -V_j(X), \quad n_{-j}(Y) = n_j(Y), \quad V_{-1-j}(Y) = V_j(Y); \quad (10)$$

in particular $n_{s^*+1}(Y) = n_{-s^*}(Y) = n_{s^*}(Y)$.

Lemma 4.3. *For odd N the functions $\sigma = A'S'$, $\pi_f = A'_{f_1}A'_{f_2}$ and $\pi_g = A'_{g_1}A'_{g_2}$ are constant.*

Proof. Periods. As in Section 3, A' is invariant under $u \mapsto u + 2\Delta$ and $u \mapsto u + 2K$ and changes sign under $u \mapsto u + 2iK'$; by the properties of ρ and (4) the same holds for S' . By Lemma 4.2(i), $u \mapsto u + 2K$ exchanges A'_{f_1} and A'_{f_2} , and A'_{g_1} and A'_{g_2} , while $u \mapsto u + 2iK'$ maps $(A'_{f_1}, A'_{f_2}, A'_{g_1}, A'_{g_2})$ to $(-A'_{f_1}, -A'_{f_2}, -A'_{g_2}, -A'_{g_1})$. So σ, π_f, π_g have the period lattice L , and it suffices to show that they have no poles on $\mathbb{C}/L$. By (a)–(c) above, poles can only occur over X and Y .

σ at X . A' has at most a simple pole at X , coming from V_{s^*} . The function S' is holomorphic at X : the terms $\rho(n_{-1}, n_0)$ and $\rho(n_0, n_1)$ tend to $\rho(n_{-1}(X), R_n)$ and $\rho(R_n, n_1(X))$ by homogeneity, and the other terms are holomorphic. By (10) the term j and the term $-1 - j$ of $S'(X)$ are $\rho(n_j, n_{j+1})$ and $\rho(-n_{j+1}, -n_j) = -\rho(n_j, n_{j+1})$ (for $j \in \{-1, 0\}$ use the limits just given), and the fixed point $j = s^*$ of $j \mapsto -1 - j$ gives $\rho(n_{s^*}, -n_{s^*}) = 0$. So $S'(X) = 0$ and σ is holomorphic at X .

σ at Y . A' is holomorphic at Y , and $A'(Y) = 0$: by (10) the terms j and $-2 - j$ of $2A'(Y)$ are $V_j \times V_{j+1}$ and $V_{j+1} \times V_j$, and the fixed point $j = -1$ gives $V_{-1} \times V_0 = 0$, as $V_{-1}(Y) = V_0(Y)$. The function S' has at most a simple pole at Y . Indeed, only the terms $\rho(n_{s^*-1}, n_{s^*})$, $\rho(n_{s^*}, n_{s^*+1})$, $\rho(n_{s^*+1}, n_{s^*+2})$ are singular; the first and the last contain one isotropic vector, whose square has a simple zero, so they have at most simple poles. For the middle one, $2s^*\Delta = 2K\tau - \Delta$ and (4), (5) give $n_{s^*}(Y + \varepsilon) = \pm\nu(\varepsilon - \Delta)/(m \operatorname{cn}(\varepsilon - \Delta))$ and $n_{s^*+1}(Y + \varepsilon) = \pm\nu(\varepsilon + \Delta)/(m \operatorname{cn}(\varepsilon + \Delta))$ with $\nu(x) = (\operatorname{dn} x/a, im'/b)$ and the same sign. By homogeneity $\rho(n_{s^*}, n_{s^*+1}) = \rho(\nu(\varepsilon - \Delta), \nu(\varepsilon + \Delta))$. Here $\nu(\varepsilon - \Delta) \cdot \nu(\varepsilon + \Delta) = \operatorname{dn}(\varepsilon - \Delta) \operatorname{dn}(\varepsilon + \Delta)/a^2 - m'^2/b^2$ is even in ε and vanishes at $\varepsilon = 0$, because $\operatorname{dn}^2 \Delta/a^2 = b'^2/(a'^2b^2) = m'^2/b^2$ by (3), so it is $O(\varepsilon^2)$; $\nu(\varepsilon - \Delta) \times \nu(\varepsilon + \Delta) = \frac{im'}{ab}(\operatorname{dn}(\varepsilon - \Delta) - \operatorname{dn}(\varepsilon + \Delta)) = O(\varepsilon)$; and $\nu(\varepsilon \mp \Delta) \cdot \nu(\varepsilon \mp \Delta)$ have simple zeros at $\varepsilon = 0$. So the middle term is $O(\varepsilon)$, and σ is holomorphic at Y .

π_f and π_g . By Lemma 4.2(ii) and (a)–(c), every foot $Q_M(u_j)$, $M \in \{f_1, f_2, g_1, g_2\}$, is holomorphic at X , and at Y only the feet on the two isotropic lines n_{s^*}, n_{s^*+1} can be singular. These two lines coincide at Y , so by Lemma 4.2(iv) exactly one focus $f \in \{f_1, f_2\}$ lies on both, and the other one, $\tilde{f}$, on neither. For $\tilde{f}$, Lemma 4.2(iii) shows that $Q_{\tilde{f}}(u_{s^*})$ and $Q_{\tilde{f}}(u_{s^*+1})$ have at most simple poles at Y whose residues are multiples of the same vector $n_{s^*}(Y) = n_{s^*+1}(Y)$; their cross product has no ε^{-2} term, so $A'_{\tilde{f}}$ has at most a simple pole at Y . For f all feet are holomorphic at Y . Since $Q_f(v)$ depends on v only through $n(v)$, and $n(K + iK' + x)$ is even in x by (5), the meromorphic function $x \mapsto Q_f(K + iK' + x)$ is even; hence $Q_f(u_{-j}) = Q_f(u_j)$ at $u = Y$, also for the two feet on the isotropic lines, whose values there are limits. The terms j and $-1 - j$ of $2A'_f(Y)$ cancel, and the fixed point $j = s^*$ gives $Q_f(u_{s^*}) \times Q_f(u_{s^*+1}) = 0$. So $A'_f(Y) = 0$ and $\pi_f = A'_f A'_{\tilde{f}}$ is holomorphic at Y . The same argument with g_1, g_2 applies to π_g . $\square$

The constancy of π_f is row $k_{605,a}$ of [RGK21, v11] (row k_{606} of the journal version, according to [Kri26c]); it was proved first by Kriebel [Kri26c, Theorem 1], for every coprime winding number, and independently in [Fer26c].

