

Four-Periodic Counterexamples to Two Elliptic Billiard Invariants

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Abstract

We give exact primitive four-periodic elliptic billiards contradicting the claimed constancy of $(A'/A) \prod_i \sin(\theta_i/2)$ for periods divisible by four. Here A is the original orbit area, A' its outer tangent-polygon area, and θ_i its internal angles. The same examples refute an older even-period assertion about $A'A''$, where A'' is the caustic-contact polygon area. An explicit continuous family proves that both examples have the same ellipse and confocal elliptic caustic, and gives the sharp ranges of both expressions on that family. The first assertion is retained in the final source table; the second was removed and its code renumbered. Previously published adjacent area identities are credited. This note is a version-qualified correction, with no claim of absolute novelty.

1 Expressions, definitions and source versions

For a counterclockwise billiard polygon $P = (P_1, \dots, P_N)$ in an ellipse, let P'_i be the intersection of the ellipse tangents at P_i, P_{i+1} , and let P''_i be the contact of that chord with the fixed confocal caustic. All indices are cyclic. Use the signed area

$$\mathcal{A}(V_1, \dots, V_N) = \frac{1}{2} \sum_{i=1}^N \det(V_i, V_{i+1})$$

and write $A = \mathcal{A}(P)$, $A' = \mathcal{A}(P')$, $A'' = \mathcal{A}(P'')$. The angles θ_i are the internal angles of P , not of P' .

Table 2 of [1], and also Table 2 on page 345 of the final article [2], lists $k_{107} = k_{103}k_{105}$ for $N \equiv 0 \pmod{4}$, where $k_{103} = A'/A$ and $k_{105} = \prod_i \sin(\theta_i/2)$. We denote this expression by K . The older table also lists $k_{111} = A'A''$ for even N , denoted below by B . That row is absent from the final table: the final code k_{111} instead denotes $A'A''/A^2 = 1$ for odd N . We do not challenge that different odd-period identity or assert a reason for the old row's deletion.

The frozen dataset records AMR-050-0001 and AMR-050-0005 use respectively the expressions K and the old B . The counterexamples below address those expressions, not an unqualified invariant number.

2 A continuous family of genuine four-periodic billiards

Fix $a > b > 0$ and put

$$M = \text{diag}(a^2, b^2), \quad D = a^2b^2, \quad s = a^2 + b^2, \quad \lambda = D/s.$$

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The outer ellipse is $x^\top M^{-1}x = 1$. For $t \in \mathbb{R}$ set

$$n = (\cos t, \sin t), \quad m = (-\sin t, \cos t), \quad h^2 = n^\top M n, \quad k^2 = m^\top M m, \quad d = n^\top M m,$$

with $h, k > 0$, and let

$$P = M n / h, \quad Q = M m / k. \quad (1)$$

We keep t distinct from the internal-angle notation θ_i . In the positively oriented orthonormal coordinates (n, m) ,

$$M = \begin{pmatrix} h^2 & d \\ d & k^2 \end{pmatrix}, \quad P = (h, d/h), \quad Q = (d/k, k), \quad h^2 + k^2 = s, \quad h^2 k^2 - d^2 = D. \quad (2)$$

In particular $hk > |d|$.

Lemma 1. *The ordered polygon $(P, Q, -P, -Q)$ is a convex primitive four-periodic billiard with the fixed confocal elliptic caustic of shape matrix $C = M - \lambda I$. Its contacts lie strictly inside its four segments. The polygons vary continuously with t .*

Proof. Equation (1) puts the vertices on the outer ellipse, with outward unit normals $n, m, -n, -m$. In the rotated coordinates,

$$Q - P = \frac{hk - d}{hk}(-h, k), \quad P + Q = \frac{hk + d}{hk}(h, k).$$

Both coefficients are positive. Since $h^2 + k^2 = s$, the incoming unit direction at P is $(h, k)/\sqrt{s}$ and the outgoing direction is $(-h, k)/\sqrt{s}$. Reflection in its tangent line $x = h$ interchanges these directions. At Q , reflection in $y = k$ sends $(-h, k)/\sqrt{s}$ to $(-h, -k)/\sqrt{s}$. Central negation proves the remaining two reflection equations.

Because $0 < \lambda < b^2$, the fixed matrix C is positive definite. Its semiaxes in the original coordinates are

$$a_c = a^2/\sqrt{s}, \quad b_c = b^2/\sqrt{s}.$$

They satisfy $a_c^2 - b_c^2 = a^2 - b^2$, so the caustic is confocal. In the rotated coordinates, the first chord has normal $w_+ = (k, h)$ and offset $u_+ = hk + d > 0$. Its caustic support satisfies

$$w_+^\top C w_+ = 2h^2 k^2 + 2hkd - \lambda s = (hk + d)^2 = u_+^2.$$

Thus it is a caustic tangent, with contact $R_+ = C w_+ / u_+$. The second chord has $w_- = (k, -h)$ and $u_- = d - hk < 0$; the same calculation gives $w_-^\top C w_- = u_-^2$ and contact $R_- = C w_- / u_-$. Direct substitution gives

$$R_+ = P + \frac{k^2}{s}(Q - P), \quad R_- = Q + \frac{h^2}{s}(-P - Q).$$

The two parameters lie in $(0, 1)$; the other contacts are their negatives. Finally $\det(P, Q) = D/(hk) > 0$. Hence the orbit is a convex central parallelogram with four distinct vertices, and has no smaller period. All coordinates are continuous in t . This proves common-family membership directly, without inferring it from two isolated samples. $\square$

