

On Seven Conjectures of Fuchs on Billiard Trajectories in Regular Polygons and Geodesics on the Regular Dodecahedron

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Abstract

In a problem contribution (Arnold Math. J. 7 (2021)), D. Fuchs stated seven conjectures on the billiard trajectories in a regular n -gon which join two vertices (short trajectories) and on their $n - 2$ types $A_0, \dots, A_{n-3}$, on closed trajectories, and on geodesics on the regular dodecahedron. We decide all seven. The tool is the double n -gon: the types are the orbits of saddle connections under an explicit group of affine automorphisms, and they are computed by an angular invariant. Conjecture 2.6 (types of parallel short trajectories) and Conjecture 2.7 (ratios of their lengths) are proved for all $n \geq 4$, and Conjecture 2.3 (types of the vertices of a reachable n -gon) for all $n \geq 5$. Conjecture 2.5 (reachable points on the two lines through a unitary pair) is true as printed for $n = 5$ and false for every $n \geq 6$; it is proved for all $n \geq 5$ with λ^2 and $\lambda^2 - 1$, $\lambda = 2 \cos \frac{\pi}{n}$, in place of $\lambda + 1$ and λ , which are the constants given by the source's own argument for $n = 6$. Conjecture 2.4 (every reachable point is a vertex of infinitely many reachable n -gons) is false for every $n \geq 5$: a reachable point whose type is not A_0 is a vertex of at most one reachable n -gon, and an explicit point is a vertex of none; exactly the points of type A_0 are vertices of infinitely many. Conjecture 3.2 is proved: a short geodesic of type A_0 on the regular dodecahedron never ends at graph distance 2 from its first vertex; its ends are exchanged by a half-turn about the midpoint of an edge, an argument which Athreya, Aulicino and Hooper used for closed saddle connections and Troubetzkoy for the other regular polyhedra. Conjecture 1.7 (after cancellation of common factors, the quotient of the lengths of a closed trajectory and of its preclosed part is never n) is false for every odd n with at least two distinct prime factors; the smallest case is $n = 15$, for the trajectories parallel to a side. It is true for odd prime powers, and for even n when it is read for the trajectories in general position in their band. These quotients were determined in all periodic directions by Veech (1992), through the monodromy of the covering of the double n -gon by the surface of the billiard; the answer to Conjecture 1.7 follows from his formula by a short computation which is not in his paper. We also give a second derivation of the formula for odd n , and a proof of the counterexample which uses neither surfaces nor homology. For prime n the conjecture was known: this case follows from a theorem of McMullen, for $n = 5$ from results of Davis, Fuchs and Tabachnikov and of Davis and Lelièvre, and it is also a consequence of Veech's formula. The proofs are elementary. This is an unrefereed note.

1 Introduction

In the problem contribution [Fuc21], Fuchs reports on computer experiments with billiard trajectories in a regular n -gon and with geodesics on the surface of the regular dodecahedron. The outcome is formulated as seven conjectures, numbered 1.7, 2.3, 2.4, 2.5, 2.6, 2.7 and 3.2 in [Fuc21]; we keep these numbers. They are the records AMR-049-0001 to AMR-049-0007 of the UnsolvedMath collection

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[ula26], where all seven are listed as open. In this note all seven are decided: four are proved, one is proved after a correction of two constants, and two are disproved, together with the statements which hold instead (Table 1).

1.1 The source and its conventions

We recall the definitions of [Fuc21], with the conventions which fix them. Let $n \geq 3$ and

$$\omega := \frac{(n-2)\pi}{n}, \quad \lambda_k := \frac{\sin((k+1)\pi/n)}{\sin(\pi/n)} \quad (k \in \mathbb{Z}), \quad \lambda := \lambda_1 = 2 \cos \frac{\pi}{n}, \quad S := \sin \frac{2\pi}{n}.$$

Thus ω is the interior angle of the regular n -gon, λ_k is the length of a diagonal which spans $k+1$ sides of the regular n -gon with side 1, and $\lambda_0 = 1$, $\lambda_2 = \lambda^2 - 1$, $\lambda\lambda_k = \lambda_{k-1} + \lambda_{k+1}$ [Fuc21, Proposition 1.3]; moreover $S = \lambda \sin \frac{\pi}{n}$. For $a, b \in \mathbb{C}$ we write $\det(a, b) := \text{Im}(\bar{a}b)$.

The initial n -gon. $P_0 \subset \mathbb{C}$ is the regular n -gon with side 1 which lies in the closed upper half-plane and has the vertices

$$X_0 = O = 0, \quad X_1, \dots, X_{n-1} = 1 \quad \text{in clockwise order}, \quad X_{j+1} - X_j = e^{i(\omega - 2\pi j/n)}$$

[Fuc21, §2.1, §2.2]. We put $\xi := X_{n-1} = 1$ and $\eta := X_1 = e^{i\omega}$. Then $\det(\xi, \eta) = S$,

$$|X_{k+1}| = \lambda_k, \quad \arg X_{k+1} = \frac{(n-2-k)\pi}{n} \quad (0 \leq k \leq n-2), \quad (1)$$

and, comparing the coordinates in the basis ξ, η ,

$$X_2 = \xi + (\lambda^2 - 1)\eta, \quad X_{n-2} = \eta + (\lambda^2 - 1)\xi. \quad (2)$$

The sides are $e_s := [X_s, X_{s+1}]$, with indices modulo n .

Short trajectories and developments. A *short trajectory* is a billiard trajectory in the regular n -gon which goes from a vertex to a vertex and meets no vertex in between [Fuc21, §1.2]. The two vertices may coincide, and the sides and the diagonals are short trajectories. We write N for the number of segments of a short trajectory. For a short trajectory which starts at O , replace every reflection of the trajectory in a side by the reflection of the whole n -gon in that side. The trajectory becomes a straight segment OB , its *development*, which passes through a chain of polygons $P^{(0)} = P_0, P^{(1)}, \dots, P^{(N-1)}$, each the mirror image of the preceding one in a common side, and ends at a vertex B of $P^{(N-1)}$ (Figure 1). The points B which arise in this way are the *reachable points* [Fuc21, §2.1]. They lie in the closed sector $\{0 \leq \arg z \leq \omega\}$, and a short trajectory from O is determined by its reachable point.

The angles α and β . Let the boundary of the n -gon be oriented counterclockwise. For an oriented short trajectory, $\alpha \in [0, \omega]$ is the angle at its first vertex between the boundary edge which leaves this vertex and the trajectory, and $\pi - \beta$ is the angle at its last vertex between the boundary edge which leaves that vertex and the trajectory [Fuc21, Fig. 8]; thus $\beta \in [\frac{2\pi}{n}, \pi]$. For a trajectory from O the edge which leaves O is the side OX_{n-1} , and α is the slope angle of the development. The source states that one of $\alpha + \beta$, $\alpha - \beta$ is a multiple of $\frac{2\pi}{n}$, and adds: “the sign is $-$, if N is even, and $+$, if N is odd” [Fuc21, §2.2]. Here N counts the segments of the trajectory, not its reflections; the rule is proved in Lemma 2.3.

Types. Definition 2.1 of [Fuc21] assigns to every short trajectory one of $n-2$ types A_k , where k is a residue modulo $n-2$, so that $A_{n-2} = A_0$. It lists, separately for even and for odd n , four conditions: two of the form $\beta - \alpha = \dots$ and two of the form $\beta + \alpha = \dots$, each with an inequality for α . Together with the rule just quoted, the definition reads as follows.

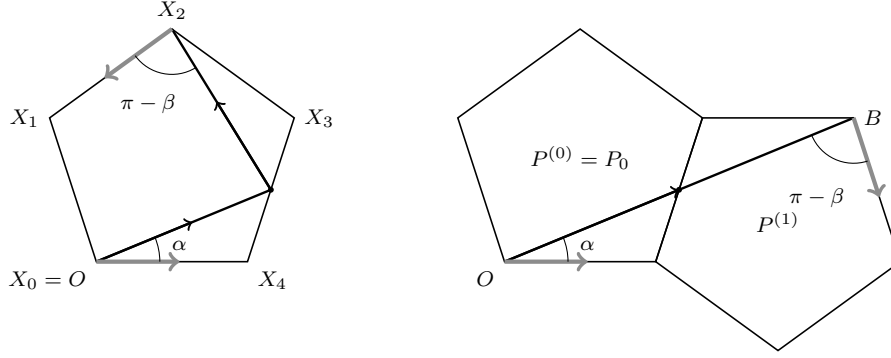


Figure 1: Left: the initial pentagon P_0 with a short trajectory from O with $N = 2$ segments, and the angles α and $\pi - \beta$ of [Fuc21, Fig. 8]; the grey arrows are the boundary edges which leave the two end vertices in the counterclockwise orientation. Right: its development OB . Since N is even, $P^{(1)}$ is a mirror image of P_0 : the image of the edge which leaves the end vertex is the edge which leaves B clockwise in $P^{(1)}$. Here $\beta - \alpha = \frac{2\pi}{5}$, and the type is A_0 .

Definition 1.1 (types [Fuc21, Definition 2.1]). A short trajectory with N segments and angles α, β has *type* A_k , where

$$\begin{aligned} \text{if } N \text{ is odd:} \quad & \alpha + \beta = \frac{2\pi j}{n} \quad \text{with } 1 \leq j \leq n-1, & k \equiv n-1-j \pmod{n-2}; \\ \text{if } N \text{ is even:} \quad & \beta - \alpha = \frac{2\pi j}{n} \quad \text{with } -(n-4) \leq 2j \leq n, & k \equiv j-1 \pmod{n-2}. \end{aligned}$$

The type of a reachable point is the type of the short trajectory from O with this development.

Remark 1.2. (a) The printed conditions are $\beta - \alpha = \frac{2(k+1)\pi}{n}$ for the small values of k , $\beta - \alpha = \frac{2(k+3-n)\pi}{n}$ for the large ones, and $\beta + \alpha = \frac{2(n-k-1)\pi}{n}$; these are the two lines of Definition 1.1. The printed inequalities for α are equivalent to $\beta \leq \pi$ in the cases with small k and to $\beta \geq \frac{2\pi}{n}$ in the cases with large k , so they hold automatically. The two tables which follow the definition in [Fuc21] agree with Definition 1.1.

(b) The rule on the parity of N is part of the definition. If the direction of a short trajectory from O is not a multiple of $\frac{\pi}{n}$, then only one of $\alpha + \beta, \alpha - \beta$ is a multiple of $\frac{2\pi}{n}$, and nothing has to be chosen. The other short trajectories from O are the sides and diagonals X_0X_{k+1} (Section 2.4); they have $N = 1$, $\alpha = \frac{(n-2-k)\pi}{n}$ and $\beta = \frac{(n-k)\pi}{n}$, hence $\alpha + \beta = \frac{2(n-k-1)\pi}{n}$, and their type is A_k , as the source says in its example [Fuc21, §2.2, Fig. 9]. But they also satisfy $\beta - \alpha = \frac{2\pi}{n}$, which without the rule would give the type A_0 for every k . The source remarks that the type A_0 is characterized by $\beta - \alpha = \frac{2\pi}{n}$; this holds for the short trajectories with an even number of segments. Precisely, by Definition 1.1, for $n \geq 4$:

$$\text{type } A_0 \iff N = 1 \text{ and the trajectory is a side, or } N \text{ is even and } \beta - \alpha = \frac{2\pi}{n}. \quad (3)$$

Indeed, for odd N the type is A_0 only if $n-1-j$ is 0 or $n-2$, that is, $j = n-1$ or $j = 1$; then $(\alpha, \beta) = (\omega, \pi)$ or $(0, \frac{2\pi}{n})$, and the trajectory runs along a side. (For $n = 3$ there is only one type, and (3) does not hold.)

Reachable n -gons and unitary pairs. A *reachable n -gon* is an image gP_0 , $g \in \text{SL}(2, \mathbb{R})$, all of whose vertices other than O are reachable points [Fuc21, §2.4]. Since g is linear and preserves the orientation, its vertices in clockwise order are $v_0 = O$ and $v_j := gX_j$ for $1 \leq j \leq n-1$. A *unitary pair* is a pair (u, v) of reachable points of type A_0 with $\det(u, v) = S (= \det(\xi, \eta))$.

Parallel trajectories. Two billiard trajectories in the regular n -gon are *parallel* if, for segments of them, the sum or the difference of the angles which they form with a side is a multiple of $\frac{2\pi}{n}$ [Fuc21, §1.1]. For short trajectories from O with slope angles α and α' this means $\alpha' = \frac{\ell\pi}{n} + \varepsilon\alpha$ with $\varepsilon = \pm 1$ and $\ell \in \mathbb{Z}$, where ℓ is even if n is even [Fuc21, §2.5].

Closed and preclosed trajectories. A *closed* trajectory avoids the vertices and repeats itself. A piece of a trajectory is *preclosed* if its two ends lie on sides, divide these sides in the same ratio and make the same angles with them, the sides and the trajectory being taken with their orientations [Fuc21, §1.7.1]; in other words, if the end of the piece is the image of its beginning under a rotation of the n -gon. It is *simple* if no proper part of it is preclosed. Closed trajectories which are reflected by the same sides in the same order are *strongly parallel*; a maximal family of strongly parallel closed trajectories will be called a *band*. A simple preclosed piece, repeated d times, is the closed trajectory, and d divides n [Fuc21, Proposition 1.5]; we call d the *factor*. It is the same for all trajectories of a band, with the possible exception of one trajectory in the middle of the band (Section 8.1).

The dodecahedron. A *short geodesic* on the surface of the regular dodecahedron goes from a vertex to a vertex and meets no vertex in between. Unfolding the faces along it gives the development of a short trajectory in the regular pentagon, and the type of the geodesic is the type of this trajectory [Fuc21, §3]; see Section 7. The *distance* of two vertices is their distance in the graph of the dodecahedron.

1.2 The seven conjectures

In this notation the conjectures of [Fuc21] are the following statements.

- 1.7** In every class of parallel closed trajectories, divide the factors of the bands by their greatest common divisor. Then no quotient is equal to n .
- 2.3** Let $n \geq 5$. The vertices $v_1, \dots, v_{n-1}$ of a reachable n -gon have the types $A_0, A_1, \dots, A_{n-3}, A_0$, in this order.
- 2.4** Every reachable point is a vertex of reachable n -gons; in parentheses the source adds that there are infinitely many of them.
- 2.5** Let $n \geq 5$ and let (u, v) be a unitary pair. (a) For every integer m for which it lies in the upper half-plane, $u + m(\lambda + 1)v$ is a reachable point of type A_0 . (b) With the same proviso, $u + (\lambda + m(\lambda + 1))v$ is a reachable point of type A_1 . (c) There are no other reachable points on the line $u + tv$. (d) The reachable points on the line $v + tu$ are described in the same way; their types are A_0 and A_{n-3} .
- 2.6** If the short trajectory from O with slope angle α has type A_k , then the parallel short trajectory from O with slope angle $\frac{\ell\pi}{n} + \varepsilon\alpha$, $\varepsilon = \pm 1$, has type $A_{\varepsilon k - \ell}$. Here ℓ is even if n is even.
- 2.7** The lengths of parallel short trajectories of types A_k and A_ℓ are in the ratio $\sin \frac{(k+1)\pi}{n} : \sin \frac{(\ell+1)\pi}{n}$.
- 3.2** A short geodesic of type A_0 on the regular dodecahedron which begins at a vertex v never ends at a vertex at distance 2 from v .

1.3 Results

Let $P'_0 := -P_0$. The *double n -gon* D_n is the flat surface obtained from P_0 and P'_0 by gluing every side e_s of P_0 to the parallel side $-e_s$ of P'_0 by a translation (Section 2.1). A *saddle connection* is an oriented geodesic segment of D_n from a vertex to a vertex without a vertex in its interior. Developments of

short trajectories from O become saddle connections which start in the corner of P_0 at O . The surface D_n has a rotation ϱ (by $\frac{2\pi}{n}$ if n is even, by $\frac{\pi}{n}$ if n is odd), the central symmetry $\mu(z) = -z$, and for every $k \in \mathbb{Z}$ a twist τ_k , which is a shear in each of the cylinders into which D_n decomposes in the direction $\frac{k\pi}{n}$ (Sections 2.3 and 2.4). Let G_0 be the group of affine automorphisms of D_n generated by ϱ , μ and all τ_k .