4.4 Proof of Theorem 2

By Lemma 4.3, σ , π_f , π_g are constant. At a real member, $A' > 0$ and all turning angles of P' lie in $(0, \pi)$ (Lemma 2.2), so by Lemma 4.1(d) $S' = \sum_j \sin 2\theta'_j = \sigma/A'$ and $K' = \sum_j \rho(n_j, n_{j+1})V_j/S'$ whenever $S' \neq 0$. Hence K' is defined at every member if $\sigma \neq 0$ and at none if $\sigma = 0$. If $\sigma \neq 0$, Lemma 4.1(c) gives

$$A'_{K'} = \frac{2(\pi_g - \pi_f)}{c^2 S'} = \frac{2(\pi_g - \pi_f)}{c^2 \sigma} A',$$

which is nonzero at every member if $\pi_g \neq \pi_f$ and zero at every member otherwise, and $A'/A'_{K'} = c^2 \sigma / (2(\pi_g - \pi_f))$. For $N = 3$, P' is a triangle with angles θ'_j , so $\sum_j \sin 2\theta'_j = 4 \prod_j \sin \theta'_j > 0$; K' is its circumcentre, whose barycentric coordinates are proportional to $\sin 2\theta'_j$, and the pedal triangle of the circumcentre is the medial triangle, of area $A'/4$. $\square$

Remark 4.4. At a real member A'_{g_2} is the complex conjugate of A'_{g_1} , so $\pi_g = |A'_{g_1}|^2 \geq 0$. We have not proved $\sigma \neq 0$ or $\pi_g \neq \pi_f$ for every odd family. The independent verification runs of Section 7 computed both for all odd $N \leq 15$ with every coprime winding number at up to 49 aspect ratios $b/a \in [0.02, 0.98]$, and for 25 odd families with $N \leq 21$ at 47 aspect ratios: no zero and no sign change within a family was found, although the values become very small for the flattest ellipses and the largest winding numbers; after division by A'^2 , both have the sign of $\tau/N - 1/4$ in all these families. For the inner polygon (row k_{503}) a five-periodic family on which the Steiner pedal area vanishes identically is known [Fer26e].

5 The evolute area: row k_{701}

5.1 Circumcentres on the ellipse

Put $z(v) = -\operatorname{sn} v + i \operatorname{cn} v$ and $w(v) = -\operatorname{sn} v - i \operatorname{cn} v$. Then $zw = 1$ and $P(v) = (\frac{a}{2}(z+w), \frac{b}{2i}(z-w))$; for real v , $z(v) = e^{i\varphi}$ with φ the eccentric anomaly of $P(v)$. By (1), $z(v+2K) = -z(v)$, $z(v+2iK') = w(v) = 1/z(v)$, $z' = i \operatorname{dn} v z$,

$$z(iK' + x) = -\frac{1 - \operatorname{dn} x}{m \operatorname{sn} x}, \quad z(3iK' + x) = -\frac{1 + \operatorname{dn} x}{m \operatorname{sn} x}, \quad \operatorname{dn}(3iK' + x) = i \operatorname{cs} x. \quad (11)$$

So z is holomorphic and nonzero except for simple zeros at $iK' + 2K\mathbb{Z} + 4iK'\mathbb{Z}$ and simple poles at $3iK' + 2K\mathbb{Z} + 4iK'\mathbb{Z}$, and $z(3iK' + x)$ is odd in x , with $z(3iK' + x) = \varrho/x + O(x)$, $\varrho = -2/m$.

Lemma 5.1. *If a circle $x^2 + y^2 - 2x_0x - 2y_0y + \gamma_0 = 0$ meets $\mathcal{E}$ in the points with eccentric anomalies $\varphi_1, \dots, \varphi_4$ (with multiplicity), then $z_1 z_2 z_3 z_4 = 1$ for $z_k = e^{i\varphi_k}$, and $x_0 = \frac{c^2}{4a} \sum_k \cos \varphi_k$, $y_0 = -\frac{c^2}{4b} \sum_k \sin \varphi_k$.*

Proof. Substituting $x = \frac{a}{2}(z+z^{-1})$, $y = \frac{b}{2i}(z-z^{-1})$ and multiplying by z^2 gives $\frac{c^2}{4}z^4 + (-ax_0 + iby_0)z^3 + (\frac{a^2+b^2}{2} + \gamma_0)z^2 + (-ax_0 - iby_0)z + \frac{c^2}{4} = 0$, whose roots are the z_k . So $\prod_k z_k = 1$, $\sum_k z_k = 4(ax_0 - iby_0)/c^2$ and $\sum_k z_k^{-1} = 4(ax_0 + iby_0)/c^2$, which gives x_0 and y_0 . $\square$

For three distinct points of $\mathcal{E}$ with z -values z_1, z_2, z_3 , the circle through them meets $\mathcal{E}$ again at $z_4 = 1/(z_1 z_2 z_3)$. For a member of a family let $z_j = z(u + jh)$, $w_j = 1/z_j$, $h = 2\Delta$, and

$$Z_j = z_{j-1} + z_j + z_{j+1} + \frac{1}{z_{j-1} z_j z_{j+1}}, \quad W_j = w_{j-1} + w_j + w_{j+1} + z_{j-1} z_j z_{j+1}.$$

By Lemma 5.1 the evolute vertex is $E_j = (\frac{c^2}{8a}(Z_j + W_j), \frac{ic^2}{8b}(Z_j - W_j))$. For points of the form $X_j = (\alpha(Z_j + W_j), \beta(Z_j - W_j))$ one has $X_j \times X_{j+1} = 2\alpha\beta(W_j Z_{j+1} - Z_j W_{j+1})$; applied to P_j and to

E_j this gives

$$\begin{aligned} A &= \frac{ab}{4i} \Phi, & \Phi &= \sum_j (w_j z_{j+1} - z_j w_{j+1}), \\ A_{ev} &= \frac{ic^4}{64ab} F, & F &= \sum_j T_j, & T_j &= W_j Z_{j+1} - Z_j W_{j+1}, \end{aligned} \tag{12}$$

so that $A_{ev}/A = -\frac{c^4}{16a^2b^2} F/\Phi$. Thus Theorem 3 amounts to $F = \kappa\Phi$ with a constant κ .