3 Areas, internal angles and sharp ranges

Theorem 2. *On the family in Lemma 1,*

$$A = \frac{2D}{hk}, \quad A' = 4hk, \quad A'' = \frac{4hkD}{s^2}, \quad \prod_{i=1}^4 \sin(\theta_i/2) = \frac{h^2 k^2}{s^2}.$$

Consequently

$$K = \frac{2(D + d^2)^2}{Ds^2}, \quad B = \frac{16D(D + d^2)}{s^2}. \quad (3)$$

Both are nonconstant for every $a > b > 0$. Their sharp ranges on this family are

$$\frac{2D}{s^2} \leq K \leq \frac{s^2}{8D}, \quad \frac{16D^2}{s^2} \leq B \leq 4D. \quad (4)$$

Proof. The original central parallelogram has signed area $2 \det(P, Q) = 2D/(hk)$. Its consecutive outer tangent intersections are $(h, k), (-h, k), (-h, -k), (h, -k)$, so $A' = 4hk$. The contact polygon is $(R_+, R_-, -R_+, -R_-)$. Since

$$\det C = D - \lambda s + \lambda^2 = \lambda^2, \quad \det(w_+, w_-) = -2hk, \quad u_+ u_- = -D,$$

its area is $2 \det(R_+, R_-) = 4hkD/s^2$.

The internal angle at P is between the negative incoming direction $(-h, -k)/\sqrt{s}$ and outgoing direction $(-h, k)/\sqrt{s}$. Its cosine is $(h^2 - k^2)/s$, and its positive half-angle sine is $k/\sqrt{s}$. At Q the positive half-angle sine is $h/\sqrt{s}$. The opposite angles repeat these values, giving the asserted product. Multiplying the relevant quantities proves (3).

Now $d = (b^2 - a^2) \sin t \cos t$, so

$$0 \leq d^2 \leq (a^2 - b^2)^2/4, \quad D + (a^2 - b^2)^2/4 = s^2/4.$$

Both expressions in (3) are strictly increasing functions of d^2 . Their endpoints occur at $t = 0$ and $t = \pi/4$; continuity gives every value between them. The two endpoints differ because $a > b$ implies $s^2 > 4D$. This proves (4). $\square$

4 A completely rational counterexample

Take $a = 4, b = 3$. The outer ellipse is $x^2/16 + y^2/9 = 1$ and the common caustic is $x^2/(256/25) + y^2/(81/25) = 1$. At $t = 0$ the orbit is

$$(4, 0), (0, 3), (-4, 0), (0, -3);$$

at $t = \pi/4$ it is

$$(16/5, 9/5), (-16/5, 9/5), (-16/5, -9/5), (16/5, -9/5).$$

Lemma 1 connects them through primitive billiards with exactly the same caustic. All original and derived areas are positive:

	axis diamond	rectangle
A	24	576/25
A'	48	50
A''	6912/625	288/25
$\prod_i \sin(\theta_i/2)$	144/625	1/4
K	288/625	625/1152
B	331776/625	576

The differences are $58849/720000 > 0$ for K and $28224/625 > 0$ for B . Thus $N = 4$, which satisfies both stated parity conditions, already contradicts both constancy assertions. No self-intersection, hyperbolic caustic, repeated shorter orbit, infinite tangent intersection or vanishing area quotient is involved. Reversing traversal reverses both derived areas and leaves their product unchanged; absolute areas also leave these examples unchanged.

5 Prior results and verification scope

Garcia and Reznik [3, Proposition 4.9] already prove $A'A = 8a^2b^2$ and $AA'' = 2a^4b^4/s^2$ for simple four-periodics. Our formulas recover both identities. They and the standard parallelogram/outer-rectangle geometry are not claimed as new. The present contribution is an explicit, reproducible counterexample with source-version distinctions and an analytic common-family certificate. No absolute priority is claimed; the removal of the old B row is not presented as an author-confirmed erratum.

The accompanying dependency-free Python checks use exact rational arithmetic for the two orbits: ellipse membership, reflection, caustic tangency inside the segments, convex order, tangent intersections, signed areas and the internal half-angle product. These finite checks support the displayed witnesses; they do not replace the continuous analytic proof or constitute proof-assistant verification. This unrefereed note was developed with AI assistance in the reasoning, source search, exact computation and drafting. It has not undergone independent human peer review.

References

- [1] D. Reznik, R. Garcia and J. Koiller, *Eighty New Invariants of N -Periodics in the Elliptic Billiard*, arXiv:2004.12497v11, 29 October 2020. [Version-specific source](#).
- [2] D. Reznik, R. Garcia and J. Koiller, *Fifty New Invariants of N -Periodics in the Elliptic Billiard*, Arnold Mathematical Journal **7** (2021), 341–355. [doi:10.1007/s40598-021-00174-y](#).
- [3] R. Garcia and D. Reznik, *Exploring self-intersected N -periodics in the elliptic billiard*, Annales Mathematicae et Informaticae, online 22 February 2022. [doi:10.33039/ami.2022.02.001](#).