Theorem A (types are orbits). *Let $n \geq 4$.*

- (a) *There is a function K on the saddle connections of D_n with values in $\mathbb{Z}/(n-2)$, defined by the positions of the two ends of the saddle connection in the corners which contain them (Section 2.2), such that the short trajectory from O which corresponds to a saddle connection γ has type $A_{K(\gamma)}$.*
- (b) *K is invariant under G_0 , and every saddle connection γ of D_n is the image under an element of G_0 of exactly one of the diagonals $\delta_k := X_0X_{k+1}$, $0 \leq k \leq n-3$, namely of the one with $k \equiv K(\gamma)$.*

Consequently two short trajectories from O have the same type if and only if the corresponding saddle connections lie in one orbit of G_0 .

This is the precise form of the remark of the source that the types should be the orbits of the Veech group [Fuc21, §2].

Theorem B (Conjecture 2.6). *Let $n \geq 4$. Let the short trajectory from O with slope angle α have type A_k , and let $\alpha' = \frac{\ell\pi}{n} + \varepsilon\alpha \in [0, \omega]$ with $\varepsilon = \pm 1$ and $\ell \in \mathbb{Z}$, where ℓ is even if n is even. Then there is a short trajectory from O with slope angle α' , and its type is $A_{\varepsilon k - \ell}$.*

Theorem C (Conjecture 2.7). *Let $n \geq 4$. For every direction θ of a saddle connection of D_n there is a number $c_\theta > 0$ such that every saddle connection γ of direction $\pm\theta$ has the length $c_\theta \lambda_{\hat{k}}$, where $\hat{k} \in \{0, \dots, n-3\}$ is the representative of $K(\gamma)$. Consequently parallel short trajectories from O of types A_k and A_ℓ , $0 \leq k, \ell \leq n-3$, have lengths in the ratio $\sin \frac{(k+1)\pi}{n} : \sin \frac{(\ell+1)\pi}{n}$.*

Theorem D (Conjecture 2.3). *Let $n \geq 5$ and $g \in \text{SL}(2, \mathbb{R})$, and suppose that the four points $gX_1, gX_2, gX_{n-2}, gX_{n-1}$ are reachable. Then gP_0 is a reachable n -gon, its vertex $v_j = gX_j$ has type A_{j-1} for $1 \leq j \leq n-1$, and g is the derivative of an element of G_0 which maps the corner of P_0 at O into itself. In particular the vertices $v_1, \dots, v_{n-1}$ of every reachable n -gon have the types $A_0, A_1, \dots, A_{n-3}, A_0$. Moreover, reachable points u, v with $\det(u, v) = S$ form a unitary pair if and only if $u = v_{n-1}$ and $v = v_1$ for a reachable n -gon.*

Theorem E (Conjecture 2.5). *Let $n \geq 5$ and let (u, v) be a unitary pair.*

- (a) *The reachable points on the line $u + \mathbb{R}v$ are exactly the points*

$$u + m\lambda^2 v \quad (\text{type } A_0) \quad \text{and} \quad u + (\lambda^2 - 1 + m\lambda^2)v \quad (\text{type } A_1), \quad m \in \mathbb{Z},$$

which lie in the open upper half-plane, and the point ξ if it lies on the line.

- (b) *The reachable points on the line $v + \mathbb{R}u$ are exactly the points $v + m\lambda^2 u$ (type A_0) and $v + (\lambda^2 - 1 + m\lambda^2)u$ (type A_{n-3}), $m \in \mathbb{Z}$, which satisfy $\det(\cdot, \eta) > 0$, and the point η if it lies on the line.*
- (c) *For $n = 5$ we have $\lambda^2 = \lambda + 1$, and (a), (b) are Conjecture 2.5 as printed. For every $n \geq 6$ and every unitary pair the printed statement fails: the points $u + (\lambda + 1)v$ and $u + \lambda v$ lie in the upper half-plane and are not reachable, and $u + \lambda^2 v$ is a reachable point which is not of one of the printed forms.*

Theorem F (Conjecture 2.4). *Let $n \geq 5$.*

- (a) *Every reachable point of type A_0 is a vertex of infinitely many reachable n -gons.*
- (b) *A reachable point of type A_k , $1 \leq k \leq n-3$, is a vertex of at most one reachable n -gon, and then it is the vertex v_{k+1} . In particular X_2 is a vertex of exactly one reachable n -gon, namely of P_0 .*
- (c) *The point $w_n := 2 + 3\zeta + \zeta^2$, $\zeta = e^{2\pi i/n}$, is a reachable point of type A_{n-3} which is a vertex of no reachable n -gon.*

Thus Conjecture 2.4 is false for every $n \geq 5$, with and without the words in parentheses, and the reachable points which are vertices of infinitely many reachable n -gons are exactly those of type A_0 .

Theorem G (Conjecture 3.2). *Let a short geodesic of type A_0 on the regular dodecahedron go from a vertex v to a vertex w . Then $w = h(v)$ for a half-turn h of the dodecahedron about the axis through the midpoints of two opposite edges; if the geodesic is not an edge, this axis passes through the midpoint of the geodesic. Consequently the distance of v and w is 1, 3, 4 or 5. In particular a short geodesic of type A_0 never ends at distance 2 from its first vertex, and it is never closed.*

That a short geodesic of type A_0 is never closed, and the use of a half-turn about the midpoint, are due to Athreya, Aulicino and Hooper [AAH22]. For the other four regular polyhedra Troubetzkoy [Tro23] showed in the same way, with the rotation by π about the axis through the midpoint of the geodesic, that the two ends of every geodesic between vertices are exchanged; see Section 1.5 and Remark 7.3.

Theorem H (Conjecture 1.7; parts (a) and (b) are due to Veech [Vee92]).

- (a) *Let n be even. The factor of every closed trajectory which is not the middle trajectory of its band divides $\frac{n}{2}$. Hence Conjecture 1.7 holds for even n , if it is read for these trajectories.*
- (b) *Let $n = 2m + 1 \geq 5$ be odd. Every class of parallel closed trajectories consists, up to the symmetries of the n -gon and the reversal of the trajectories, of m bands. Their simple preclosed pieces have the lengths $c \lambda \sin \frac{2j\pi}{n} / \sin \frac{\pi}{n}$, $1 \leq j \leq m$, with one constant $c > 0$, and the factor of the j -th band is*

$$d_j = \frac{n'}{\gcd(n', j)}$$

for a divisor n' of n which depends only on the class. Every divisor n' of n occurs. For the trajectories parallel to a side, $c = 1$ and $n' = n$, so that $d_j = n / \gcd(n, j)$.

- (c) *Let n be odd. Conjecture 1.7 holds for n if and only if n is a power of a prime. If n has at least two distinct prime factors, it fails for the trajectories parallel to a side; the smallest case is $n = 15$, where the seven bands have the factors 15, 15, 5, 15, 3, 5, 15. If n is prime, then in every class the factors are all 1 or all n .*

Parts (a) and (b) of Theorem H are contained in Veech's paper [Vee92] on the billiard in a regular polygon, where the factors appear as the degrees with which the cylinders of the surface of the billiard cover the cylinders of the double n -gon [Vee92, Proposition 8.11 and formulas (7.7), (8.16)]; see Sections 1.5 and 8.5. Part (c) is deduced from (b) by a few lines of arithmetic, which are not in [Vee92]; this deduction, and the answer to Conjecture 1.7 which it gives, are what this note adds. The last statement of (c) is Proposition 1.8 of [Fuc21]; it follows from a theorem of McMullen [McM26], see Section 1.5, and it is likewise a consequence of Veech's formula. The failure of Conjecture 1.7 for the trajectories parallel to a side also has an elementary proof which uses neither D_n nor homology (Proposition 8.2).

Conjecture	Record of [ula26]	Outcome	Theorem
1.7	AMR-049-0001	false for every odd n with at least two distinct prime factors (smallest case $n = 15$); true for odd prime powers, and for even n for the trajectories which are not middle trajectories of bands; this follows from Veech's formula for the factors [Vee92]	H
2.3	AMR-049-0002	proved for all $n \geq 5$	D
2.4	AMR-049-0003	false for every $n \geq 5$, also without the words in parentheses; exactly the points of type A_0 are vertices of infinitely many reachable n -gons	F
2.5	AMR-049-0004	as printed: true for $n = 5$, false for every $n \geq 6$; proved for all $n \geq 5$ with λ^2 and $\lambda^2 - 1$ in place of $\lambda + 1$ and λ	E
2.6	AMR-049-0005	proved for all $n \geq 4$	B
2.7	AMR-049-0006	proved for all $n \geq 4$	C
3.2	AMR-049-0007	proved	G

Table 1: The seven conjectures of [Fuc21], the corresponding records of the corpus, and their outcomes.

1.4 Remarks on the text of the source

Five features of the text of [Fuc21], and of the records of [ula26], matter for the answers.

(1) *The constants in Conjecture 2.5.* The conjecture is printed with the step $\lambda + 1$ and the offset λ . For $n = 6$ the source proves it in lattice coordinates [Fuc21, §2.4]: there the reachable points on the line $u + tv$ are found to be the points $u + kv$ with $k \equiv 0$ (type A_0) and $k \equiv 2$ (type A_1) modulo 3. So the step is $3 = \lambda^2$ and the offset is $2 = \lambda^2 - 1$, and not $1 + \sqrt{3}$ and $\sqrt{3}$. The equalities $\lambda + 1 = \lambda^2$ and $\lambda = \lambda^2 - 1$ hold exactly for $n = 5$. Theorem E is the statement with λ^2 and $\lambda^2 - 1$.

(2) *Conjecture 2.4.* The paragraph of [Fuc21, §2.4] which is headed as a proof of Conjectures 2.4 and 2.5 for $n = 6$ determines the reachable points on the line $u + tv$; it proves the items of Conjecture 2.5 and contains no argument for reachable points whose type is not A_0 . Example 6.2 shows with the formulas of the source that Conjecture 2.4 fails for $n = 6$.

(3) *Misprints.* In [Fuc21, §2.4] the vector η is printed with the opposite sign of its first coordinate; it is $e^{i\omega}$. The equivalence of the two descriptions of unitary pairs is attributed there to Theorem 3.1 of [Fuc21], which is a statement on polyhedra; this equivalence is the last statement of Theorem D. In the case of odd $n = 2m + 1$ of Definition 2.1 the first condition is printed for $k \leq m + 1$, where $k \leq m - 1$ is meant, as in the table which follows it. In two sentences of the proof of Conjecture 2.6 for $n = 6$ the text has A_2 where the list of types and the table have A_3 . Finally, for even n the names “even” and “odd” of the two classes of directions are interchanged between Section 1 and Section 2 of the source; no conjecture depends on this, and we use the names of Section 1 only in Table 5.

(4) *Conjecture 1.7 concerns bands.* A band can contain one trajectory, its middle trajectory, whose preclosed piece is half as long as that of the other trajectories of the band. For even n its factor can be n : the regular n -gon through the midpoints of the sides is a closed trajectory with a preclosed piece of one segment. If such trajectories were counted with their own factor, Conjecture 1.7 would fail for $n = 6$ and $n = 10$, in contradiction with the table of experiments of the source (Remark 8.4). Theorem H(a) refers to the other trajectories.

(5) *The records of the corpus.* The statements of the records of [ula26] are paraphrases of the conjectures. Record AMR-049-0004 has the printed constants of Conjecture 2.5, and record AMR-049-0003 states Conjecture 2.4 for infinitely many reachable n -gons. Record AMR-049-0007 defines the type A_0 of a geodesic only by the equation $\beta - \alpha = \frac{2\pi}{5}$, for angles at the two ends which it does not specify. With the angles of [Fuc21, Fig. 8] and without the rule on the parity of N , this statement is false for a simple reason: a diagonal of a face is a short geodesic with $N = 1$ and $\beta - \alpha = \frac{2\pi}{5}$ (Remark 1.2(b)), and it joins two vertices at distance 2. In the source the diagonals have the types

A_1 and A_2 . The statement proved in Theorem G is that of the source, with the type of Definition 1.1 and (3). The records name the source by a title which differs from the title of the article; the address of the file is that of [Fuc21].

1.5 Earlier work and method

The problem and all definitions are due to Fuchs [Fuc21]. For $n = 6$ the source proves, in lattice coordinates, Conjecture 2.3, the four items of Conjecture 2.5 in the form of Theorem E, and Conjecture 2.6 [Fuc21, §2.4, §2.5]; Theorems B, D and E agree with these proofs. For Conjecture 1.7 the source gives a table of experiments for $n = 6, 8, 9, 10, 12$, which is compared with the computations of this note in Table 5, and it derives from the conjecture that for prime n the factors of a class are all 1 or all n [Fuc21, Proposition 1.8]; it says that for $n = 5$ and $n = 7$ this is proved in [DFT11].

The pentagon. Davis, Fuchs and Tabachnikov [DFT11, Theorem 11, Corollary 12] show for $n = 5$ that a closed trajectory of length L on the double pentagon gives a billiard trajectory of length L or $5L$, according to a congruence modulo 5, and that the two closed billiard trajectories of one direction have lengths in the golden ratio. This is Theorem H for $n = 5$. In the arXiv version 1 of [DFT11], which is the version read for this note, the case $n = 7$ is discussed by examples, and the statement on the lengths is proved for $n = 5$ only. Davis and Lelièvre [DL18, Theorem 3.31] also obtain the lengths p and $5p$ for the pentagon.

Prime n . McMullen [McM26] studies the billiard trajectories through the midpoints of the sides of a regular n -gon in a periodic direction s . For a prime $n = p$ he proves that the closed ones among them have all the full dihedral symmetry or all only a reflection symmetry, according to the parity of a valuation of s , and that their lengths are $C(s) |\sin \frac{2\pi\ell}{n}|$ [McM26, Theorems 2.3, 2.4 and 7.1]; his Theorem 2.2 describes, for all odd n , which of these trajectories are isometric. A closed trajectory through the midpoint of a side is the middle trajectory of a band, and it is invariant under the rotations of the n -gon exactly if the factor is n (Remark 8.4). In the language of the source these theorems therefore give Proposition 1.8 of [Fuc21], which is Conjecture 1.7 for prime n . McMullen's proof uses that the first homology group of the surface is $\mathbb{Z}[\zeta_p]$ [McM26, Lemma 7.2]; this is not available for composite n , and his Section 7 assumes throughout that n is prime. Theorem H for prime n agrees with his result. The article [McM26] does not mention [Fuc21].

The dodecahedron. Athreya, Aulicino and Hooper [AAH22] classify the closed saddle connections on the dodecahedron. They show that a surface built from regular polygons, an odd number of which meet at each vertex, has no closed saddle connection whose midpoint is a Weierstrass point in their sense: the involution of the rotation group which rotates about this midpoint would reverse the saddle connection and fix its end [AAH22, Proposition 5.1, Theorem 5.2]; and they conclude that no closed saddle connection of the dodecahedron is equivalent to the short saddle connection of the double pentagon [AAH22, §10.2.1]. By Theorem A the saddle connections which are equivalent to the short one are those of type A_0 . So this is the statement that no short geodesic of type A_0 is closed, for which the source asks for a geometric proof [Fuc21, §3]. (The numbers of statements of [AAH22] are those of the arXiv version 2.) The proof of Theorem G applies the same involution to geodesics which need not be closed. For the cube, the tetrahedron, the octahedron and the icosahedron this was done by Troubetzkoy [Tro23]: there the unfolding of every geodesic between two vertices is centrally symmetric about its midpoint; after refolding, this symmetry becomes a rotation by π of the polyhedron which maps the geodesic to itself and exchanges its two ends, so that the ends are different. He notes that the proof does not carry over to the dodecahedron, because regular pentagons do not tile the plane. By Lemma 7.1 the unfolding of a short geodesic on the dodecahedron which is not an edge is centrally symmetric exactly if its type is A_0 , and for these geodesics the argument applies.

A public record on Conjecture 3.2. A record on Zenodo [Dan26] presents a short geodesic between two vertices at distance 2 as a counterexample to Conjecture 3.2. This geodesic exists, but with

Definition 2.1 of the source its type is A_1 ; see Remark 7.4.