5.2 Proof of Theorem 3 for odd $N \geq 5$ and even $N \geq 8$

Periods and poles. F and Φ are Laurent polynomials in the z_j . They are invariant under $u \mapsto u + h$ (relabelling) and $u \mapsto u + 2K$ ($z_j \mapsto -z_j$, under which every T_j and every term of Φ is invariant), and they change sign under $u \mapsto u + 2iK'$ ($z_j \mapsto 1/z_j$ exchanges Z_j and W_j). Let $L = h\mathbb{Z} + 2K\mathbb{Z} + 4iK'\mathbb{Z}$. Poles of F and Φ occur only where some z_j is 0 or ∞ , that is, on the two classes $iK' + L$ and $3iK' + L$, which are exchanged by $u \mapsto u + 2iK'$. Let $u^* = 3iK'$. Since $h = 4K\tau/N$, $z_j(u^*)$ is infinite exactly when $N \mid 2j$, and never zero. So for odd N only z_0 is infinite at u^* , and for even N exactly z_0 and $z_{N/2}$; for even N , $z_{j+N/2} = -z_j$ identically, since $(N/2)h = 2K\tau$ with τ odd. By (11) $z_{-l}(u^*) = -z_l(u^*)$; we write

$$\zeta = z_0, \quad \zeta_l = \frac{1 + \operatorname{dn}(lh)}{m \operatorname{sn}(lh)} = z_{-l}(u^*) = -z_l(u^*),$$

and $\zeta = \varrho/\varepsilon + O(\varepsilon)$ with $\varepsilon = u - u^*$.

The residue of Φ . The terms of Φ containing z_0 are $(w_{-1}z_0 - z_{-1}w_0) + (w_0z_1 - z_0w_1) = \zeta(w_{-1} - w_1) + (z_1 - z_{-1})/\zeta$. Hence, for odd N ,

$$\operatorname{Res}_{u^*} \Phi = \varrho(w_{-1} - w_1)(u^*) = \frac{2\varrho}{\zeta_1} = -\frac{4 \operatorname{sn} h}{1 + \operatorname{dn} h} \neq 0,$$

since $0 < h < 2K$; for even N the terms around $z_{N/2}$ contribute the same amount once more, because Φ is invariant under $z_j \mapsto -z_j$.

F has at most a simple pole at u^* . For $j \in \{-1, 0, 1\}$ write

$$Z_j = \zeta + \alpha_j + \beta_j/\zeta, \quad W_j = \gamma_j\zeta + \delta_j + 1/\zeta,$$

where α_j, δ_j are the sums and γ_j, β_j the products of the two other z 's, resp. w 's, among z_{j-1}, z_j, z_{j+1} . For $N \geq 5$ the other Z_j, W_j do not contain z_0 , so ζ occurs only in T_{-2}, T_{-1}, T_0, T_1 ; T_{-2} and T_1 are of degree one in ζ , and T_{-1}, T_0 of degree two, with

$$[\zeta^2](T_{-1} + T_0) = (\gamma_{-1} - \gamma_0) + (\gamma_0 - \gamma_1) = z_{-2}z_{-1} - z_1z_2 =: c_2(u),$$

which vanishes at u^* , since $z_{-l}(u^*) = -z_l(u^*)$. For odd $N \geq 5$ all coefficients are holomorphic at u^* . Hence $F = c_2\zeta^2 + c_1\zeta + O(1)$ with $c_2 = c'_2(u^*)\varepsilon + O(\varepsilon^2)$, so F has at most a simple pole at u^* and

$$\operatorname{Res}_{u^*} F = \varrho^2 c'_2(u^*) + \varrho c_1(u^*).$$

For even $N \geq 8$ the second infinite vertex $z_{N/2}$ occurs only in $T_{N/2-2}, \dots, T_{N/2+1}$, which are different from $T_{-2}, \dots, T_1$ and do not contain z_0 (as $N/2 \geq 4$); since $T_{j+N/2} = T_j$, these terms contribute the same residue once more, and the conclusion is the same.

Conclusion. Let $\kappa = \operatorname{Res}_{u^*} F / \operatorname{Res}_{u^*} \Phi$, a number that depends only on the family. Then $G = F - \kappa\Phi$ has at most simple poles, on the class of u^* with residue 0, hence none there, and by $G(u + 2iK') = -G(u)$ none on the class of iK' either. So G is a holomorphic elliptic function, hence constant, and $G(u + 2iK') = -G(u)$ forces $G = 0$. Thus $F = \kappa\Phi$, and $A_{ev} = \nu A$ with $\nu = -c^4\kappa/(16a^2b^2)$ by (12). At a real member $A > 0$ by Lemma 2.2, so $A/A_{ev} = 1/\nu$ at every member if $\nu \neq 0$, and $A_{ev} = 0$ at every member if $\nu = 0$. $\square$

Proposition 5.2. *For odd $N \geq 5$ and even $N \geq 8$,*

$$\kappa = -\frac{2}{m}(\operatorname{cs} h + \operatorname{cs} 2h) \zeta_1^2 \zeta_2 + 2 + \frac{2\zeta_1}{\zeta_2} + \zeta_1^4 - \zeta_1^2 \zeta_2^2, \quad h = 2\Delta = \frac{4K\tau}{N}.$$

As $h \rightarrow 0$, $\kappa = 6 - (2 - m^2)h^2 + O(h^4)$, locally uniformly in m . As $N \rightarrow \infty$ with τ fixed, $h \rightarrow 0$, $m \rightarrow c/a$ and $\kappa \rightarrow 6$.

Proof. By $z' = i \operatorname{dn} v z$ and (11), $z'_l(u^*) = -\operatorname{cs}(lh) z_l(u^*)$, so $z'_{\pm 1}(u^*) = \operatorname{cs}(h)\zeta_1$, $z'_{\pm 2}(u^*) = \operatorname{cs}(2h)\zeta_2$ and $c'_2(u^*) = 2(\operatorname{cs} h + \operatorname{cs} 2h)\zeta_1\zeta_2$. The coefficients of ζ are, identically,

$$\begin{aligned} [\zeta]T_{-2} &= W_{-2} - \gamma_{-1}Z_{-2} = w_{-2} + w_{-1} - z_{-2}z_{-1}(z_{-2} + z_{-1}), \\ [\zeta]T_1 &= \gamma_1Z_2 - W_2 = z_1z_2(z_1 + z_2) - w_1 - w_2, \\ [\zeta]T_{-1} &= z_{-1}^2(z_{-2} - z_1) + w_{-2} - w_1, \quad [\zeta]T_0 = z_1^2(z_{-1} - z_2) + w_{-1} - w_2. \end{aligned}$$

At u^* the first two equal $\zeta_1^{-1} + \zeta_2^{-1} - \zeta_1\zeta_2(\zeta_1 + \zeta_2)$ and the last two equal $\zeta_1^2(\zeta_1 + \zeta_2) + \zeta_1^{-1} + \zeta_2^{-1}$, so $c_1(u^*) = 4(\zeta_1^{-1} + \zeta_2^{-1}) + 2\zeta_1(\zeta_1^2 - \zeta_2^2)$. Dividing $\varrho^2 c'_2 + \varrho c_1$ by $2\varrho/\zeta_1$ gives the formula. For the expansion, insert the Maclaurin series of sn , cn , dn , whose coefficients are polynomials in m^2 [DLM26, §22.10]; the poles at $h = 0$ cancel, so κ extends analytically to $h = 0$, and one obtains $6 - (2 - m^2)h^2 + \frac{5m^4 - 2m^2 + 2}{12}h^4 + O(h^6)$. Finally let $N \rightarrow \infty$ with τ fixed, and suppose $\Delta \geq \delta > 0$ along a subsequence. Then $K = N\Delta/(2\tau) \rightarrow \infty$, so $m \rightarrow 1$ and $m' \rightarrow 0$; for $N \geq 4\tau$ we have $\Delta \leq K/2$, and since dn decreases on $[0, K]$ and $\operatorname{dn}(K/2) = \sqrt{m'}$ [DLM26, §22.5], (3) gives $(a/b)m' = ab''/(a''b) = \operatorname{dn} \Delta \geq \sqrt{m'}$, that is $m' \geq b^2/a^2$, a contradiction. So $h = 2\Delta \rightarrow 0$, hence $\lambda = b^2 \operatorname{sn}^2 \Delta \rightarrow 0$ and $m \rightarrow c/a$, and $\kappa \rightarrow 6$. $\square$

The limit $\kappa = 6$ corresponds to $A/A_{ev} \rightarrow -8a^2b^2/(3c^4)$, the ratio of the area πab of $\mathcal{E}$ to the signed area $-3\pi c^4/(8ab)$ of its evolute, which is traversed in the opposite sense.

5.3 $N = 6$, $N = 4$ and $N = 3$

Proposition 5.3. *For every centrally symmetric hexagon inscribed in $\mathcal{E}$, $F = 4\Phi$, so $A_{ev} = -\frac{c^4}{4a^2b^2}A$. In particular $\kappa = 4$ for $N = 6$, which completes the proof of Theorem 3.*

Proof. Let the vertices have the z -values $(t, p, q, -t, -p, -q)$, and put $\pi_0 = tpq$, $s = p + q$, $d = p - q$, and $\bar{t} = 1/t$, $\bar{s} = 1/p + 1/q$, $\bar{d} = 1/p - 1/q$. Then $Z_{j+3} = -Z_j$, $W_{j+3} = -W_j$, so $F = 2(T_0 + T_1 + T_2)$, and for $j = 0, 1, 2, 3$

$$Z_j = \sigma_j + \varepsilon_j/\pi_0, \quad W_j = \bar{\sigma}_j + \varepsilon_j\pi_0,$$

with $(\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3) = (-1, 1, -1, 1)$, $(\sigma_0, \sigma_1, \sigma_2, \sigma_3) = (t + d, t + s, -t + s, -t - d)$ and $\bar{\sigma}_j$ the same expressions in $\bar{t}$, $\bar{s}$, $\bar{d}$. So

$$T_j = (\bar{\sigma}_j\sigma_{j+1} - \sigma_j\bar{\sigma}_{j+1}) + \pi_0(\varepsilon_j\sigma_{j+1} - \varepsilon_{j+1}\sigma_j) + \pi_0^{-1}(\varepsilon_{j+1}\bar{\sigma}_j - \varepsilon_j\bar{\sigma}_{j+1}).$$

Summed over $j = 0, 1, 2$, the π_0 -parts give $-(\sigma_0 + \sigma_1) + (\sigma_1 + \sigma_2) + (\sigma_0 - \sigma_2) = 0$, and likewise the π_0^{-1} -parts give 0. The first parts are $2q/t - 2t/q + (s\bar{d} - \bar{s}d)$, $2s/t - 2\bar{s}t$ and $2p/t - 2t/p + (s\bar{d} - \bar{s}d)$, and $s\bar{d} - \bar{s}d = 2(q/p - p/q)$. Hence $T_0 + T_1 + T_2 = 4(s/t - \bar{s}t + q/p - p/q)$. On the other hand $\Phi = 2[(p/t - t/p) + (q/p - p/q) + (q/t - t/q)] = 2(s/t - \bar{s}t + q/p - p/q)$. So $F = 4\Phi$, and (12) gives the area relation. For $N = 6$ the trajectories are centrally symmetric (Section 3). $\square$

Remark 5.4. The pole argument of the previous subsection also works for $N = 6$. There $\zeta' = z_3 = -\zeta$ enters $T_1, \dots, T_4$, so the two windows overlap in T_1 and T_4 . With $\Pi = z_1z_2$, $\Sigma_1 = z_1 + z_2$ and

$\bar{\Pi} = w_1 w_2$, $\bar{\Sigma}_1 = w_1 + w_2$ one has $Z_1 = \zeta + \Sigma_1 + \bar{\Pi}/\zeta$, $W_1 = \Pi\zeta + \bar{\Sigma}_1 + 1/\zeta$, $Z_2 = \zeta' + \Sigma_1 + \bar{\Pi}/\zeta'$, $W_2 = \Pi\zeta' + \bar{\Sigma}_1 + 1/\zeta'$, and expanding with $\zeta' = -\zeta$,

$$T_1 = 2\zeta(\Pi\Sigma_1 - \bar{\Sigma}_1) - 2(\bar{\Sigma}_1\bar{\Pi} - \Sigma_1)/\zeta :$$

the products of ζ and ζ' cancel, and the same holds for T_4 . The coefficients of ζ^2 in $T_{-1} + T_0$ and in $T_2 + T_3$ vanish identically, since central symmetry gives $z_{-2}z_{-1} = z_1z_2$. So F has at most a simple pole at u^* also for $N = 6$; Proposition 5.3 gives the value $\kappa = 4$ directly. The formula of Proposition 5.2 also gives 4 for $N = 6$ ($h = 2K/3$, so $\zeta_2 = \zeta_1$ and $\text{cs } 2h = -\text{cs } h$), although it was derived for disjoint windows.