Veech's paper on the billiard in a regular polygon. The source cites the papers of Veech [Vee89, Vee92] and of Ward [War98] and expects that its conjectures can be proved with them. For Conjecture 1.7 the paper [Vee92] contains the decisive formula. For $n > 4$ Veech realises the translation surface $Y(n)$ of the billiard in the regular n -gon as an unbranched cyclic covering of the double n -gon, of degree $N = n$ for odd n and $N = \frac{n}{2}$ for even n , which is given by a character χ of the first homology group of the double n -gon with values in $\mathbb{Z}/N$; and he computes χ on the closed geodesics of every cylinder in every periodic direction [Vee92, Lemmas 4.3, 4.10, 8.3, 8.8 and Proposition 8.11]. A cylinder on whose closed geodesics χ has the value x is covered by $d = \gcd(x, N)$ cylinders of $Y(n)$, each N/d times as long [Vee92, (7.7), (8.16)]. In the language of the source, N/d is the factor of the corresponding band. So parts (a) and (b) of Theorem H are due to Veech; the dictionary is given in Section 8.5. Veech uses these numbers to evaluate the constant in his asymptotic formula for the number of bands of periodic trajectories of length at most t [Vee92, Theorems 1.3 and 9.25]. He does not compare the factors of the cylinders of one direction with each other, and the question of Conjecture 1.7, which was posed in 2021, does not occur in his paper. For Conjecture 1.7 this note adds the deduction of Theorem H(c) from Veech's formula, the elementary proof of the counterexample (Proposition 8.2), and a second, self-contained derivation of the formula for odd n (Section 8.6). The prime case, for which we cited McMullen and, for $n = 5$, Davis, Fuchs and Tabachnikov and Davis and Lelièvre, is also an immediate consequence of Veech's formula, although it is not stated in [Vee92]. On the types of short trajectories, on reachable n -gons, on unitary pairs and on polyhedra, that is, on Conjectures 2.3 to 2.6 and 3.2, the paper [Vee92] contains nothing; for Conjecture 2.7 see Remark 3.2. The proofs of Sections 2 to 7 do not use [Vee89, Vee92, War98]. Veech's theorem that the affine group of the surface of the regular n -gon is a lattice [Vee89] is the background of the subject. In particular the cylinders of the double n -gon in the directions of its sides and diagonals, and the affine automorphisms which twist all cylinders of such a direction at once (our τ_k), go back to [Vee89]; they are recalled from there in [Vee92, pp. 348 and 351–353]. We have not read [Vee89], and we prove what is needed in Lemmas 2.8 and 2.10.

Method. All proofs are elementary. Section 2 develops the flat geometry of D_n : an angular coordinate B on the rays at the vertices, from which K is defined; the cylinders in the directions $\frac{k\pi}{n}$, all of which have the modulus $2 \cot \frac{\pi}{n}$; and a descent which shortens a saddle connection by twists until its direction is a multiple of $\frac{\pi}{n}$. Section 3 treats parallel trajectories. Sections 4 to 6 compare the height of a saddle connection with the heights of the cylinders next to a diagonal. Section 7 rests on the central symmetry of the developments of type A_0 . Section 8 first gives the counterexample by a symmetry argument in the n -gon, then recalls Veech's formula for the factors in all directions, and derives it a second time for odd n by letting G_0 act on the first cohomology of D_n with coefficients in $\mathbb{Z}/n$. Section 9 describes the computations: every theorem of this note is proved without them, and they test the theorems and the reading of the definitions on many exact instances.

2 The double n -gon

In this section $n \geq 3$, unless $n \geq 4$ is stated.

2.1 The surface

Lemma 2.1. *The mirror image of P_0 in the line of its side e_s is, as a set, the image of P_0 under the point reflection $z \mapsto X_s + X_{s+1} - z$. Consequently the polygons of a development are*

$$P^{(j)} = c_j + (-1)^j P_0, \quad c_0 = 0,$$

and if the segment passes from $P^{(j)}$ to $P^{(j+1)}$ through the side $c_j + (-1)^j e_s$, then $c_{j+1} = c_j + (-1)^j (X_s + X_{s+1})$.

Proof. The point reflection in the midpoint of e_s is the composition of the reflection in the line of e_s and the reflection in the perpendicular bisector of e_s , and the latter maps P_0 onto itself. The same applies to every side of a translate of P_0 or of $P'_0 = -P_0$. $\square$

Let

$$D_n := (P_0 \sqcup P'_0) / \sim, \quad e_s \ni w \sim w - X_s - X_{s+1} \in -e_s \quad (s \in \mathbb{Z}/n).$$

The gluing maps are translations, so D_n is a compact translation surface with conical points at the images of the vertices: directions, lengths, and the holonomy vector of a geodesic segment are defined. Since $X_s \sim -X_{s+1}$, all $2n$ vertices of P_0 and P'_0 give one point of D_n if n is odd, of angle $2n\omega = 2(n-2)\pi$, and two points if n is even, each of angle $(n-2)\pi$. We call them the *vertices* of D_n .

By Lemma 2.1, the translations $c_j + (-1)^j P_0 \rightarrow (-1)^j P_0$ map the straight segment of a development to a geodesic segment of D_n : where the segment crosses the side $c_j + (-1)^j e_s$ at the point $c_j + (-1)^j w$, $w \in e_s$, the next polygon is $c_{j+1} - (-1)^j P_0$, and the same point is $c_{j+1} - (-1)^j (X_s + X_{s+1} - w)$, in accordance with the gluing. Thus the short trajectories from O correspond bijectively to the saddle connections which start at the vertex X_0 of P_0 in a direction of $[0, \omega]$. We write γ_B for the saddle connection with holonomy B which corresponds to the reachable point B .

2.2 Corners, rays and the invariant K

A *corner* is a pair (ε, s) with $\varepsilon \in \{+, -\}$ and $s \in \mathbb{Z}/n$: the corner of the polygon εP_0 at its vertex εX_s . Its *start ray* points to εX_{s-1} and its *end ray* to εX_{s+1} ; the interior of the corner lies counterclockwise from the start ray. A *ray* at a vertex of D_n is a triple $(\varepsilon, s, \varphi)$ with $\varphi \in [0, \omega]$: the ray in the corner (ε, s) which makes the angle φ with the start ray. The end ray of (ε, s) runs along the side εe_s , which is glued to $-\varepsilon e_s$ with $\varepsilon X_s \sim -\varepsilon X_{s+1}$; hence

$$(\varepsilon, s, \omega) = (-\varepsilon, s+1, 0), \quad (4)$$

and the corners which follow each other counterclockwise around a vertex are $(\varepsilon, s), (-\varepsilon, s+1), (\varepsilon, s+2), \dots$. We define

$$B(\varepsilon, s, \varphi) := s\omega + \varphi \in \mathbb{R}/(n-2)\pi\mathbb{Z}.$$

Lemma 2.2. (a) B is a well defined function on the rays at the vertices of D_n . Around each vertex it is continuous and increases with unit speed in the counterclockwise sense.

(b) The direction of the ray $(\varepsilon, s, \varphi)$ is $B - s\pi$ if $\varepsilon = +$ and $B - (s-1)\pi$ if $\varepsilon = -$. In particular it is congruent to B modulo π .

Proof. (a) s is defined modulo n and $n\omega = (n-2)\pi$; and B takes the same value on the two sides of (4). (b) We have $X_{s-1} - X_s = -e^{i(\omega - 2\pi(s-1)/n)} = e^{-2\pi is/n}$, because $\omega + \pi + \frac{2\pi}{n} = 2\pi$. So the start ray of $(+, s)$ has the direction $-\frac{2\pi s}{n}$ and that of $(-, s)$ the direction $\pi - \frac{2\pi s}{n}$, and $s\omega + \frac{2\pi s}{n} = s\pi$. $\square$

For a saddle connection γ let γ^+ be its initial ray and γ^- the initial ray of the reversed saddle connection γ^{-1} . Their directions differ by π . By Lemma 2.2(b),

$$K(\gamma) := \frac{B(\gamma^-) - B(\gamma^+)}{\pi} \in \mathbb{Z}/(n-2)$$

is a well defined residue, and $K(\gamma^{-1}) = -K(\gamma)$.

Lemma 2.3 (types and K). *Let a short trajectory from O have N segments, the angles α , β and the reachable point B , and let the last polygon of its development be $c + \varepsilon P_0$, $\varepsilon = (-1)^{N-1}$, with $B = c + \varepsilon X_s$. Then*

$$\alpha + \beta \equiv -\frac{2\pi s}{n} \pmod{2\pi} \quad \text{if } N \text{ is odd,} \quad (5)$$

$$\beta - \alpha \equiv \frac{2\pi(s+1)}{n} \pmod{2\pi} \quad \text{if } N \text{ is even.} \quad (6)$$

In particular the rule of the source on the parity of N holds, and the type of the trajectory is A_k with $k \equiv K(\gamma_B)$ modulo $n-2$.

Proof. Let g be the composition of the $N-1$ reflections of the development, so that $P^{(N-1)} = g(P_0)$, and let X_t be the last vertex of the trajectory in the n -gon, so that $B = g(X_t)$. In the n -gon, the boundary edge which leaves X_t in the counterclockwise orientation is $X_t X_{t-1}$, and $\pi - \beta$ is the angle, measured counterclockwise, from this edge to the last segment of the trajectory, traversed backwards. The ray γ_B^- is the image under g of the latter, up to the translation by $-c$. Clearly $B(\gamma_B^+) = \alpha$, and the direction of γ_B^- is $\alpha + \pi$.

Let N be odd. Then g preserves the orientation and maps P_0 onto $c + P_0$, so $g(X_{t-1}) = c + X_{s-1}$. The image of the edge $X_t X_{t-1}$ is the start ray of the corner $(+, s)$, and $\gamma_B^- = (+, s, \pi - \beta)$. By Lemma 2.2(b) its direction is $-\frac{2\pi s}{n} + \pi - \beta$, which gives (5). As $\frac{2\pi}{n} \leq \alpha + \beta \leq \omega + \pi$, we get $\alpha + \beta = \frac{2\pi j}{n}$ with $1 \leq j \leq n-1$, and we may take $s = n - j$. Then

$$K(\gamma_B) = \frac{s\omega + \pi - \beta - \alpha}{\pi} = s + 1 - \frac{2(s+j)}{n} = s - 1 = n - 1 - j,$$

which is the index k of Definition 1.1.

Let N be even. Then g reverses the orientation and maps P_0 onto $c - P_0$, so $g(X_{t-1}) = c - X_{s+1}$. The image of the edge $X_t X_{t-1}$ is the end ray of the corner $(-, s)$, and since g reverses the sense of angles, $\pi - \beta$ is the angle, measured clockwise, from this end ray to γ_B^- . Hence $\gamma_B^- = (-, s, \omega - \pi + \beta) = (-, s, \beta - \frac{2\pi}{n})$, with the direction $\pi - \frac{2\pi s}{n} + \beta - \frac{2\pi}{n}$, which gives (6). As $\frac{2\pi}{n} - \omega \leq \beta - \alpha \leq \pi$, we get $\beta - \alpha = \frac{2\pi j}{n}$ with $-(n-4) \leq 2j \leq n$ and $s+1-j = nq$, $q \in \mathbb{Z}$. Then

$$K(\gamma_B) = \frac{s\omega + \beta - \frac{2\pi}{n} - \alpha}{\pi} = s - \frac{2(s+1-j)}{n} = s - 2q = j - 1 + (n-2)q,$$

which is congruent to the index $k \equiv j - 1$ of Definition 1.1. $\square$

Lemma 2.4 (diagonals). *For $0 \leq k \leq n-2$ let δ_k be the segment $X_0 X_{k+1}$ of P_0 , oriented from X_0 ; so δ_0 and δ_{n-2} are the two sides at O . Then $|\delta_k| = \lambda_k$ and $K(\delta_k) = k$.*

Proof. The length is (1). By (1), $B(\delta_k^+) = \frac{(n-2-k)\pi}{n}$. The ray δ_k^- lies in the corner $(+, k+1)$ and makes with its start ray, which points to X_k , the inscribed angle $\frac{k\pi}{n}$; so $B(\delta_k^-) = (k+1)\omega + \frac{k\pi}{n}$ and $K(\delta_k) = \frac{1}{n}((k+1)(n-2) + k - (n-2-k)) = k$. $\square$

2.3 Symmetries

A homeomorphism ψ of D_n is an *affine automorphism* if it maps vertices to vertices and is affine in the flat charts; its linear part $D\psi \in \text{GL}(2, \mathbb{R})$ is then constant. It acts on the rays at the vertices; we write ψ_* for this action. The matrix $D\psi$ acts on the directions modulo π ; let $F: \mathbb{R} \rightarrow \mathbb{R}$ be a lift of this action, so that $F(x + \pi) = F(x) \pm \pi$ according to the sign of $\det D\psi$. By Lemma 2.2, $B(\psi_* r) - F(B(r))$ is a continuous function of the ray r with values in $\pi\mathbb{Z}$ modulo $(n-2)\pi$, hence

constant around each vertex. We call ψ *B-uniform* if this constant is the same at all vertices. For odd n there is only one vertex, and every affine automorphism is *B-uniform*. Compositions and inverses of *B-uniform* automorphisms are *B-uniform*.

Lemma 2.5. *If ψ is B-uniform, then $K(\psi\gamma) = K(\gamma)$ for every saddle connection γ if ψ preserves the orientation, and $K(\psi\gamma) = -K(\gamma)$ if it reverses the orientation.*

Proof. Let $a \in \mathbb{R}$ represent $B(\gamma^+)$; then $a + K(\gamma)\pi$ represents $B(\gamma^-)$. With the constant C of ψ we get $B(\psi_*\gamma^-) - B(\psi_*\gamma^+) = F(a + K(\gamma)\pi) + C - F(a) - C = \pm K(\gamma)\pi$. $\square$

Lemma 2.6. *The following isometries of D_n are B-uniform affine automorphisms.*

- (a) *The central symmetry $\mu(z) = -z$, which exchanges P_0 and P'_0 . It acts on the rays by $(\varepsilon, s, \varphi) \mapsto (-\varepsilon, s, \varphi)$, and $B \circ \mu_* = B$.*
- (b) *The rotation ϱ . For even n it rotates P_0 and P'_0 by $\frac{2\pi}{n}$ about their centres, acts by $(\varepsilon, s, \varphi) \mapsto (\varepsilon, s - 1, \varphi)$, and $B \circ \varrho_* = B - \omega$. For odd $n = 2m + 1$ it rotates P_0 by $\frac{\pi}{n}$ about its centre onto P'_0 , and P'_0 onto P_0 ; it acts by $(\varepsilon, s, \varphi) \mapsto (-\varepsilon, s + m, \varphi)$, and $B \circ \varrho_* = B + m\omega$.*
- (c) *Every reflection σ which maps P_0 onto itself, extended to P'_0 by $z \mapsto -\sigma(-z)$. If $\sigma(X_s) = X_{c-s}$ for all s , it acts by $(\varepsilon, s, \varphi) \mapsto (\varepsilon, c - s, \omega - \varphi)$, and $B \circ \sigma_* = (c + 1)\omega - B$. For the reflection s_0 in the bisector of the angle at O , which is linear, $c = 0$ and $B \circ s_{0*} = \omega - B$.*

The group generated by ϱ and μ acts transitively on the $2n$ corners. For odd n , ϱ^2 acts by $(\varepsilon, s, \varphi) \mapsto (\varepsilon, s - 1, \varphi)$ and $\varrho^n = \mu$.

Proof. Each of these maps is an isometry of $P_0 \sqcup P'_0$ which permutes the two polygons; we check that it respects the gluing and compute its action on the corners.

(a) If $w \in e_s$, then $\mu(w) = -w \in -e_s$, and the partner $w - X_s - X_{s+1} \in -e_s$ of w is mapped to $w' := X_s + X_{s+1} - w \in e_s$, whose partner is $w' - X_s - X_{s+1} = -w$. The corner of εP_0 at εX_s is mapped to the corner of $-\varepsilon P_0$ at $-\varepsilon X_s$, start ray to start ray.

(b) Let c_0 be the centre of P_0 . Let n be even and $R = R_{2\pi/n}$ the rotation of the plane by $\frac{2\pi}{n}$; then $\varrho(z) = R(z - c_0) + c_0$ on P_0 and $\varrho(z) = R(z + c_0) - c_0$ on P'_0 , and $\varrho(\pm X_s) = \pm X_{s-1}$. For $w \in e_s$ the difference of $\varrho(w)$ and of the image of its partner is $R(X_s + X_{s+1} - 2c_0) + 2c_0 = X_{s-1} + X_s$, as the gluing of e_{s-1} requires. Let $n = 2m + 1$. Since $-R_{\pi/n} = R_{(m+1)2\pi/n}$, the map $z \mapsto R_{\pi/n}(z - c_0) - c_0$ sends X_s to $-X_{s-m-1} = -X_{s+m}$ and P_0 onto P'_0 ; likewise $z \mapsto R_{\pi/n}(z + c_0) + c_0$ sends $-X_s$ to X_{s+m} and P'_0 onto P_0 . For $w \in e_s$ we have $\varrho(w) = -X_{s+m} + R_{\pi/n}(w - X_s)$, and the partner $w - X_s - X_{s+1} = -X_{s+1} + (w - X_s)$ is mapped to $X_{s+m+1} + R_{\pi/n}(w - X_s)$; the difference $X_{s+m} + X_{s+m+1}$ is the one required for the side e_{s+m} . In both cases ϱ preserves the orientation, so it maps start rays to start rays, and the formulas for B follow from the definition; for odd n , $m\omega - \frac{\pi}{n} = (m - 1)\pi$.