Proposition 5.5. *For every parallelogram inscribed in $\mathcal{E}$ whose consecutive vertices have the eccentric anomalies φ_1, φ_2 ,*

$$A_{ev} = -\frac{c^4}{4a^2b^2} \sin^2(\varphi_1 + \varphi_2) A.$$

The rhombus with vertices $(\pm a, 0)$, $(0, \pm b)$ and the rectangle with vertices $(\pm a^2, \pm b^2)/\sqrt{a^2 + b^2}$ are members of the 4-periodic family, whose caustic parameter is $\lambda = a^2b^2/(a^2 + b^2)$; for the rhombus $A_{ev}/A = -c^4/(4a^2b^2)$, and for the rectangle $A_{ev} = 0$. So A/A_{ev} is not constant on the 4-periodic family.

Proof. Let the z -values be $(t, p, -t, -p)$ and $x = tp$. Then $Z_0 = t - 1/(p^2t)$, $W_0 = 1/t - p^2t$, $Z_1 = p - 1/(pt^2)$, $W_1 = 1/p - pt^2$, $Z_{j+2} = -Z_j$, $W_{j+2} = -W_j$, so $T_0 = T_1 = T_2 = T_3$ and a short computation gives $F = 4T_0 = -4(p^2 - t^2)(x^2 - 1)^2/x^3$, while $\Phi = 4(p^2 - t^2)/x$. Hence $F/\Phi = -(x - 1/x)^2 = 4\sin^2(\varphi_1 + \varphi_2)$ for $t = e^{i\varphi_1}$, $p = e^{i\varphi_2}$, and (12) gives the formula. The side $x/a + y/b = 1$ of the rhombus is tangent to $\mathcal{C}_\lambda$ if and only if $a''^2/a^2 + b''^2/b^2 = 1$, that is $\lambda = a^2b^2/(a^2 + b^2)$; the sides $y = \pm b^2/\sqrt{a^2 + b^2}$ and $x = \pm a^2/\sqrt{a^2 + b^2}$ of the rectangle are tangent to $\mathcal{C}_\lambda$ for the same λ . Both quadrilaterals are therefore 4-periodic trajectories with this caustic, traversed counterclockwise, hence members of one family (Lemma 2.1). For the rhombus $\varphi_1 + \varphi_2 = \pi/2$, for the rectangle $\varphi_1 + \varphi_2 = \pi$. $\square$

At u^* the function F of Proposition 5.5 has a pole of order three ($t \rightarrow \infty$), so the pole argument fails for $N = 4$, as it must. For $N = 3$ the three perpendicular bisectors of the triangle concur, all evolute vertices coincide, and $A_{ev} = 0$, as the source notes. For $N = 4$ the source restricts row k_{701} to $N > 4$ without a statement on A/A_{ev} ; it notes that $A'_{ev} = 0$ for $N = 4$, where P' is a rectangle [RGK21, v11, Section 3.6].

Remark 5.6. We have not proved $\kappa \neq 0$. In all families computed $\kappa \neq 0$, with $\kappa > 0$ for $\tau/N < 1/3$ and $\kappa < 0$ for $\tau/N > 1/3$, so the sign of A/A_{ev} depends on the family; for instance $\kappa = 3.2410\dots$ for $(N, \tau) = (5, 1)$ and $\kappa = -0.8694\dots$ for $(5, 2)$ on $a = 1$, $b = 4/5$. The independent verification runs of Section 7 found no sign change of κ within any family for $N \leq 16$ and every coprime winding number, at up to 49 aspect ratios $b/a \in [0.02, 0.98]$; for $\tau = 1$, κ lies in $[3.23, 5.70]$ there, and for $\tau/N > 1/3$ it tends to 0 only as $b/a \rightarrow 0$. Evaluating the formula of Proposition 5.2 with $h = 4K\varrho_0$ for a continuous rotation number $\varrho_0 \in (0.02, 0.4975)$ and five moduli, one of the runs found a single zero, at $\varrho_0 = 1/3$; this zero is exact for every modulus, since $h = 4K/3$ gives $\zeta_2 = -\zeta_1$ and $\text{cs } 2h = -\text{cs } h$. Since $\tau/N = 1/3$ with $\text{gcd}(\tau, N) = 1$ forces $N = 3$, this suggests $\nu \neq 0$ for every family with $N \geq 5$; it is not a proof. If $\nu = 0$ on some family, then $A_{ev} = 0$ at every member and row k_{701} is vacuous there rather than false.

6 Focal antipedal sums: row k_{603}

Row k_{603} states $\sum_i q_{1,i}^* = \sum_i q_{2,i}^*$ for every N , where $q_{j,i}^*$ is the distance from f_j to the i -th vertex of the antipedal polygon of P with respect to f_j . Kriebel proved it [Kri26d, Theorem 1] for every

closed trajectory with a confocal elliptic caustic, stars and repeated traversals included, by showing that on each side the difference of the two focal distances is a fixed multiple of the increment of the y -coordinate. We record this identity in the notation of Section 2; we claim nothing new for this row.

Lemma 6.1 ([Kri26d, Lemma 1]). *Let f, B_1, B_2 be three non-collinear points and Q^* the intersection of the lines through B_k perpendicular to $B_k - f$. Then $|Q^* - f| = |B_1 - f| |B_2 - f| / h$, where h is the distance from f to the line $B_1 B_2$.*

Proof. B_1 and B_2 lie on the circle with diameter fQ^* , so $|Q^* - f| = 2R$ with R the circumradius of the triangle fB_1B_2 , and $2R = |B_1 - f| |B_2 - f| |B_1 - B_2| / (2 \text{ area}) = |B_1 - f| |B_2 - f| / h$. $\square$

Proposition 6.2. *For the side from $P(t - \Delta)$ to $P(t + \Delta)$ of a member (t real) let q_1^*, q_2^* be the two focal antipedal distances and $y_{\pm} = b \operatorname{cn}(t \pm \Delta)$. Then*

$$q_1^* - q_2^* = C (y_+ - y_-), \quad C = \frac{c (a^2 b^2 - \lambda (a^2 + b^2))}{ab \sqrt{\lambda} (b^2 - \lambda)}.$$

Summing over the sides of a closed trajectory gives $\sum_i q_{1,i}^* = \sum_i q_{2,i}^*$.

Proof. The focal distances of $P(v)$ are $d_1 = a - c \operatorname{sn} v$, $d_2 = a + c \operatorname{sn} v$. The side lies on $\ell(t) \cdot X = 1$ (Lemma 2.1), with $\ell \cdot f_{1,2} = \pm m \operatorname{sn} t$ and $\ell \cdot \ell = \operatorname{dn}^2 t / b'^2$; so the distances of f_1, f_2 from it are $h_{1,2} = b'' (1 \mp m \operatorname{sn} t) / \operatorname{dn} t > 0$. Put $s = \operatorname{sn} t$, $s_{\pm} = \operatorname{sn}(t \pm \Delta)$ and $D = 1 - m^2 s^2 \operatorname{sn}^2 \Delta$. The addition theorems give $s_+ + s_- = 2s \operatorname{cn} \Delta \operatorname{dn} \Delta / D$, $s_+ s_- = (s^2 - \operatorname{sn}^2 \Delta) / D$ and $y_+ - y_- = -2bs \operatorname{sn} \Delta \operatorname{dn} t \operatorname{dn} \Delta / D$. By Lemma 6.1,

$$\begin{aligned} q_1^* - q_2^* &= \frac{\operatorname{dn} t}{b''} \left(\frac{(a - cs_-)(a - cs_+)}{1 - ms} - \frac{(a + cs_-)(a + cs_+)}{1 + ms} \right) \\ &= \frac{2s \operatorname{dn} t}{b'' D} \cdot \frac{ma^2 D + mc^2 (s^2 - \operatorname{sn}^2 \Delta) - 2ac \operatorname{cn} \Delta \operatorname{dn} \Delta}{1 - m^2 s^2}. \end{aligned}$$

With $\alpha = ma^2 - mc^2 \operatorname{sn}^2 \Delta - 2ac \operatorname{cn} \Delta \operatorname{dn} \Delta$ the last numerator is $\alpha + ms^2 (c^2 - m^2 a^2 \operatorname{sn}^2 \Delta)$, and this equals $\alpha (1 - m^2 s^2)$ because $c^2 + m^2 a^2 - m^2 (a^2 + c^2) \operatorname{sn}^2 \Delta - 2acm \operatorname{cn} \Delta \operatorname{dn} \Delta = 0$; indeed, after multiplication by $a''^2 b^2 / c^2$ and insertion of (3) the left side becomes $\lambda (a^2 - b^2 - c^2) = 0$. Hence $q_1^* - q_2^* = 2\alpha s \operatorname{dn} t / (b'' D) = -\alpha (y_+ - y_-) / (bb'' \operatorname{sn} \Delta \operatorname{dn} \Delta)$, and (3) turns $-\alpha / (bb'' \operatorname{sn} \Delta \operatorname{dn} \Delta)$ into C . $\square$

This is the edge identity of [Kri26d, Section 3] (whose foci $(\pm c, 0)$ are labelled the other way, so that his coefficient is $-C$). The coefficient vanishes exactly for $\lambda = a^2 b^2 / (a^2 + b^2)$, the caustic of the 4-periodic family, where the two distances agree side by side.

7 Computations and reproducibility

The package that accompanies this note contains four groups of programs with their recorded outputs; its `README.md` gives the command lines and running times. No proof depends on a computation.

The exact program of this note, `check_note.py` (Python with `mpmath` and `sympy`; about 100 seconds; 150 checks), tests the statements as printed. (i) On trajectories built from the reflection law alone, without Jacobi functions, at 50 digits, six members per family on two ellipses: Theorem 1 for $(N, \tau) = (6, 1), (10, 1), (10, 3), (14, 3)$ and the opposite behaviour of Remark 3.3 for $(4, 1), (8, 1), (8, 3), (12, 5)$; Theorem 2 with its closed form for six odd families, and the stationarity of $M \mapsto A'_M$ at the source's centroid; Theorem 3 for eight families with $5 \leq N \leq 8$, Proposition 5.5 and $N = 3$; Proposition 6.2. (ii) In exact arithmetic: every entry of Proposition 3.2 and the cosines

in its proof, the two members of [Kri26a], the rhombus and rectangle of Proposition 5.5, and two wrong inputs, which are rejected. (iii) Propositions 5.3 and 5.5 and Remark 5.4 symbolically, and Proposition 5.3 on random hexagons. (iv) The sign rule of Remark 5.6 on 24 pairs of a family and an ellipse. (v) With Jacobi functions: Lemmas 2.1 and 3.1, the residues of A at X and of A' at Y (contour integrals), $(A'/A)S$ at complex phases, the closed form of κ and its expansion, the identity used in Proposition 6.2, and Lemma 4.1(b), (c) at complex points.

The programs with which the results were first obtained (original/): a toolkit for the parametrisation with an independent check of the reflection law at every vertex, a screen of the four rows over 18 families on two ellipses, numerical checks of the divisor counts (local orders by logarithmic slopes), of the residues, of the lemmas of Sections 4 to 6 and of the closed form of κ , and the exact computation of Proposition 3.2. *Two independent verification runs*, each with its own programs and recorded outputs, are in the folders `verification_run_A/` and `verification_run_B/`, and the programs of a *third verification run*, which examined the text of this note, are in `independent_run_2/`.