(c) σ maps e_s onto e_{c-s-1} , exchanging the ends. For $w \in e_s$ the partner $w - X_s - X_{s+1}$ is mapped to $-\sigma(X_s + X_{s+1} - w)$. The map $z \mapsto \sigma(X_s + X_{s+1} - z)$ is the composition of σ with a point reflection, and so is $z \mapsto X_{c-s} + X_{c-s-1} - \sigma(z)$; both send X_s to X_{c-s-1} and X_{s+1} to X_{c-s} , so they coincide, and the image of the partner is $\sigma(w) - X_{c-s-1} - X_{c-s}$, the partner of $\sigma(w)$. As σ reverses the orientation, it maps the start ray of (ε, s) to the end ray of $(\varepsilon, c - s)$ and a ray at the angle φ from the former to the ray at the angle φ from the latter, measured clockwise.

In each case $B \circ \psi_* - F \circ B$ is given by one formula at all corners, so the maps are *B-uniform*. The last statements follow from the formulas for the action on the corners; for odd n , ϱ^n and μ are isometries with the same action on the corners, hence equal. $\square$

2.4 Cylinders in the base directions and twists

A *base direction* is a direction $\frac{k\pi}{n}$, $k \in \mathbb{Z}$.

Lemma 2.7. *The saddle connections of D_n in a base direction are exactly the sides and diagonals of P_0 and of P'_0 in that direction. More precisely, every ray at a vertex of D_n whose direction is a base direction runs along a side or a diagonal of P_0 or P'_0 .*

Proof. From a vertex of the regular n -gon the other $n - 1$ vertices are seen in the directions which make the angles $\frac{j\pi}{n}$, $0 \leq j \leq n - 2$, with the start ray of the corner. By Lemma 2.2(b) the direction of every start ray is a multiple of $\frac{\pi}{n}$. So a ray $(\varepsilon, s, \varphi)$ has a base direction if and only if $\varphi \in \frac{\pi}{n}\mathbb{Z}$, and then it points to a vertex of εP_0 . $\square$

The chords of P_0 in a base direction are the sides and diagonals $X_a X_b$ with a fixed residue $a + b$ modulo n . They cut P_0 into *strips*: for $a \in \mathbb{Z}/n$ and $2 \leq j \leq n - 1$ let Σ be the convex hull of $X_a, X_{a+1}, X_{a+j-1}, X_{a+j}$. It is bounded by the parallel chords $c_1 = X_a X_{a+j}$ and $c_2 = X_{a+1} X_{a+j-1}$ and by the sides e_a and e_{a+j-1} ; if $j = 2$, then c_2 is a point and Σ is a triangle, and if $j = n - 1$, then c_1 is a side.

Lemma 2.8 (cylinders). *Let Σ be a strip as above. The image Z of $\Sigma \cup (-\Sigma)$ in D_n is a flat cylinder: the strip between the lines of c_1 and c_2 , divided by the translation by $\ell \hat{u}$, where $\hat{u}$ is the unit vector in the direction from X_a to X_{a+j} and*

$$\ell = |c_1| + |c_2| = \lambda \lambda_{j-2} \quad \text{is its circumference,} \quad h = \sin \frac{(j-1)\pi}{n} = \lambda_{j-2} \sin \frac{\pi}{n} \quad \text{its height.} \quad (7)$$

Its boundary components are $c_1 \cup (-c_2)$ and $c_2 \cup (-c_1)$. In the strip between the two lines, the lifts of the vertices are the points $X_a + i\ell \hat{u}$ and $X_{a+j} + i\ell \hat{u}$ on the line of c_1 , and $X_{a+1} + i\ell \hat{u}$ and $X_{a+j-1} + i\ell \hat{u}$ on the line of c_2 , $i \in \mathbb{Z}$. In particular every cylinder of D_n in a base direction has the modulus $\ell/h = 2 \cot \frac{\pi}{n}$.

Proof. The polygon $-\Sigma + X_a + X_{a+1}$ is the image of Σ under the point reflection in the midpoint of the side e_a ; this reflection exchanges the lines of c_1 and c_2 . So Σ and this polygon, which are glued along e_a , form a parallelogram between the two lines. On the line of c_1 it has the side which consists of the image of c_2 , followed by c_1 , of length $|c_2| + |c_1|$; its two other sides are e_{a+j-1} and the image of e_{a+j-1} , which are parallel and are glued to each other by a translation along the lines. This gives the cylinder, its circumference $|c_1| + |c_2|$ and the positions of the vertices. The height is the distance of the two lines; as the side e_a makes the inscribed angle $\frac{(j-1)\pi}{n}$ with c_1 , it is $\sin \frac{(j-1)\pi}{n}$. Finally $|c_1| + |c_2| = \lambda_{j-1} + \lambda_{j-3} = \lambda \lambda_{j-2}$ by (1) and the relation $\lambda \lambda_k = \lambda_{k-1} + \lambda_{k+1}$. $\square$

If c_1 is a side of P_0 , then c_1 and $-c_1$ are the same saddle connection of D_n , which then belongs to both boundary components of Z ; this is harmless in what follows. The following two facts are used in Sections 4 to 6.

Lemma 2.9 (crossing a cylinder). *Let Z be a cylinder as in Lemma 2.8, with the direction $\hat{u}$ and the height h . The height of a geodesic segment with the holonomy vector z is $|\det(\hat{u}, z)|$.*

- (a) *Let δ be a saddle connection in the boundary of Z , oriented so that Z lies on its right. Every ray at the initial vertex of δ which arises from δ^+ by a clockwise rotation through an angle in $(0, \pi)$ enters the interior of Z . The same holds with “left” and “counterclockwise”.*

- (b) Let γ be a geodesic segment without vertices in its interior which ends at a vertex, and suppose that γ starts at a point of the boundary of Z in a direction which points into the interior of Z . Then the height of γ is at least h , and the initial part of γ of height h runs through Z from one boundary component to the other. If the height of γ is equal to h and γ starts at a vertex, then γ is the image of a segment in the strip of Lemma 2.8 which joins a lift of a vertex on one of the two lines to a lift of a vertex on the other line.
- (c) Conversely, every segment in the strip which joins lifts of vertices on the two lines is the lift of a saddle connection.

Proof. The closed strip of Lemma 2.8 is mapped onto Z by a map which is a local isometry on the open strip and, at every point of the two lines, on the half-disc which lies in the strip.

(a) Take the lift of δ on that one of the two lines for which the strip lies on its right. At the lift of the initial vertex the rays in question are the images of the rays which point into the open strip.

(b) The lift of γ which starts on one of the two lines is a straight segment. As long as it runs in the open strip it meets no lift of a vertex, and γ ends at a vertex; so it reaches the other line, and its part up to there has the height h . If the height of γ is h , then γ ends there, at a lift of a vertex; and if γ starts at a vertex, its lift starts at a lift of a vertex.

(c) The interior of such a segment lies in the open strip, which contains no lift of a vertex. $\square$

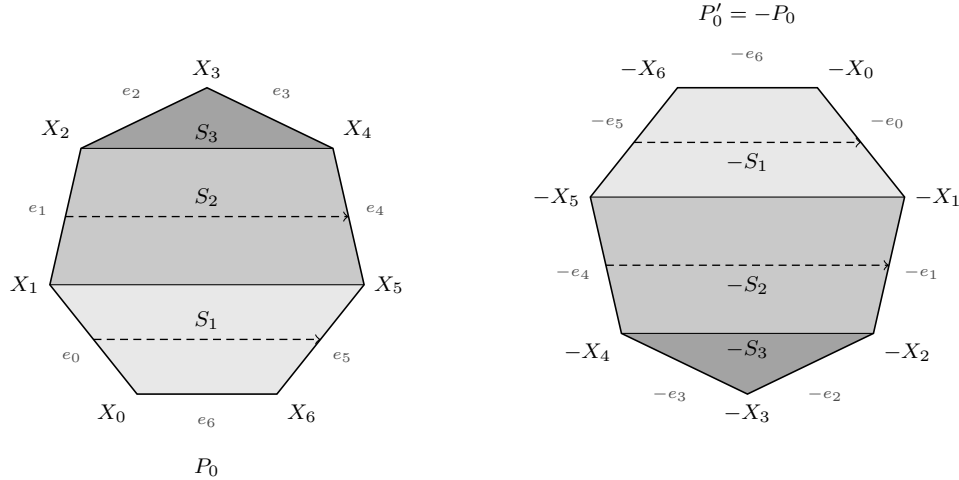


Figure 2: The double heptagon D_7 : the side e_s of P_0 is glued to the side $-e_s$ of P'_0 by a translation. The horizontal chords cut P_0 into the strips S_1, S_2, S_3 , and $C_j = S_j \cup (-S_j)$ are the three horizontal cylinders; the dashed lines are closed geodesics in C_1 and C_2 . All three cylinders have the modulus $2 \cot \frac{\pi}{7}$.

Lemma 2.10 (twists). *Let $k \in \mathbb{Z}$ and $T := \begin{pmatrix} 1 & 2 \cot(\pi/n) \\ 0 & 1 \end{pmatrix}$. There is an affine automorphism τ_k of D_n with the derivative $T_k := R_{k\pi/n} T R_{k\pi/n}^{-1}$ which fixes every saddle connection of the direction $\frac{k\pi}{n}$ pointwise. It preserves the orientation, it is B -uniform, and $K(\tau_k \gamma) = K(\gamma)$ for every saddle connection γ . In the strip of a cylinder Z of this direction, as in Lemma 2.8, a lift of τ_k fixes one of the two lines pointwise and translates the other one by $\pm \ell \hat{u}$.*

Proof. In the strip of a cylinder Z of the direction $\frac{k\pi}{n}$ use the coordinate x along $\hat{u}$ and the coordinate y across. The shear $(x, y) \mapsto (x + 2 \cot \frac{\pi}{n} (y - y_0), y)$ fixes the line $y = y_0$ and translates the line at

the distance h from it by $\pm 2 \cot \frac{\pi}{n} \cdot h = \pm \ell$, by Lemma 2.8. So it induces a map of the closed cylinder which is the identity on both boundary components. By Lemma 2.7 the chords of the direction $\frac{k\pi}{n}$ cut P_0 and P'_0 into strips, so the closed cylinders of this direction cover D_n , and these maps fit together to an affine automorphism τ_k with the derivative T_k . It fixes every vertex, and at every vertex it fixes the rays of the direction $\frac{k\pi}{n}$ modulo π . With the lift F of T_k which fixes $\frac{k\pi}{n}$, and hence $\frac{k\pi}{n} + \pi\mathbb{Z}$, the constant of τ_k is therefore 0 at every vertex, and Lemma 2.5 applies. $\square$

The horizontal cylinders. For later use we fix the notation in the direction 0. Let $L_j := X_j X_{n-1-j}$, so that L_0 is the bottom side of P_0 , let S_j be the strip between L_{j-1} and L_j , and $C_j := S_j \cup (-S_j)$. By Lemma 2.8, C_j has the height $\sin \frac{2j\pi}{n}$ and the circumference

$$\ell_j = \lambda \frac{\sin(2j\pi/n)}{\sin(\pi/n)}. \quad (8)$$

Here $1 \leq j \leq m$ if $n = 2m + 1$, where S_m is a triangle, and $1 \leq j \leq m - 1$ if $n = 2m$ (Figure 2).

2.5 Finiteness and descent

Lemma 2.11 (finiteness). *Let $n \geq 4$. For every $L > 0$ the surface D_n has only finitely many saddle connections of length at most L .*

Proof. Let a segment of a development cross three consecutive sides s_1, s_2, s_3 at the interior points q_1, q_2, q_3 ; here s_1, s_2 are sides of one polygon and s_2, s_3 sides of the next one. If s_1 and s_2 are not adjacent, then $|q_1 q_2| \geq d_0$, the least distance of two non-adjacent sides of P_0 ; the same holds for s_2 and s_3 . Otherwise s_1, s_2 have a common vertex V and s_2, s_3 a common vertex V' . If $V' = V$, then s_1 and s_3 are segments from V which lie on different sides of s_2 and enclose the angle $2\omega \geq \pi$; here $n \geq 4$ is used. But a line which does not pass through V meets only rays from V which lie in an open half-plane. So V' is the other end of s_2 , and s_1, s_3 are two disjoint segments of length 1 which start at the two ends of s_2 on different sides of it and make the angle ω with it. This configuration is unique up to isometry, and the distance d_1 of s_1 and s_3 is positive. Hence $|q_1 q_3| \geq \min(d_0, d_1)$ in all cases, a segment of length at most L crosses at most $2L/\min(d_0, d_1) + 2$ sides, and from each of the $2n$ corners only finitely many chains of polygons of this length exist. $\square$

Lemma 2.12 (descent). *Let $n \geq 4$. The elements of G_0 are B -uniform affine automorphisms which preserve the orientation. Every saddle connection of D_n is mapped by an element of G_0 onto one of the segments $\delta_0, \dots, \delta_{n-2}$ of Lemma 2.4.*

Proof. The first statement holds for the generators by Lemmas 2.6 and 2.10. Let γ be a saddle connection with the direction θ . If θ is a base direction, then γ is a side or a diagonal of P_0 or P'_0 by Lemma 2.7, and an element of the group generated by ϱ and μ moves its initial corner to $(+, 0)$; the image starts at X_0 in a base direction and is one of the δ_k . Otherwise $\frac{k\pi}{n} < \theta < \frac{(k+1)\pi}{n}$ for one $k \in \mathbb{Z}$. In the orthonormal frame whose first vector has the direction $\frac{k\pi}{n}$ the holonomy of γ is (x, y) with $y > 0$ and $x > y \cot \frac{\pi}{n}$, and the holonomy of $\tau_k^{-1}\gamma$ is $(x - 2y \cot \frac{\pi}{n}, y)$, where $|x - 2y \cot \frac{\pi}{n}| < x$. So $\tau_k^{-1}\gamma$ is strictly shorter than γ . We repeat this. By Lemma 2.11 the lengths cannot decrease infinitely often, so after finitely many steps a base direction is reached. $\square$

Lemma 2.13 (parallel saddle connections exist). *Let $n \geq 4$ and let θ be the direction of a saddle connection of D_n . Then every ray at a vertex of D_n whose direction is congruent to θ modulo π is the initial ray of a saddle connection.*

Proof. Let γ have the direction θ and let $\psi \in G_0$ map γ onto some δ_k (Lemma 2.12). Then ψ maps the given ray to a ray of a base direction, which by Lemma 2.7 runs along a side or a diagonal β . The saddle connection $\psi^{-1}(\beta)$ starts along the given ray. $\square$

2.6 Types are orbits

Proof of Theorem A. (a) is Lemma 2.3, with the function K of Section 2.2. (b) By Lemmas 2.5 and 2.12, K is invariant under G_0 , and every saddle connection γ is the image of some δ_k , $0 \leq k \leq n-2$, under an element of G_0 ; then $K(\gamma) = K(\delta_k) = k$ by Lemma 2.4. It remains to show that δ_{n-2} and δ_0 lie in one orbit. The symmetry μ maps δ_{n-2} , the side from X_0 to X_{n-1} , to the segment from $-X_0$ to $-X_{n-1}$ in P'_0 . By the gluing of e_{n-1} , which is the translation by -1 , this is the side e_{n-1} of P_0 , traversed from X_{n-1} to X_0 . The rotation of P_0 with $X_s \mapsto X_{s+1}$, which is ϱ^{-1} for even n and ϱ^{-2} for odd n (Lemma 2.6), maps it to the segment from X_0 to X_1 , which is δ_0 . Hence every saddle connection lies in the orbit of δ_k for exactly one $k \in \{0, \dots, n-3\}$, namely for $k \equiv K(\gamma)$. The last statement of the theorem follows from (a) and (b). $\square$

We shall use Theorem A in the following form: if w is a reachable point of type A_k , $0 \leq k \leq n-3$, then there is $\psi \in G_0$ with $\psi(\delta_k) = \gamma_w$, and then $D\psi \cdot X_{k+1} = w$.

3 Parallel short trajectories: Conjectures 2.6 and 2.7

In this section $n \geq 4$.

Lemma 3.1. *Let γ, γ' be saddle connections of D_n whose directions are congruent modulo π , and let $B(\gamma'^+) = B(\gamma^+) + j\pi$ with $j \in \mathbb{Z}$. Then $K(\gamma') = K(\gamma) - 2j$.*

Proof. First let the direction be a base direction $\frac{k\pi}{n}$. The n axes of symmetry of P_0 have the directions $\frac{\pi}{2} + \frac{i\pi}{n}$, $i \in \mathbb{Z}$; one of them is the bisector of the angle at O . Let σ be the reflection in the axis which is perpendicular to the direction $\frac{k\pi}{n}$, extended to D_n as in Lemma 2.6(c). A chord of P_0 which is perpendicular to the axis is mapped to itself, with its ends exchanged; the same holds in P'_0 . By Lemma 2.7 every saddle connection γ of this direction is such a chord, so $\gamma^- = \sigma_*\gamma^+$, and Lemma 2.6(c) gives $B(\gamma^-) = C - B(\gamma^+)$ with a constant C which depends only on σ . Hence $K(\gamma) = (C - 2B(\gamma^+))/\pi$ for all saddle connections of this direction, and the claim follows.

In general, let $\psi \in G_0$ map γ onto one of the δ_k (Lemma 2.12). Then $\psi\gamma$ and $\psi\gamma'$ have the same base direction modulo π , $K(\psi\gamma) = K(\gamma)$ and $K(\psi\gamma') = K(\gamma')$. If a represents $B(\gamma^+)$ and F is a lift for ψ , then $B(\psi_*\gamma'^+) - B(\psi_*\gamma^+) = F(a + j\pi) - F(a) = j\pi$, because ψ is B -uniform and preserves the orientation. So the first case applies. $\square$

Proof of Theorem B. Let γ be the saddle connection of the given short trajectory; so $\gamma^+ = (+, 0, \alpha)$ and $K(\gamma) = k$ by Lemma 2.3.

Case 1: $\varepsilon = -1$ and $\ell = n - 2$, that is, $\alpha' = \omega - \alpha$. The reflection s_0 acts on the corner $(+, 0)$ by $(+, 0, \varphi) \mapsto (+, 0, \omega - \varphi)$. So $s_0\gamma$ is a saddle connection from X_0 with the slope angle α' , and $K(s_0\gamma) = -k \equiv \varepsilon k - \ell$ by Lemmas 2.5 and 2.6.

Case 2: $\varepsilon = +1$. Let $\chi := \varrho^{-\ell/2}$ if n is even and $\chi := \varrho^{-\ell}$ if n is odd; this is the rotation of D_n by $-\frac{\ell\pi}{n}$. The ray $r' := (+, 0, \alpha')$ has the direction $\alpha' = \alpha + \frac{\ell\pi}{n}$, so χ_*r' has the direction α of γ . By Lemma 2.13, χ_*r' is the initial ray of a saddle connection β . Then $\delta := \chi^{-1}\beta$ is a saddle connection with $\delta^+ = r'$: the short trajectory with the slope angle α' exists, and its type is $K(\delta) = K(\beta)$. By Lemma 2.6(b),

$$B(\beta^+) = \alpha' + \frac{\ell}{2}\omega = \alpha + \frac{\ell}{2}\pi \quad (n \text{ even}), \quad B(\beta^+) = \alpha' - \ell m\omega = \alpha - \ell(m-1)\pi \quad (n = 2m+1),$$

where we used $\frac{\ell\pi}{n} + \frac{\ell}{2}\omega = \frac{\ell}{2}\pi$ and $1 - m(n-2) = -(m-1)n$. Now Lemma 3.1, applied to γ and β , gives $K(\beta) = k - \ell$ for even n , and $K(\beta) = k + 2\ell(m-1) = k + \ell(n-3) \equiv k - \ell$ modulo $n-2$ for odd n .

Case 3: $\varepsilon = -1$ and ℓ arbitrary. Write $\alpha' = \frac{(\ell-(n-2))\pi}{n} + (\omega - \alpha)$. By Case 1 the short trajectory with the slope angle $\omega - \alpha$ exists and has type A_{-k} . The number $\ell - (n - 2)$ is even if n and ℓ are even. By Case 2 the short trajectory with the slope angle α' exists and has type $A_{-k-(\ell-(n-2))} = A_{-k-\ell}$. $\square$

Proof of Theorem C. Let θ be a base direction. By Lemma 2.7 a saddle connection γ of the direction $\pm\theta$ is a side or a diagonal of P_0 or P'_0 , and an element of the group generated by ϱ and μ , which is an isometry in G_0 , maps it onto some δ_k , $0 \leq k \leq n - 2$. So $|\gamma| = \lambda_k$ and $K(\gamma) = k$ by Lemmas 2.4 and 2.5; as $\lambda_{n-2} = \lambda_0$, this is $\lambda_{\hat{k}}$, and $c_\theta = 1$. For an arbitrary direction θ of a saddle connection there is $\psi \in G_0$ which maps a saddle connection of this direction onto some δ_k (Lemma 2.12). The linear map $D\psi$ maps the direction $\pm\theta$ to a base direction and multiplies the lengths of all vectors of this direction by one factor, and ψ preserves K . This gives the first statement.

Let now two parallel short trajectories from O have the slope angles α and $\alpha' = \frac{\ell\pi}{n} + \varepsilon\alpha$, where ℓ is even if n is even, the types A_k and $A_{k'}$ with $0 \leq k, k' \leq n - 3$, and the saddle connections γ and γ' . If $\varepsilon = +1$, the rotation χ of the proof of Theorem B is an isometry in G_0 , and $\chi\gamma'$ has the direction α ; hence $|\gamma'| = |\chi\gamma'| = c_\alpha\lambda_{k'}$ and $|\gamma| = c_\alpha\lambda_k$. If $\varepsilon = -1$, then $s_0\gamma'$ has the same length as γ' , the slope angle $\omega - \alpha' = \frac{(n-2-\ell)\pi}{n} + \alpha$ and $K(s_0\gamma') = -k'$; the number $n - 2 - \ell$ is even if n is even. By the previous case its length is $c_\alpha\lambda_{k''}$, where $k'' = 0$ if $k' = 0$ and $k'' = n - 2 - k'$ otherwise, and $\lambda_{n-2-k'} = \lambda_{k'}$ because $\sin \frac{(n-1-k')\pi}{n} = \sin \frac{(k'+1)\pi}{n}$. $\square$

Remark 3.2. Let θ be the direction of a saddle connection. By Lemmas 2.13 and 3.1 the saddle connections γ_j which start along the rays with $B = \theta + j\pi$ satisfy $K(\gamma_j) = K(\gamma_0) - 2j$. For odd n every type occurs in every such direction. For even n only the types A_k with k of one parity occur; this is Proposition 1.4 of [Fuc21], and it divides the directions of saddle connections into two classes. Together with Theorem C this gives the numbers and the ratios of the lengths of parallel short trajectories which are listed in [Fuc21, §1.5, §1.6].

A related earlier statement concerns closed geodesics instead of saddle connections: in every periodic direction the circumferences of the cylinders of the double n -gon are proportional to the numbers $\sin \frac{(2l-1)\pi}{n}$, or to the numbers $\sin \frac{2l\pi}{n}$, by [Vee92, (7.6), (7.15)], where the second formula is quoted from [Vee89]; compare Lemma 2.8. The statement of Theorem C for short trajectories, with their types, is not in [Vee92].

4 Reachable n -gons: Conjecture 2.3

In Sections 4 to 6 we assume $n \geq 5$. Recall that γ_w denotes the saddle connection of a reachable point w ; its initial ray is $(+, 0, \arg w)$.

Lemma 4.1. (a) *Let w, w' be reachable points with $\det(w, w') > 0$. Then $\gamma_{w'}^+$ arises from γ_w^+ by a counterclockwise rotation through an angle in $(0, \pi)$.*

(b) *Let ψ be an affine automorphism of D_n which preserves the orientation, and let the ray r' arise from the ray r by a counterclockwise rotation through an angle in $(0, \pi)$. Then ψ_*r' arises from ψ_*r by a counterclockwise rotation through an angle in $(0, \pi)$, and ψ_* maps the rays between r and r' onto the rays between ψ_*r and ψ_*r' .*

Proof. (a) Both rays lie in the corner $(+, 0)$, and the angle is $\arg w' - \arg w$. (b) ψ_* preserves the counterclockwise order of the rays at a vertex. The directions of the rays between r and r' fill a sector which is smaller than a half-plane, and $D\psi$ maps it onto a sector which is smaller than a half-plane. $\square$

Proof of Theorem D. Let $u := g\xi = gX_{n-1}$ and $v := g\eta = gX_1$. Then $\det(u, v) = S > 0$, and since u, v lie in the closed sector, $0 \leq \arg u < \arg v \leq \omega$.

Step 1: $K(\gamma_v) \in \{0, n-3\}$. Let $k \in \{0, \dots, n-3\}$ represent $K(\gamma_v)$. By Theorem A there is $\psi \in G_0$ with $\psi(\delta_k) = \gamma_v$; let $A := D\psi$, so that $AX_{k+1} = v$. Put $\gamma' := \psi^{-1}(\gamma_u)$ and $u' := A^{-1}u$, its holonomy vector; then $\det(u', X_{k+1}) = S$. By Lemma 4.1, γ' starts at X_0 along a ray which arises from δ_k^+ by a clockwise rotation through an angle in $(0, \pi)$. Let Z be the cylinder of the strip between the chords $X_{k+1}X_0$ and $X_{k+2}X_{n-1}$; in the notation of Lemma 2.8 this is the strip with $a = k+1$ and $j = n-k-1$. It lies on the right of δ_k , and its height is $\sin \frac{(n-k-2)\pi}{n} = \sin \frac{(k+2)\pi}{n}$. By Lemma 2.9(a), γ' enters the interior of Z , and by Lemma 2.9(b) its height S/λ_k is at least the height of Z :

$$\sin \frac{\pi}{n} \sin \frac{2\pi}{n} \geq \sin \frac{(k+1)\pi}{n} \sin \frac{(k+2)\pi}{n}, \quad \text{that is,} \quad \cos \frac{(2k+3)\pi}{n} \geq \cos \frac{3\pi}{n}.$$

As $\frac{3\pi}{n} \leq \frac{(2k+3)\pi}{n} \leq 2\pi - \frac{3\pi}{n}$, this holds only for $k = 0$ and for $k = n-3$, and then with equality. So, by Lemma 2.9(b), γ' is the image of a segment in the strip of Z which joins X_0 to a lift of a vertex on the other line.

Step 2: $k \neq n-3$. Suppose that $k = n-3$; then $\delta_k = X_0X_{n-2}$, the strip is the triangle $X_{n-2}X_{n-1}X_0$, the cylinder Z has the height $\sin \frac{\pi}{n}$ and the circumference $\lambda = |X_{n-2}|$, and the lifts of the vertices on the other line are the points $\xi + iX_{n-2}$, $i \in \mathbb{Z}$. So $u' = \xi + iX_{n-2}$. By (2) the point $v_{n-2} = gX_{n-2} = v + (\lambda^2 - 1)u$, which is reachable by hypothesis, satisfies $\det(u, v_{n-2}) > 0$ and $\det(v_{n-2}, v) > 0$. Hence $\gamma'' := \psi^{-1}(\gamma_{v_{n-2}})$ also starts at X_0 along a ray which arises from δ_k^+ by a clockwise rotation through an angle in $(0, \pi)$, and it enters Z . Its holonomy vector is $X_{n-2} + (\lambda^2 - 1)u'$, and its height $(\lambda^2 - 1) \sin \frac{\pi}{n}$ is larger than $\sin \frac{\pi}{n}$, because $\lambda^2 > 2$ for $n \geq 5$. So γ'' leaves Z through a point of the other boundary component which is not a vertex. This component consists of a vertex and of the saddle connection $-(X_{n-2}X_0)$ of P'_0 , and on the other side of this saddle connection lies $-\Sigma_2$, where Σ_2 is the strip of P_0 between X_0X_{n-2} and X_1X_{n-3} . Its cylinder has the height $\sin \frac{3\pi}{n}$ by Lemma 2.8. The remaining part of γ'' enters this cylinder and has the height

$$(\lambda^2 - 2) \sin \frac{\pi}{n} = 2 \cos \frac{2\pi}{n} \sin \frac{\pi}{n} = \sin \frac{3\pi}{n} - \sin \frac{\pi}{n},$$

which is positive and smaller than $\sin \frac{3\pi}{n}$ for $n \geq 5$. This contradicts Lemma 2.9(b). Hence $K(\gamma_v) = 0$.

Step 3: $K(\gamma_u) = 0$. Let $\tilde{g} := s_0 g s_0 \in \text{SL}(2, \mathbb{R})$. Since $s_0 X_j = X_{n-j}$, we have $\tilde{g} X_j = s_0(v_{n-j})$. The reflection s_0 is an automorphism of D_n which maps the corner $(+, 0)$ onto itself, so it maps reachable points to reachable points, and $\gamma_{s_0 u} = s_0(\gamma_u)$. Thus $\tilde{g} X_1, \tilde{g} X_2, \tilde{g} X_{n-2}, \tilde{g} X_{n-1}$ are reachable, and Steps 1 and 2 for $\tilde{g}$ give $K(\gamma_{s_0 u}) = 0$, because $\tilde{g} \eta = s_0 u$. By Lemma 2.5, $K(\gamma_{s_0 u}) = -K(\gamma_u)$.

Step 4: g is the derivative of an element of G_0 . Now $k = 0$, $\psi(\delta_0) = \gamma_v$ and $A\eta = v$. The cylinder Z of Step 1 belongs to the strip between the side X_1X_0 and the chord X_2X_{n-1} ; it has the height S and the circumference $\lambda\lambda_{n-3} = \lambda^2$. By Lemma 2.8 and (2) the lifts of the vertices on the line $\xi + \mathbb{R}\eta$ are the points

$$\xi + t\eta, \quad t \in \Lambda := \lambda^2\mathbb{Z} \cup (\lambda^2\mathbb{Z} - 1). \quad (9)$$

For $t \in \Lambda$ let σ_t be the saddle connection whose lift is the segment from X_0 to $\xi + t\eta$ (Lemma 2.9(c)). The twist τ_{n-2} of the direction ω of η has a lift which fixes the line $\mathbb{R}\eta$ and translates the line $\xi + \mathbb{R}\eta$ by $\pm\lambda^2\eta$ (Lemma 2.10). The segments $\delta_{n-2} = X_0X_{n-1}$ and $\delta_1 = X_0X_2$ lie in the strip and end at ξ and at $\xi + (\lambda^2 - 1)\eta$. Hence, for $i \in \mathbb{Z}$,

$$\sigma_{i\lambda^2} = \tau_{n-2}^{\pm i}(\delta_{n-2}), \quad K(\sigma_{i\lambda^2}) = 0; \quad \sigma_{i\lambda^2-1} = \tau_{n-2}^{\pm(i-1)}(\delta_1), \quad K(\sigma_{i\lambda^2-1}) = 1. \quad (10)$$

By Step 1, $\gamma' = \sigma_t$ for some $t \in \Lambda$, and $K(\gamma') = K(\gamma_u) = 0$ by Step 3. As $1 \not\equiv 0$ modulo $n-2$, we get $\gamma' = \tau_{n-2}^j(\delta_{n-2})$ for some $j \in \mathbb{Z}$. Let $\Psi := \psi \circ \tau_{n-2}^j \in G_0$. Then $\Psi(\delta_{n-2}) = \gamma_u$ and $\Psi(\delta_0) = \gamma_v$, because τ_{n-2} fixes δ_0 ; so $D\Psi$ maps ξ to u and η to v , that is, $D\Psi = g$.

Step 5: all vertices. By Lemma 4.1(b), Ψ_* maps the rays $(+, 0, \varphi)$, $0 \leq \varphi \leq \omega$, which are the rays between δ_{n-2}^+ and δ_0^+ , onto the rays between γ_u^+ and γ_v^+ , and these lie in the corner $(+, 0)$. Hence for $1 \leq j \leq n-1$ the saddle connection $\Psi(\delta_{j-1})$ starts in the corner $(+, 0)$ and has the holonomy vector gX_j : the point $v_j = gX_j$ is reachable, and its type is $K(\delta_{j-1}) = j-1$ by Lemmas 2.3 and 2.4. Moreover Ψ maps the corner of P_0 at O into itself.

The last statement. If $u = v_{n-1}$ and $v = v_1$ for a reachable n -gon, then u and v have type A_0 by what was proved. Conversely let (u, v) be a unitary pair and $g \in \text{SL}(2, \mathbb{R})$ the linear map with $g\xi = u$, $g\eta = v$. Then $K(\gamma_v) = 0$, and in Step 1 with $k = 0$ the height of γ' is S , the height of Z ; so $\gamma' = \sigma_t$ for some $t \in \Lambda$. Since $K(\gamma_u) = 0$, Steps 4 and 5 apply and show that gP_0 is a reachable n -gon. $\square$

Remark 4.2. The proof shows more: if u, v are reachable points with $\det(u, v) = S$, then the pair of their types is (A_0, A_0) , (A_1, A_0) or (A_0, A_{n-3}) . Indeed, by Step 1, $K(\gamma_v) \in \{0, n-3\}$. If $K(\gamma_v) = 0$, then $\gamma' = \sigma_t$ and $K(\gamma_u) \in \{0, 1\}$ by (10). If $K(\gamma_v) = n-3$, then the lift of γ' joins X_0 to $\xi + iX_{n-2}$, so γ' is the image of δ_{n-2} under a power of τ_1 and $K(\gamma_u) = 0$. The computations find exactly these three pairs of types (Section 9).