Verification. The results were first obtained, with proofs and with the programs in `original/`. Two independent verification runs, both AI-assisted, followed; each wrote its own programs, and neither read the other's report. The first run re-derived the parametrisation without the literature and every step of the proofs; it supplied the proof that the stationary point of the pedal area is the source's centroid (Lemma 4.1(b)), the identity of Proposition 5.3 and the edge identity of Proposition 6.2, recomputed Proposition 3.2 exactly, and tested all four rows at real and complex phases on three ellipses, the local orders at the special points, and the degenerate cases by scans over the aspect ratio. The second run built all trajectories from the reflection law only, rebuilt the members of Proposition 3.2 and of [Kri26a] exactly from their first vertices, rejected perturbed inputs, confirmed the divisor table of Theorem 1 at complex phases, tested the four rows on 60 pairs of a family and an ellipse, scanned σ , $\pi_g - \pi_f$ and κ , read the journal version of [RGK21] and the deposits, repeated the literature search, checked the bibliography, and re-ran the original programs, reproducing their recorded outputs. Both runs confirmed every statement; neither found a mathematical error. They required thirteen corrections, some of them overlapping, all of which were made: a cosine in the proof of Proposition 3.2 ($\sqrt{2}/3$, not $5/9$); the condition $\sigma \neq 0$ in Theorem 2; that the source's families are simple; a proof, instead of a numerical check, that the source's centroid is the stationary point; the sign rule of κ and the restriction to fixed τ in its limit, in place of an incorrect range; the positivity of A at every member; the attribution of the case $N = 4$; a derivation of the residue of A' at Y and of the cancellation for $N = 6$, which the second run had confirmed only numerically (Section 3, Remark 5.4); and four bibliographic points (the numbering of Stachel's theorem, [GR22] for the numerical tables, title and pages of [CC21], the active caption of the source's Figure 1). When this note was written, the proof of Lemma 3.1, Proposition 5.5, the proof of the limit in Proposition 5.2, the constant C of Proposition 6.2 and the program `check_note.py` were added, and the residue of A' at Y , the cancellation for $N = 6$ and the identification of the stationary point were written out.

A third verification run, also AI-assisted, then examined the text of the note, these parts first. It re-derived every proof line by line; compared the definitions, the rows and the cited statements with the source (arXiv v11), with Stachel's two papers and with the deposits cited, reading Kriebel's deposits on k_{108} and k_{603} in full and the paper of Garcia and Reznik in its journal version and in all three arXiv versions; listed Kriebel's newest records again; and repeated the literature search. Its three programs, in `independent_run_2/`, check the statements as printed: in exact arithmetic (every entry of Proposition 3.2 and of Kriebel's example, Propositions 5.3 and 5.5 and Remark 5.4, the coefficients in the proofs of Theorem 3 and Proposition 5.2, Lemma 4.1, the identities of Proposition 6.2, and the expansion of κ to order h^4 from series of sn , cn , dn generated by their differential equations); in plane geometry at 60 digits, on trajectories built by the tangent construction (Theorem 1 and the

opposite behaviour of Remark 3.3, also for $N = 12$ and 16 with every winding number, on 54 pairs of a family and an ellipse; Theorem 2 on 33 and Theorem 3 on 36 such pairs; the limit, the range and the sign rule of κ ; row k_{603} with Lemma 6.1 and Proposition 6.2; Propositions 5.3 and 5.5 on random inscribed polygons); and with Jacobi functions at complex phases (the divisors of Theorems 1 and 2 by the argument principle, the residues by contour integrals, F/Φ against the closed form of κ). It found no error in a theorem, proposition or lemma and required ten corrections, which were made: the exact provenance of Kriebel’s example; the description of the numerical tables of [GR22], which are in an appendix file of its arXiv source that is not compiled; credit to the author’s earlier notes for three ingredients (Section 1); the reason for an equality of feet in the proof of Lemma 4.3; the exactness of the zero of κ at rotation number $1/3$; the punctuation of (2); and the record of this run in this paragraph, in the list of programs above, in “What was read” and in the package. It also re-ran every program of the package from the archive; all recorded outputs were reproduced, that of `check_note.py` apart from its running time.

Scope and priority. Row k_{108} is false as printed for every family with $N \equiv 2 \pmod{4}$ (shown first by Kriebel for one family [Kri26a]), and its product form holds. Row k_{502} holds for every odd N on every family on which the ratio is defined at one member; whether this is every family is open. Row k_{701} holds for every $N \geq 5$ in the form $A_{ev} = \nu A$; it fails for $N = 4$, is vacuous for $N = 3$, and $\nu \neq 0$ is open. Row k_{603} is Kriebel’s theorem [Kri26d]. The rows are due to Reznik, Garcia and Koiller [RGK21]; the parametrisation is Stachel’s [Sta22, Theorem 2], the reflection property, the angle formula and the central symmetry are printed in [Sta21], the constancy of π_f is Kriebel’s [Kri26c] (also [Fer26c]), and the constancy of the pedal area on circles about the curvature centroid is Steiner’s theorem, as recalled in [RGK21]. To our knowledge the product identity of Theorem 1(a) with the non-constancy of the printed quotient on every family, Theorem 2 and Theorem 3, with Propositions 5.3 and 5.5, are not in print.

What was read: [RGK21] in the arXiv version v11 (the tables with the four rows and with k_{107} , the definitions, the list of symbols, the method section and the figure captions) and Tables 2, 6, 7 and 8 of the journal version; [Sta22, Theorem 2] with its proof and [Sta21, Lemmas 2.1, 2.2 and Corollary 4.2], in the arXiv versions cited (the journal numbering of [Sta22] was not checked, as the publisher’s site refused automated access; [Kri26a, Kri26c] cite the journal version as Theorem 4.3); the statements of Theorems 1–3 and Corollary 6.2 of [AST22] in arXiv v2; the journal version of [GR22] (its appendices and Table 1) and the sources and compiled versions of its arXiv versions v1–v3; [Kri26a, Kri26d] in full and the statements and the cited parts of [Kri26c]; the statements and the scope of [Fer26g, Fer26e, Fer26h, Fer26c]; and the descriptions of the other deposits cited. [CC21] and [AFI⁺17] were not read; what we say about them is taken from [RGK21]. The proofs of the cited theorems were not checked; the parametrisation, the angle formula and the central symmetry are proved here, while the reflection property and two classical facts on the foci (in Lemma 3.1) are quoted.

The searches were made on 15 October 2026 by the two verification runs, repeated in part during the writing and, except Semantic Scholar, by the third verification run on 16 October 2026: arXiv (elliptic billiards and their invariants, evolute polygons, Steiner curvature centroids, antipedal polygons), the 19 works citing the journal version of [RGK21] in OpenAlex and the 31 citing records in Semantic Scholar, Crossref, zbMATH Open, Zenodo (all 131 records of A. Kriebel, the newest of them listed again at the end of the writing and on 16 October 2026; the author’s records; queries for elliptic billiards, k_{108} , k_{502} , k_{701} , the record numbers, evolutes and curvature centroids), and one web search in each of the second and the third run. No printed or deposited proof of the identity of Theorem 1(a), of row k_{502} or of row k_{701} was found in these sources; the closest are the deposits [Kri26a] (one counterexample to k_{108} , no repaired identity), [Fer26g, Fer26e] (rows k_{501} and k_{503} , which exclude k_{502}),

[Fer26h] (rows k_{702} and k_{703}) and [Kri26c, Fer26c] (row $k_{605,a}$). A search that finds nothing is not a proof of novelty.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

References

- [AFI⁺17] Maxim Arnold, Dmitry Fuchs, Ivan Izmetiev, Serge Tabachnikov, and Emmanuel Tsukerman. Iterating evolutes and involutes. *Discrete & Computational Geometry*, 58(1):80–143, 2017. doi:10.1007/s00454-017-9890-y.
- [AST22] Arseniy Akopyan, Richard Schwartz, and Serge Tabachnikov. Billiards in ellipses revisited. *European Journal of Mathematics*, 8(4):1313–1327, 2022. Statement numbers as in arXiv:2001.02934v2. arXiv:2001.02934, doi:10.1007/s40879-020-00426-9.
- [CC21] Ana C. Chavez-Caliz. More about areas and centers of Poncelet polygons. *Arnold Mathematical Journal*, 7(1):91–106, 2021. arXiv:2004.05404, doi:10.1007/s40598-020-00154-8.
- [DLM26] DLMF. NIST digital library of mathematical functions. <https://dlmf.nist.gov/>, 2026. Version 1.2.8 of 2026-09-15.
- [Fer26a] Alper Ferudun. Antipedal areas of N -periodics in the elliptic billiard: row k401 fails at the centre and holds at the foci. Zenodo deposit, published 2026-10-15, unrefereed, 2026. doi:10.5281/zenodo.23403151.
- [Fer26b] Alper Ferudun. Central symmetry and seven area identities for even N -periodics in the elliptic billiard. Zenodo deposit, published 2026-10-15, unrefereed, 2026. doi:10.5281/zenodo.23396998.
- [Fer26c] Alper Ferudun. Five focal area invariants of N -periodics in the elliptic billiard. Zenodo deposit, published 2026-10-15, unrefereed, 2026. doi:10.5281/zenodo.23404834.
- [Fer26d] Alper Ferudun. Four-periodic counterexamples to two elliptic billiard invariants. Zenodo deposit, published 2026-10-01, unrefereed, 2026. doi:10.5281/zenodo.23075934.
- [Fer26e] Alper Ferudun. Inner Steiner pedal ratios and exceptional zero-area families. Zenodo deposit, published 2026-10-01, unrefereed, 2026. doi:10.5281/zenodo.23090907.
- [Fer26f] Alper Ferudun. Pedal areas of periodic elliptic billiard trajectories with respect to an arbitrary point. Zenodo deposit, published 2026-10-15, unrefereed, 2026. doi:10.5281/zenodo.23400541.
- [Fer26g] Alper Ferudun. Steiner-centroid pedal area ratios in elliptic billiards. Zenodo deposit, published 2026-10-01, unrefereed, 2026. doi:10.5281/zenodo.23089778.
- [Fer26h] Alper Ferudun. A telescoping formula for evolute areas in elliptic billiards. Zenodo deposit, published 2026-09-30, unrefereed, 2026. doi:10.5281/zenodo.23061834.

- [GR22] Ronaldo Garcia and Dan Reznik. Exploring self-intersected N -periodics in the elliptic billiard. *Annales Mathematicae et Informaticae*, 55:49–75, 2022. arXiv:2011.06640v3 is titled “Invariants of self-intersected N -periodics in the elliptic billiard”; the numerical values of k_{107} and k_{108} are in an appendix file of its arXiv source that is not included in the compiled arXiv versions or in the journal version. arXiv:2011.06640, doi:10.33039/ami.2022.02.001.
- [Kri26a] Alec Kriebel. An exact six-period correction to the printed elliptic billiard assertion k108. Zenodo deposit, published 2026-10-08, unrefereed, 2026. doi:10.5281/zenodo.23271172.
- [Kri26b] Alec Kriebel. A four-period counterexample to the conjectured elliptic billiard invariant k107. Zenodo deposit, published 2026-10-07, unrefereed, 2026. doi:10.5281/zenodo.23231145.
- [Kri26c] Alec Kriebel. The outer focal-pedal area product in odd elliptic billiards. Zenodo deposit, published 2026-10-15, unrefereed, 2026. doi:10.5281/zenodo.23398175.
- [Kri26d] Alec Kriebel. A telescoping proof of the focal antipedal sum invariant. Zenodo deposit, published 2026-10-06, unrefereed, 2026. doi:10.5281/zenodo.23191247.
- [RGK21] Dan Reznik, Ronaldo Garcia, and Jair Koiller. Fifty new invariants of N -periodics in the elliptic billiard. *Arnold Mathematical Journal*, 7(3):341–355, 2021. arXiv:2004.12497v11 (2020) is titled “Eighty new invariants of N -periodics in the elliptic billiard”; table numbers as in v11. arXiv:2004.12497, doi:10.1007/s40598-021-00174-y.
- [Sta21] Hellmuth Stachel. The geometry of billiards in ellipses and their Poncelet grids. *Journal of Geometry*, 112(3):Article 40, 2021. Statement numbers as in arXiv:2105.03362v2. arXiv:2105.03362, doi:10.1007/s00022-021-00606-2.
- [Sta22] Hellmuth Stachel. On the motion of billiards in ellipses. *European Journal of Mathematics*, 8(4):1602–1622, 2022. Statement numbers as in arXiv:2105.03624v2. arXiv:2105.03624, doi:10.1007/s40879-021-00524-2.
- [ula26] ulamai. UnsolvedMath. Hugging Face dataset, <https://huggingface.co/datasets/ulamai/UnsolvedMath>, 2026. Version 1.6.0; records AMR-050-0002, AMR-050-0028, AMR-050-0032 and AMR-050-0039.