5 Lines through a unitary pair: Conjecture 2.5

Proof of Theorem E. (a) By Theorem D and its proof there is $\Psi \in G_0$ with $\Psi(\delta_{n-2}) = \gamma_u$, $\Psi(\delta_0) = \gamma_v$ and $D\Psi = g$, where $g\xi = u$ and $g\eta = v$. Let Z , Λ and σ_t be as in Step 4 of that proof.

Let $w = u + tv$ be a reachable point on the line. Then $\det(w, v) = S > 0$, so by Lemma 4.1 the saddle connection $\Psi^{-1}(\gamma_w)$ starts at X_0 along a ray which arises from δ_0^+ by a clockwise rotation through an angle in $(0, \pi)$. It enters Z (Lemma 2.9(a)), its holonomy vector is $g^{-1}w = \xi + t\eta$, and its height is S , the height of Z . By Lemma 2.9(b), $t \in \Lambda$ and $\Psi^{-1}(\gamma_w) = \sigma_t$. By (10) the type of w is A_0 if $t \in \lambda^2\mathbb{Z}$ and A_1 if $t \in \lambda^2\mathbb{Z} - 1$.

Conversely let $t \in \Lambda$. By what was just shown, $u + tv$ is reachable if and only if the saddle connection $\Psi(\sigma_t)$, whose holonomy vector is $u + tv$, starts in the corner $(+, 0)$. Its initial ray arises from $\gamma_v^+ = (+, 0, \arg v)$ by a clockwise rotation through an angle $\theta' \in (0, \pi)$ (Lemma 4.1), and it has the direction $\arg v - \theta' \in (\arg v - \pi, \arg v)$. It lies in the corner $(+, 0)$ if and only if $\theta' \leq \arg v$. As $\arg v \leq \omega < \pi$, this means that $u + tv$ lies in the open upper half-plane or on the positive real axis; in the latter case $\Psi(\sigma_t)$ starts along the ray $(+, 0, 0)$, so it is the side δ_{n-2} and $u + tv = \xi$.

(b) The reflection s_0 maps reachable points of type A_k to reachable points of type A_{-k} (Lemmas 2.5 and 2.6), and $\det(s_0v, s_0u) = \det(u, v)$. So (s_0v, s_0u) is a unitary pair, and by (a) the reachable points on the line $s_0v + \mathbb{R}s_0u$ are the points $s_0v + ts_0u$, $t \in \Lambda$, which lie in the open upper half-plane or are equal to ξ , with the types A_0 and A_1 . Applying s_0 , which exchanges ξ and η and maps the open upper half-plane onto $\{z : \det(z, \eta) > 0\}$, we obtain (b), with the types A_0 and $A_{-1} = A_{n-3}$.

(c) For $n = 5$, λ is the golden ratio and $\lambda^2 = \lambda + 1$, so that $\Lambda = (\lambda + 1)\mathbb{Z} \cup (\lambda + (\lambda + 1)\mathbb{Z})$. Let $n \geq 6$. Then $\frac{1+\sqrt{5}}{2} < \lambda < 2$, hence

$$0 < \lambda < \lambda + 1 < \lambda^2 < 2\lambda + 1, \quad \lambda \neq \lambda^2 - 1, \quad \lambda + 1 \neq \lambda^2 - 1.$$

The only element of Λ in the interval $(0, \lambda^2)$ is $\lambda^2 - 1$; so $\lambda \notin \Lambda$ and $\lambda + 1 \notin \Lambda$. As $\text{Im } v > 0$ and $\text{Im } u \geq 0$, the points $u + (\lambda + 1)v$ and $u + \lambda v$ lie in the open upper half-plane, and by (a) they are not reachable; these are the cases $m = 1$ of the printed item (a) and $m = 0$ of the printed item (b). By (a), $u + \lambda^2v$ is a reachable point of type A_0 , and λ^2 lies strictly between $\lambda + 1$ and $2\lambda + 1$; so it is none of the printed values $m(\lambda + 1)$ and $\lambda + m(\lambda + 1)$, and the printed item (c) fails. $\square$

Remark 5.1. For $m = -1$ the printed item (b) of Conjecture 2.5 gives the point $u - v$, which is the point $t = -1$ of the second family of Theorem E(a). So this instance of the printed statement is

correct whenever $u - v$ lies in the upper half-plane. In the computations for $n = 7, 9, 11$ the values $m = -1$ in item (b) and $m = 0$ in item (a) are the only ones for which a printed point is reachable (Section 9).

6 Reachable points and reachable n -gons: Conjecture 2.4

We write $f_\xi(z) := \det(z, \eta)/S$ and $f_\eta(z) := \det(\xi, z)/S$ for the coordinates of $z \in \mathbb{C}$ in the basis ξ, η ; both are non-negative on the closed sector, hence on all reachable points. By (1),

$$X_{k+1} = a\eta + b\xi, \quad a = \frac{\lambda_k \sin \frac{(k+2)\pi}{n}}{S}, \quad b = \frac{\lambda_k \sin \frac{k\pi}{n}}{S} \quad (0 \leq k \leq n-2). \quad (11)$$

Lemma 6.1. *Let $1 \leq k \leq n-3$, let w be a reachable point and $\Psi \in G_0$ with $\Psi(\delta_k) = \gamma_w$, and put $g := D\Psi$, $u := g\xi$, $v := g\eta$. If $g'P_0$ is a reachable n -gon with the vertex w , then there is $i \in \mathbb{Z}$ with*

$$g'\eta = v + ic_1w, \quad g'\xi = u - ic_2w, \quad c_1 := \frac{\lambda\lambda_{k-1}}{\lambda_k}, \quad c_2 := \frac{\lambda\lambda_{k+1}}{\lambda_k}.$$

Moreover $ac_1 = bc_2 = \kappa := \sin \frac{k\pi}{n} \sin \frac{(k+2)\pi}{n} / \sin^2 \frac{\pi}{n} > 1$.

Proof. By Theorem A, w has type A_k , and by Theorem D the only vertex of type A_k of a reachable n -gon is v_{k+1} . So $w = g'X_{k+1}$, and $u' := g'\xi$, $v' := g'\eta$ are reachable points of type A_0 with $\det(u', v') = S$ and $w = av' + bu'$. Also $w = gX_{k+1} = av + bu$. Let $\hat{X} := X_{k+1}/\lambda_k$.

The left side. Let Z^L be the cylinder of the strip between X_0X_{k+1} and X_1X_k ($a = 0$, $j = k+1$ in Lemma 2.8). It lies on the left of δ_k , has the height $\sin \frac{k\pi}{n}$ and the circumference $\ell_L = \lambda\lambda_{k-1}$, and the lifts of the vertices on its other line are $X_1 + i\ell_L\hat{X}$ and $X_k + i\ell_L\hat{X}$, $i \in \mathbb{Z}$. As $\det(w, v') = bS > 0$, Lemma 4.1 shows that $\Psi^{-1}(\gamma_{v'})$ starts at X_0 along a ray which arises from δ_k^+ by a counterclockwise rotation through an angle in $(0, \pi)$; so it enters Z^L . Its height is $\det(w, v')/\lambda_k = bS/\lambda_k = \sin \frac{k\pi}{n}$, the height of Z^L . By Lemma 2.9(b) its holonomy vector $g^{-1}v'$ is one of the points $X_1 + i\ell_L\hat{X}$, $X_k + i\ell_L\hat{X}$. The twist τ of the direction of X_{k+1} has a lift which fixes the line $\mathbb{R}\hat{X}$ and translates the other line by $\pm\ell_L\hat{X}$ (Lemma 2.10), and the segments $\delta_0 = X_0X_1$ and $\delta_{k-1} = X_0X_k$ lie in the strip. So the saddle connections with the lifts from X_0 to $X_1 + i\ell_L\hat{X}$ are the $\tau^{\pm i}(\delta_0)$ and have $K = 0$, and those with the lifts to $X_k + i\ell_L\hat{X}$ are the $\tau^{\pm i}(\delta_{k-1})$ and have $K = k-1$, which is not 0 modulo $n-2$ if $k \geq 2$ (for $k = 1$ the two families coincide). Since $K(\gamma_{v'}) = 0$, we get $g^{-1}v' = \eta + i\ell_L\hat{X} = \eta + ic_1X_{k+1}$ for some $i \in \mathbb{Z}$, that is, $v' = v + ic_1w$.

The right side. Let Z^R be the cylinder of the strip between $X_{k+1}X_0$ and $X_{k+2}X_{n-1}$, on the right of δ_k , as in Step 1 of the proof of Theorem D; its height is $\sin \frac{(k+2)\pi}{n}$, its circumference $\ell_R = \lambda\lambda_{k+1}$, and the lifts of the vertices on its other line are $X_{n-1} + i\ell_R\hat{X}$ and $X_{k+2} + i\ell_R\hat{X}$. As $\det(u', w) = aS > 0$, the saddle connection $\Psi^{-1}(\gamma_{u'})$ starts at X_0 into Z^R , with the height $aS/\lambda_k = \sin \frac{(k+2)\pi}{n}$. As before, its lift ends at $X_{n-1} + i'\ell_R\hat{X}$, where $K = K(\delta_{n-2}) = 0$, or at $X_{k+2} + i'\ell_R\hat{X}$, where $K = K(\delta_{k+1}) = k+1$, which is not 0 modulo $n-2$ if $k \leq n-4$ (for $k = n-3$ the two families coincide). Hence $u' = u - i'c_2w$ for some $i' \in \mathbb{Z}$.

Conclusion. From $av' + bu' = w = av + bu$ we get $aic_1 = bi'c_2$. By (11) and $S = \lambda \sin \frac{\pi}{n}$,

$$ac_1 = \frac{\lambda\lambda_{k-1} \sin \frac{(k+2)\pi}{n}}{S} = \kappa, \quad bc_2 = \frac{\lambda\lambda_{k+1} \sin \frac{k\pi}{n}}{S} = \kappa,$$

so $i' = i$. Finally $2 \sin \frac{k\pi}{n} \sin \frac{(k+2)\pi}{n} = \cos \frac{2\pi}{n} - \cos \frac{(2k+2)\pi}{n} \geq \cos \frac{2\pi}{n} - \cos \frac{4\pi}{n}$, because $\frac{4\pi}{n} \leq \frac{(2k+2)\pi}{n} \leq 2\pi - \frac{4\pi}{n}$; and with $c := \cos \frac{2\pi}{n} \in (0, 1)$ we have $c - \cos \frac{4\pi}{n} - (1 - c) = 2c(1 - c) > 0$. Hence $2 \sin \frac{k\pi}{n} \sin \frac{(k+2)\pi}{n} > 1 - c = 2 \sin^2 \frac{\pi}{n}$, and $\kappa > 1$. $\square$

Proof of Theorem F. (a) Let $w \neq \xi$ be a reachable point of type A_0 ; then $\arg w > 0$. By Theorem A there is $\psi \in G_0$ with $\psi(\delta_0) = \gamma_w$; let $A := D\psi$, so $A\eta = w$. For $i \in \mathbb{Z}$ the saddle connection $\psi(\sigma_{i\lambda^2})$, with σ_t as in (10), has $K = 0$ and the holonomy vector $u_i := A\xi + i\lambda^2 w$, and $\det(u_i, w) = S$. By Lemma 4.1 its initial ray arises from $\gamma_w^+ = (+, 0, \arg w)$ by a clockwise rotation through an angle $\theta_i \in (0, \pi)$. As $i \rightarrow +\infty$, the direction of u_i tends to that of w , so $\theta_i \rightarrow 0$. For all large i we have $\theta_i < \arg w$; then the ray lies in the corner $(+, 0)$, the point u_i is reachable of type A_0 , (u_i, w) is a unitary pair, and by Theorem D the points u_i and w are the vertices v_{n-1} and v_1 of a reachable n -gon. The points u_i are pairwise distinct. For $w = \xi$ we apply this to $s_0\xi = \eta$ and use that s_0 maps reachable n -gons to reachable n -gons.

(b) Let w be a reachable point of type A_k , $1 \leq k \leq n-3$, which is a vertex of a reachable n -gon gP_0 . By Theorem D, $w = gX_{k+1}$ and $g = D\Psi$ for some $\Psi \in G_0$ with $\Psi(\delta_k) = \gamma_w$; and $u := g\xi$, $v := g\eta$ are reachable. Let $g'P_0$ be a reachable n -gon with the vertex w , and let i be as in Lemma 6.1. Suppose $i \geq 1$. Then, with $w = av + bu$,

$$0 \leq f_\xi(g'\xi) = f_\xi(u) - ic_2(af_\xi(v) + bf_\xi(u)) = (1 - i\kappa)f_\xi(u) - ic_2af_\xi(v).$$

Both terms on the right are ≤ 0 , so $f_\xi(u) = f_\xi(v) = 0$: the points u and v are multiples of η , which contradicts $\det(u, v) = S$. Suppose $i \leq -1$. Then $0 \leq f_\eta(g'\eta) = (1 + i\kappa)f_\eta(v) + ic_1bf_\eta(u)$ gives $f_\eta(u) = f_\eta(v) = 0$, again a contradiction. Hence $i = 0$ and $g' = g$. The point X_2 has type A_1 by Lemma 2.4 and is a vertex of the reachable n -gon P_0 .

(c) Let $e := e^{i\pi/n}$, so that $X_{n-2} = \lambda e$ by (1). Let D be the diagonal of P'_0 from $-X_1$ to $-X_{n-1}$. It is the image of $\delta_{n-3} = X_0X_{n-2}$ under $\Psi_1 := \mu \circ \varrho'$, where $\varrho' \in G_0$ is the rotation of D_n with $X_s \mapsto X_{s+1}$; so $K(D) = n-3$. The holonomy vector of D is $\eta - \xi$, of length λ and direction $\pi - \frac{\pi}{n}$. Since $-X_1 \sim X_0$, the saddle connection D starts at the vertex X_0 , in the corner $(-, 1)$ and at the angle $\frac{\pi}{n}$ from its start ray, whose direction is $\pi - \frac{2\pi}{n}$. The corner $(-, 1)$ follows the corner $(+, 0)$ counterclockwise, so D^+ arises from the ray $r_0 := (+, 0, \frac{\pi}{n}) = \delta_{n-3}^+$ by the counterclockwise rotation through ω .

Let $\gamma := \tau_1(D)$. The twist τ_1 fixes r_0 and the ray r_π which arises from r_0 by the counterclockwise rotation through π (Lemma 2.10), and it preserves the counterclockwise order of the rays. As $\omega < \pi$, the ray γ^+ lies between r_0 and r_π ; the rays between r_0 and r_π are distinguished by their directions, which fill the interval $(\frac{\pi}{n}, \frac{\pi}{n} + \pi)$. The holonomy vector of γ is

$$T_1(\eta - \xi) = \eta - \xi + 2 \cot \frac{\pi}{n} \det(e, \eta - \xi) e = \eta - \xi + \lambda^3 e = \eta - \xi + \lambda^2 X_{n-2},$$

because $\det(e, \eta - \xi) = \lambda \sin \frac{2\pi}{n}$ and $2 \cot \frac{\pi}{n} \sin \frac{2\pi}{n} = \lambda^2$. With $\eta = -\zeta^{-1}$, $X_{n-2} = 1 + \zeta$ and $\lambda^2 = 2 + \zeta + \zeta^{-1}$ this vector is $2 + 3\zeta + \zeta^2 = w_n$, and by (2)

$$w_n = (\lambda^4 - \lambda^2 - 1)\xi + (\lambda^2 + 1)\eta. \quad (12)$$

Both coefficients are positive, and $\det(e, w_n) = \det(e, \eta - \xi) > 0$. So $\frac{\pi}{n} < \arg w_n < \omega$, and γ^+ is the ray $(+, 0, \arg w_n)$. Hence $\gamma = \gamma_{w_n}$: the point w_n is reachable, and its type is $K(\gamma) = K(D) = n-3$.

Let $\Psi_0 := \tau_1 \circ \Psi_1 \in G_0$; then $\Psi_0(\delta_{n-3}) = \gamma_{w_n}$, and $g_0 := D\Psi_0 = T_1 \circ (-R_{-2\pi/n})$ satisfies

$$g_0\xi = T_1(\eta) = \eta + (\lambda^2 - 1)X_{n-2}, \quad g_0\eta = T_1(e^{-4\pi i/n}) = e^{-4\pi i/n} - \lambda_4 X_{n-2},$$

since $2 \cot \frac{\pi}{n} \det(e, \eta) = \lambda\lambda_2$ and $2 \cot \frac{\pi}{n} \det(e, e^{-4\pi i/n}) = -\lambda\lambda_4$. Suppose that $g'P_0$ is a reachable n -gon with the vertex w_n . Lemma 6.1 with $k = n-3$, where $c_1 = \lambda^2 - 1$ and $c_2 = 1$, gives an $i \in \mathbb{Z}$ with $g'\xi = g_0\xi - iw_n$ and $g'\eta = g_0\eta + i(\lambda^2 - 1)w_n$. We have $f_\eta(\eta) = f_\eta(X_{n-2}) = 1$, $f_\eta(w_n) = \lambda^2 + 1$, $f_\eta(e^{-4\pi i/n}) = -2 \cos \frac{2\pi}{n} = -(\lambda^2 - 2)$ and $\lambda_4 = \lambda^4 - 3\lambda^2 + 1$. Therefore

$$\begin{aligned} 0 \leq f_\eta(g'\xi) &= \lambda^2 - i(\lambda^2 + 1), & \text{which forces } i \leq 0; \\ 0 \leq f_\eta(g'\eta) &= -(\lambda^4 - 2\lambda^2 - 1) + i(\lambda^4 - 1), & \text{which forces } i \geq 1, \end{aligned}$$

because $\lambda^4 - 2\lambda^2 - 1 > 0$ for $\lambda^2 > 1 + \sqrt{2}$, and $\lambda^2 \geq \frac{3+\sqrt{5}}{2}$ for $n \geq 5$. This is a contradiction.

The last statement of the theorem follows from (a), (b) and (c). $\square$

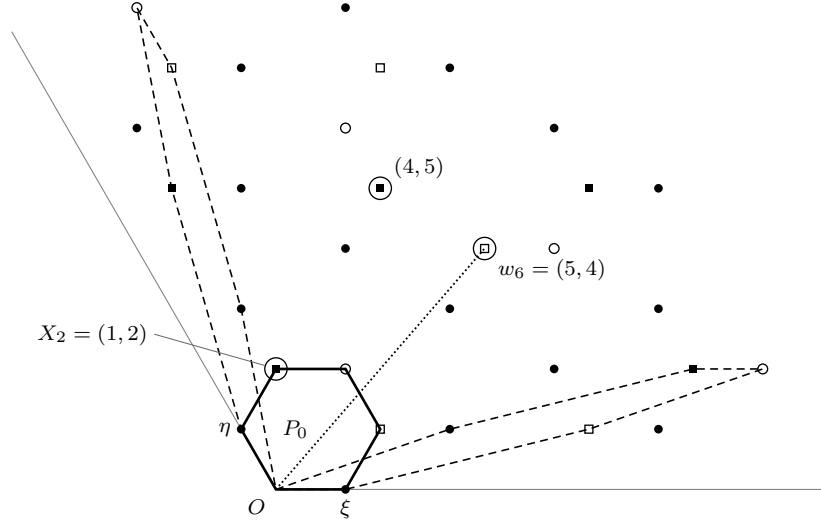


Figure 3: The regular hexagon: the 31 reachable points of length at most 7.3, with their types A_0 (black disc), A_1 (black square), A_2 (circle), A_3 (square); the initial hexagon P_0 and two further reachable hexagons (dashed). The point $X_2 = (1, 2)$ is a vertex of P_0 and of no other reachable hexagon; the points $(4, 5)$ and $w_6 = (5, 4)$ are vertices of none. Coordinates as in Example 6.2; the dotted segment is the development of the short trajectory of w_6 .

Example 6.2 ($n = 6$ in the coordinates of the source). For $n = 6$ the source uses the coordinates (p, q) of the point $p\xi + q\eta$ and gives the following description [Fuc21, §2.3]. The vertices of all hexagons are the points (p, q) with integers $p, q \geq 0$ and $p+q \not\equiv 2$ modulo 3; such a point is reachable if $\gcd(p, q) = 1$, or if $\gcd(p, q) = 2$ and $p+q \equiv 1$ modulo 3; its type is A_1 if $(p, q) \equiv (1, 2)$ modulo 3, A_2 if p and q are even, A_3 if $(p, q) \equiv (2, 1)$ modulo 3, and A_0 otherwise. A reachable hexagon with $v_1 = (p, q)$ and $v_5 = (p', q')$ has $p'q - pq' = 1$, the further vertices $v_2 = 2v_1 + v_5$, $v_3 = 2v_1 + 2v_5$, $v_4 = v_1 + 2v_5$, and v_1, v_5 are of type A_0 [Fuc21, §2.4]. With these formulas alone one sees that Conjecture 2.4 fails for $n = 6$ (Figure 3).

The point $X_2 = (1, 2)$ is reachable of type A_1 , so it is not v_1 or v_5 , and it is not v_3 , whose coordinates are even. If $(1, 2) = 2v_1 + v_5$ with non-negative coordinates, then $v_1 = (0, 1)$ and $v_5 = (1, 0)$, which is P_0 . If $(1, 2) = v_1 + 2v_5$, then $v_5 = (0, 1)$ and $v_1 = (1, 0)$, but then $p'q - pq' = -1$. So X_2 is a vertex of exactly one reachable hexagon.

The point $w = (4, 5)$ is reachable of type A_1 . If $w = 2v_1 + v_5$ with $v_1 = (a, b)$, then $a, b \leq 2$ and the condition $p'q - pq' = 1$ reads $4b - 5a = 1$, with the integer solutions $(a, b) \in (3, 4) + \mathbb{Z}(4, 5)$; none has $0 \leq a, b \leq 2$. If $w = v_1 + 2v_5$ with $v_5 = (a, b)$, the condition reads $5a - 4b = 1$, so $(a, b) \in (1, 1) + \mathbb{Z}(4, 5)$; the only solution with $0 \leq a, b \leq 2$ is $(1, 1)$, which is not a vertex of a hexagon. The cases $w = v_3$, $w = v_1$, $w = v_5$ are excluded as for X_2 . So $(4, 5)$ is a vertex of no reachable hexagon. Its mirror image in the bisector at O is $(5, 4) = 5\xi + 4\eta = w_6$, in accordance with (12).

Remark 6.3. For $5 \leq n \leq 12$ an exhaustive exact search (Section 9) finds that, among the reachable points of length at most 9 whose type is not A_0 , between 37 and 54 per cent are vertices of no reachable n -gon and the others of exactly one, and that the shortest reachable points which are vertices of none are w_n and its mirror image in the bisector at O ; the short trajectory of w_n has three segments.

7 The dodecahedron: Conjecture 3.2

7.1 The type of a short geodesic

Let $\mathcal{D}$ be the surface of the regular dodecahedron with edges of length 1, and let every face be oriented counterclockwise as seen from outside. Let c be a short geodesic which starts at a vertex v in a face f . Place f in the plane as the initial pentagon P_0 , with v at O and with its orientation, and unfold the faces which c crosses one after the other into the plane. This gives a chain of regular pentagons $P^{(0)} = P_0, \dots, P^{(N-1)}$, each the mirror image of the preceding one in a common side, and c becomes a straight segment OB which ends at a vertex B of $P^{(N-1)}$ and meets no other vertex. This is the development of a short trajectory in the regular pentagon, and the type of c is the type of this trajectory [Fuc21, §3]. By (3), the geodesic c has type A_0 if and only if

$$c \text{ is an edge, or } N \text{ is even and } \beta - \alpha = \frac{2\pi}{5}, \quad (13)$$

where α, β are the angles of the billiard trajectory in the oriented pentagon (Section 1.1). If the faces are oriented clockwise instead, the types A_1 and A_2 are exchanged and the type A_0 is not changed, by Lemmas 2.5 and 2.6(c). The next lemma states the second condition of (13) without angles: the last face of the unfolding is the image of the first one under the point reflection in the midpoint of the geodesic.

Lemma 7.1 (central symmetry). *Let $n \geq 4$, and let a short trajectory from O have $N \geq 2$ segments, the development $P^{(0)}, \dots, P^{(N-1)}$ and the reachable point B . The following statements are equivalent.*

- (i) *The type of the trajectory is A_0 .*
- (ii) *N is even and $P^{(N-1)} = B - P_0$, that is, $P^{(N-1)} = \iota(P_0)$ for the point reflection $\iota(z) := B - z$ in the midpoint of OB .*

If they hold, then $\iota(P^{(j)}) = P^{(N-1-j)}$ for $0 \leq j \leq N-1$, and the midpoint $\frac{B}{2}$ of the segment OB is the midpoint of the common side of the two middle polygons $P^{(N/2-1)}$ and $P^{(N/2)}$.

Proof. By (3), for $N \geq 2$ the type is A_0 if and only if N is even and $\beta - \alpha = \frac{2\pi}{n}$. Let N be even and, as in Lemma 2.3, $P^{(N-1)} = c - P_0$ and $B = c - X_s$. By (6), $\beta - \alpha \equiv \frac{2\pi(s+1)}{n}$ modulo 2π . As $-\frac{(n-4)\pi}{n} \leq \beta - \alpha \leq \pi$, the equation $\beta - \alpha = \frac{2\pi}{n}$ holds if and only if $s \equiv 0$ modulo n , that is, if and only if $B = c$ and $P^{(N-1)} = B - P_0$. This shows that (i) and (ii) are equivalent.

Assume (ii). The polygons $\iota(P^{(N-1)}) = P_0, \iota(P^{(N-2)}), \dots, \iota(P^{(0)})$ form a chain in which each polygon is the mirror image of the preceding one in a common side, and the segment $\iota(OB) = OB$ passes through them in this order on its way from $\iota(B) = O$ to $\iota(O) = B$. The development of a segment which starts at O in P_0 is determined by the segment. Hence $\iota(P^{(N-1-j)}) = P^{(j)}$ for all j . In particular ι exchanges the two middle polygons; so it maps their common side to itself, and its fixed point $\frac{B}{2}$ is the midpoint of this side. $\square$

7.2 Half-turns of the dodecahedron

We use the dodecahedron with the 20 vertices

$$(\pm 1, \pm 1, \pm 1), \quad (0, \pm \phi^{-1}, \pm \phi), \quad (\pm \phi^{-1}, \pm \phi, 0), \quad (\pm \phi, 0, \pm \phi^{-1}), \quad \phi = \frac{1 + \sqrt{5}}{2},$$

in which two vertices are joined by an edge if and only if their distance is $2\phi^{-1}$. We shall use $\phi^2 = \phi + 1$, $\phi^{-1} = \phi - 1$ and $\phi^2 + \phi^{-2} = 3$. For every edge, the rotation by π about the line through the centre and the midpoint of the edge maps the dodecahedron to itself. Opposite edges give the same rotation, so there are 15 such *half-turns*.

Lemma 7.2. *For every vertex v of the dodecahedron and every half-turn h , the distance of v and $h(v)$ is 1, 3, 4 or 5. More precisely, for a fixed vertex v this distance is 1 for three half-turns, 3 for six, 4 for three and 5 for three of them.*

Proof. The rotation group of the dodecahedron is transitive on the vertices, permutes the half-turns by conjugation and preserves the distances. So we may take $v = (1, 1, 1)$. Its neighbours form the set $N_1 = \{(0, \phi^{-1}, \phi), (\phi^{-1}, \phi, 0), (\phi, 0, \phi^{-1})\}$. The neighbours of $(0, \phi^{-1}, \phi)$ are v , $(-1, 1, 1)$ and $(0, -\phi^{-1}, \phi)$; with the cyclic permutations of the coordinates, which fix v , we find the set of the vertices at distance 2 from v :

$$N_2 = \{(-1, 1, 1), (1, -1, 1), (1, 1, -1), (0, -\phi^{-1}, \phi), (-\phi^{-1}, \phi, 0), (\phi, 0, -\phi^{-1})\}.$$

The 20 vertices are v , the points of N_1 , N_2 , $-N_2$, $-N_1$, and $-v$. The map $x \mapsto -x$ is a symmetry of the dodecahedron, so $-N_1$ and $-N_2$ are the sets of the vertices at distance 1 and 2 from $-v$. The vertex $(1, -1, -1)$ of $-N_2$ is adjacent to $(\phi, 0, -\phi^{-1}) \in N_2$, and $(-\phi, 0, \phi^{-1}) \in -N_2$ is adjacent to $(-1, 1, 1) \in N_2$; with the cyclic permutations it follows that $-N_2$ is the set of the vertices at distance 3 from v . Consequently $-N_1$ is the set of the vertices at distance 4, and $-v$ is at distance 5.

The midpoint of the edge from $(0, \phi^{-1}, \phi)$ to $(0, -\phi^{-1}, \phi)$ is $(0, 0, \phi)$, and the midpoint of the edge from v to $(0, \phi^{-1}, \phi)$ is $\frac{1}{2}(1, \phi, \phi^2)$, a multiple of $(\phi^{-1}, 1, \phi)$. The changes of the signs of the coordinates and the cyclic permutations of the coordinates are symmetries of the dodecahedron, and the images of these two edges under them are all 30 edges. Hence the axes of the half-turns are spanned by the three vectors $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ and by the twelve vectors $a = (s_1\phi^{-1}, s_2, s_3\phi)$ and their cyclic permutations, where the signs $s_i = \pm 1$ are taken up to a common factor -1 . The half-turn about the axis spanned by a is $h_a(x) = \frac{2(x \cdot a)}{a \cdot a} a - x$. For $a = (1, 0, 0)$ we get $h_a(v) = (1, -1, -1)$. For $a = (s_1\phi^{-1}, s_2, s_3\phi)$ we have $a \cdot a = 4$, and:

(s_1, s_2, s_3)	$v \cdot a$	$h_a(v) = \frac{v \cdot a}{2} a - v$	in the set
$(+, +, +)$	2ϕ	$(0, \phi^{-1}, \phi)$	N_1
$(-, +, +)$	2	$(-\phi, 0, \phi^{-1})$	$-N_2$
$(+, -, +)$	$2\phi^{-1}$	$(-\phi^{-1}, -\phi, 0)$	$-N_1$
$(+, +, -)$	0	$(-1, -1, -1)$	$\{-v\}$

The cyclic permutations of the vectors a give the cyclic permutations of these images. So the fifteen points $h(v)$ are the three points of N_1 , the six points of $-N_2$, the three points of $-N_1$, and three times the point $-v$. $\square$

The program `dodecahedron_halfturns.py` of the package (Section 9) repeats this computation exactly, in $\mathbb{Q}(\sqrt{5})$, for all 20 vertices, and prints the matrix of the 20×20 distances and the 15 half-turns as permutations of the vertices.

7.3 Proof of Theorem G

Proof of Theorem G. Let c be a short geodesic of type A_0 from v to w with N segments. If $N = 1$, then c is an edge by (13), and the half-turn about the midpoint of this edge exchanges v and w . Let $N \geq 2$. Then N is even, and by Lemma 7.1 the midpoint of the segment OB of the unfolding is the midpoint of the common side of the two middle polygons. So the midpoint $\hat{M}$ of c is the midpoint of an edge E of the dodecahedron, and E separates the two faces F, F' which c traverses before and after $\hat{M}$. Let h be the half-turn about the axis through $\hat{M}$. It is an isometry of the surface $\mathcal{D}$ which maps E to itself and exchanges F and F' ; in the unfolding of $F \cup F'$ into the plane it is the rotation by π about the image of $\hat{M}$. Let $c: [-\frac{L}{2}, \frac{L}{2}] \rightarrow \mathcal{D}$ be the parametrization of the geodesic by arc length

with $c(0) = \hat{M}$. Then $t \mapsto h(c(-t))$ is a geodesic which passes through $\hat{M}$ at $t = 0$ with the same direction as c . A geodesic on $\mathcal{D}$ is determined by a point and a direction as long as it meets no vertex, and c meets no vertex for $|t| < \frac{L}{2}$. Hence $h(c(-t)) = c(t)$ for $|t| < \frac{L}{2}$, and by continuity $h(v) = w$. Now Lemma 7.2 gives the distances. $\square$

Remark 7.3 (earlier work). The argument with the half-turn is that of Athreya, Auricino and Hooper [AAH22, Proposition 5.1]. They apply it to a closed saddle connection whose midpoint is a Weierstrass point in their sense, for which it gives $h(v) = v$, and this is impossible, because a half-turn fixes no vertex. The statement of Theorem G that a short geodesic of type A_0 is never closed is thus their result [AAH22, Theorem 5.2, §10.2.1]; see Section 1.5. Troubetzkoy [Tro23] applies the rotation by π about the axis through the midpoint to the geodesics between two vertices of the cube, the tetrahedron, the octahedron and the icosahedron, whose unfoldings are all centrally symmetric, and concludes, as in the proof above, that it exchanges the two ends. What Theorem G adds is Lemma 7.1, which identifies the short geodesics of type A_0 on the dodecahedron with those whose unfolding is centrally symmetric, so that this argument applies to them, and the evaluation of the distances in Lemma 7.2, that is, the statement on the distance 2.

7.4 The census of the source and a public record

The source supports Conjecture 3.2 by a census [Fuc21, §3]: for a fixed face f , an edge e of f and a vertex v of e , it lists the short geodesics of length less than 120 which start at v in f at an angle less than $\frac{3\pi}{10}$ with e , and counts them by the type and by the distance of the end vertex from v . Table 2 compares this table with the exact enumeration. The zeros of the column A_0 at the distances 0 and 2, which are the content of Theorem G, agree, and so do the proportions; but the totals are 3750 and 3702. The following can be said about this difference. First, two independent exact enumerations, by the program of this note and by a program of the second verification run, agree in all twelve entries, and variants of the rule (including the direction of the bisector, or excluding the edge e) change the total by at most one. Secondly, no bound for the length reproduces the table of the source: in the exact enumeration the entry in the column A_0 at the distance 5 exceeds 330 for every bound above 119.22, while for this bound the entry at the distance 1 is 655; the smallest sum of the absolute differences of the twelve entries, over all bounds up to 130, is 18, for a bound near 120.86 with 3752 geodesics. Thirdly, the numbers $N(L)$ of geodesics of length less than L in this enumeration are 927, 1644, 2573, 3129, 3702, 4013, 4356 for $L = 60, 80, 100, 110, 120, 125, 130$, so that $N(L)/L^2$ lies between 0.2568 and 0.2586, whereas $3750/120^2 = 0.2604$. We do not know how the difference arises. No statement of this note depends on the census.

distance of the end vertex	0	1	2	3	4	5	total
type A_0 , source [Fuc21]	0	672	0	778	342	330	2122
type A_0 , exact enumeration	0	662	0	766	336	332	2096
types A_1 and A_2 , source [Fuc21]	128	211	529	313	322	125	1628
types A_1 and A_2 , exact enumeration	126	206	523	304	323	124	1606

Table 2: Short geodesics of length less than 120 on the dodecahedron which start at a fixed vertex v in a fixed face at an angle in $[0, \frac{3\pi}{10})$ with a fixed edge, by the distance of the end vertex from v : the table of [Fuc21, §3], with the total 3750, and the exact enumeration of this note, with the total 3702.

Remark 7.4 (a public record on Conjecture 3.2). The record [Dan26], a software record on Zenodo which reproduces a release of a repository on GitHub, presents a short geodesic on the dodecahedron between two vertices at distance 2 as a counterexample to Conjecture 3.2. The facts are as follows; the programs are part of the package (Section 9).

(a) *The geodesic exists.* In the labelling of the record it runs from the vertex 0 to the vertex 2 through 16 faces, it crosses 15 edges, and its squared length is $\frac{1}{2}(307 + 137\sqrt{5})$. With the face cycles of the record, the exact program of this note and, independently, the program of the second verification run find exactly one short geodesic with this sequence of faces; it crosses the same edges, has $N = 16$ segments, the slope angle $\alpha \approx 0.028\pi$ and exactly this squared length, and its ends are at distance 2.

(b) *With Definition 2.1 its type is A_1 .* Here N is even, and in the notation of Lemma 2.3 the last polygon of the unfolding is $c - P_0$ and the end point is $c - X_1$. By (6), $\beta - \alpha = \frac{4\pi}{5}$, which is the condition for the type A_1 . In terms of Lemma 7.1: the last polygon is not $B - P_0$, and the unfolding is not centrally symmetric; the segment meets the fifteen edges at the fractions $t_1 < \dots < t_{15}$ of its length with $t_8 \approx 0.584$ in place of $\frac{1}{2}$, and $t_i + t_{16-i}$ ranges from 0.976 to 1.192 in place of 1. Short geodesics of the types A_1 and A_2 which end at distance 2 are frequent (Table 2).

(c) *The convention of the record.* As far as can be seen from the file of the record which contains the data of the geodesic, the equation $\beta - \alpha = \frac{2\pi}{5}$ is derived there from the directions of the first edge, from the vertex 0 to the vertex 16, and of the edge from the end vertex 2 to the vertex 10. The latter is the edge which leaves the end vertex in the counterclockwise orientation of the last face; in the unfolding the angle from this edge to the last segment, traversed backwards, is $\alpha + \frac{2\pi}{5}$. In Definition 2.1 the angle β belongs to the billiard trajectory in the oriented pentagon. After an odd number of crossings the last face of the unfolding is a mirror image of that pentagon; the image of the boundary edge which leaves the last vertex of the trajectory is the other edge at the end vertex, here the edge from 2 to 16, and the sense of the angle is reversed (proof of Lemma 2.3, Figure 1). In the notation of Lemma 2.3, for even N and the end point $c - X_s$, Definition 2.1 has $\beta - \alpha \equiv \frac{2\pi(s+1)}{n}$, and the relation used in the record amounts to $\frac{2\pi s}{n}$ in its place.

(d) *Which convention is that of the source.* The two conventions give different types to every short trajectory with an even number of segments. The source describes the types of all reachable points for $n = 4$ and $n = 6$ by explicit arithmetic conditions [Fuc21, §2.3]. Definition 2.1, as read in this note, reproduces these descriptions for all reachable points in the computed ranges (6875 points for $n = 4$ and 7605 points for $n = 6$, of length at most 120). With the other convention every reachable point with even N which was tested receives another type than in the source, or none (280 of 280 points for $n = 4$, 574 of 574 for $n = 6$). For the dodecahedron, the column A_0 of Table 2 would become (46, 77, 162, 108, 117, 20) with the other convention, or (0, 1, 0, 0, 0, 0) if e is the other edge of f at v .

Hence the geodesic of the record is not of type A_0 in the sense of the source. The record refutes the statement in which the type is defined with the other convention for β , and not Conjecture 3.2 as it is posed in [Fuc21]. As said in Section 1.4, the text of the corpus record does not specify the angles. Of the record, three text files were consulted, namely its description, its statement of the problem and the file with the data of the geodesic and the argument, and the metadata; its other files, among them a paper and the programs, were not examined.

8 Closed trajectories: Conjecture 1.7

8.1 The reading of the conjecture

A *phase point* is a pair $x = (p, d)$ of a point p of the boundary of P_0 which is not a vertex and a unit vector d which points into P_0 . The billiard map $\mathcal{T}$ sends x to the phase point of the next reflection; it is not defined if the ray from p in the direction d ends at a vertex. The symmetries of P_0 act on the phase points and commute with $\mathcal{T}$. Let r be the rotation of P_0 with $r(X_s) = X_{s+1}$; the rotations r^a , $a \in \mathbb{Z}/n$, act freely on the phase points. A trajectory is a sequence $x_i = \mathcal{T}^i x_0$, $i \in \mathbb{Z}$; it is closed if $x_j = x_0$ for some $j \geq 1$. By the definition recalled in Section 1.1, the piece from x_0 to x_j is preclosed if and only if $x_j = r^a x_0$ for some a ; a reflection of P_0 is excluded, because it reverses the ratio in

which the point divides the oriented side.

Let x_0 lie on a closed trajectory and let $j_0 \geq 1$ be the least number with $x_{j_0} = r^a x_0$ for some a . Since $\mathcal{T}$ commutes with r , we have $x_{ij_0} = r^{ia} x_0$, and if $x_j = r^b x_0$, then j_0 divides j . So the piece from x_0 to x_{j_0} is the simple preclosed piece, the closed trajectory consists of its images under the powers of r^a , and it closes after exactly d pieces, where

$$d = \text{the order of } r^a = \frac{n}{\gcd(n, a)} \quad (14)$$

is the factor.

To compare the trajectories of a band we unfold. Let L be the oriented line through p_0 in the direction d_0 , and let $g_0 = \text{id}$, $g_{i+1} = g_i \circ \nu_i$, where ν_i is the reflection in the line of the side of P_0 which contains the point of x_{i+1} . Then L passes through the polygons $Q_i = g_i(P_0)$, and $x_i = g_i^{-1}(q_i, d_0)$, where q_i is the point at which L enters Q_i . Hence

$$x_j = r^a x_0 \iff g_j \circ r^a \text{ maps the oriented line } L \text{ to itself, and } q_0 \text{ to } q_j. \quad (15)$$

The closed trajectories which are strongly parallel to the given one unfold to the lines parallel to L which pass through the same polygons Q_i ; so a band is an open strip of parallel lines, bounded by two lines through vertices of the polygons Q_i .

Lemma 8.1. *For every band there are an even number j_* and a residue a_* such that, for every trajectory of the band with at most one exception, the simple preclosed piece has j_* segments and ends at $x_{j_*} = r^{a_*} x_0$. These trajectories have simple preclosed pieces of the same length and the same factor $n/\gcd(n, a_*)$, which we call the factor of the band. The exceptional trajectory, if there is one, is the middle line of the band; its simple preclosed piece has the odd number $\frac{j_*}{2}$ of segments and ends at $r^{a_1} x_0$ with $2a_1 = a_*$. For odd n its factor is the factor of the band.*

Proof. Let $x_j = r^a x_0$ with j even. Then $g_j \circ r^a$ preserves the orientation of the plane, so by (15) it is a translation along L . It maps every line L' of the band to itself, and by (15) $x'_j = r^a x'_0$ for the trajectory of L' . Let $x_j = r^a x_0$ with j odd. Then $g_j \circ r^a$ is a glide reflection with the axis L , or the reflection in L . For another line L' of the band and a residue a' , the linear part of $g_j \circ r^{a'}$ is that of $g_j \circ r^a$, composed with a rotation; it fixes the direction of L only if $a' = a$, and $g_j \circ r^a$ maps L' to its mirror image in L . So for odd j at most the trajectory of the axis of $g_j \circ r^a$ is preclosed after j segments. This isometry maps the chain of the polygons Q_i to itself, hence the strip of the band to itself, and its axis is the middle line of the strip.

Let j_* be the least even number such that $g_{j_*} \circ r^{a_*}$ is a translation along L for some a_* . By the first part, j_* and a_* are the same for all lines of the band, and a trajectory whose simple preclosed piece does not have j_* segments has a simple preclosed piece with an odd number j_1 of segments, ending at $r^{a_1} x_0$; it is the middle line. For this trajectory $x_{2j_1} = r^{2a_1} x_0$ and j_1 divides j_* , which gives $j_* = 2j_1$ and $a_* = 2a_1$. The lengths are equal, because the simple preclosed piece unfolds to the segment of L' between q'_0 and its image under the translation. For odd n the residues a_1 and $2a_1$ have the same order in $\mathbb{Z}/n$. $\square$

With Lemma 8.1, Conjecture 1.7 reads: *in every class of parallel closed trajectories, let g be the greatest common divisor of the factors of its bands; then no band has the factor ng , that is, $d/g \neq n$ for all bands.* The statements of the source on preclosed and closed trajectories, in particular its Proposition 1.5(b) and its table, refer to the factors of the bands in this sense; see Remark 8.4 and Table 5.

8.2 The trajectories parallel to a side

Recall the horizontal chords $L_j = X_j X_{n-1-j}$, the strips S_j between L_{j-1} and L_j , and the numbers ℓ_j of (8).

Proposition 8.2. *Let $n = 2m + 1 \geq 5$ be odd and $1 \leq j \leq m$. Let p be a point of the side e_{j-1} which is not a vertex, and consider the billiard trajectory in P_0 which starts at p in the horizontal direction, into the strip S_j . It is closed. If p is not the midpoint of e_{j-1} , its simple preclosed piece consists of two segments and has the length $\ell_j = \lambda \sin \frac{2j\pi}{n} / \sin \frac{\pi}{n}$, and its factor is*

$$d_j = \frac{n}{\gcd(n, j)}.$$

If p is the midpoint, the simple preclosed piece is one segment, of length $\frac{1}{2}\ell_j$, and the factor is again d_j .

Proof. Let σ be the reflection of P_0 in its vertical axis of symmetry. Then $\sigma(X_s) = X_{n-1-s}$ and $\sigma(e_s) = e_{n-2-s}$; in particular $\sigma(e_{j-1}) = e_{n-1-j}$, and the horizontal chord from p ends at $\sigma(p)$. For a phase point $x = (q, d)$ with $q \in e_s$ let $t(x) \in (0, 1)$ be given by $q = X_s + t(x)(X_{s+1} - X_s)$, and let $\vartheta(x) \in (0, \pi)$ be the angle from the vector $X_{s+1} - X_s$ to d , measured clockwise. A rotation r^a maps x to the phase point on e_{s+a} with the same pair (t, ϑ) , and the reflection σ maps it to the phase point on e_{n-2-s} with the pair $(1-t, \pi - \vartheta)$. The law of reflection says: if a segment arrives at the point q' in the direction d' , then the next phase point is at q' and has the angle $\pi - \vartheta(q', -d')$.

Let $x_0 = (p, 1)$, with $t(x_0) = t$ and $\vartheta(x_0) = \vartheta$. The first segment arrives at $\sigma(p) \in e_{n-1-j}$ in the direction 1, and $(\sigma(p), -1) = \sigma(x_0)$ has the pair $(1-t, \pi - \vartheta)$. So x_1 is the phase point on e_{n-1-j} with the pair $(1-t, \vartheta)$. Let $R := r^{n-2j}$, the rotation with $R(e_{j-1}) = e_{n-1-j}$, and let $x_0^* := (p^*, 1)$, where p^* is the mirror image of p in the midpoint of e_{j-1} ; it has the pair $(1-t, \vartheta)$ on e_{j-1} . Thus

$$\mathcal{T}(x_0) = x_1 = R x_0^*.$$

The same holds with the roles of p and p^* exchanged: $\mathcal{T}(x_0^*) = R x_0$. Hence $x_2 = \mathcal{T}(R x_0^*) = R \mathcal{T}(x_0^*) = R^2 x_0$, and all $\mathcal{T}^i x_0$ are defined. If $x_1 = r^a x_0$ for some a , then r^a maps e_{j-1} to e_{n-1-j} , so $r^a = R$, and $R x_0 = R x_0^*$ gives $p = p^*$. Hence for $p \neq p^*$ the simple preclosed piece consists of the first two segments, and by (14) the factor is the order of $R^2 = r^{-4j}$, which is $n / \gcd(n, 4j) = n / \gcd(n, j)$. The second segment is the image under R of the chord from p^* . If p divides e_{j-1} in the ratio $t : (1-t)$, the chords from p and from p^* have the lengths $(1-t)|L_{j-1}| + t|L_j|$ and $t|L_{j-1}| + (1-t)|L_j|$, with the sum $|L_{j-1}| + |L_j| = \ell_j$ (Lemma 2.8). For $p = p^*$ we have $x_1 = R x_0$, and the factor is the order of $R = r^{-2j}$, which is also $n / \gcd(n, j)$. $\square$

Corollary 8.3. *Let n be odd with at least two distinct prime factors. Then Conjecture 1.7 is false for n : in the class of the closed trajectories parallel to a side, the greatest common divisor of the factors of the bands is 1, and the band in the strip S_1 has the factor n .*

Proof. The trajectories of Proposition 8.2 with p in the interior of e_{j-1} , p not the midpoint, belong to this class, and for each j they lie in one band, with the factor $d_j = n / \gcd(n, j)$. Let q be a prime factor of n and q^e the largest power of q which divides n . As $n \neq q^e$ and n is odd, $q^e \leq \frac{n}{3} \leq m$, and $d_{q^e} = n / q^e$ is not divisible by q . So the greatest common divisor of the factors of the class is divisible by no prime factor of n ; it is 1. And $d_1 = n$. $\square$

For $n = 15$ the factors are listed in Table 3; see also Figure 4. The same happens for $n = 21, 33, 35, 39, 45, \dots$

j	1	2	3	4	5	6	7
length ℓ_j of the simple preclosed piece	3.8271	6.9924	8.9487	9.3577	8.1487	5.5306	1.9563
rotation $R^2 = r^{-4j}$: residue $-4j$ modulo 15	11	7	3	14	10	6	2
factor $d_j = 15/\gcd(15, j)$	15	15	5	15	3	5	15
segments of the closed trajectory	30	30	10	30	6	10	30

Table 3: The seven bands of trajectories parallel to a side of the regular 15-gon with side 1 (Proposition 8.2). The greatest common divisor of the factors is 1, and four factors are equal to 15.

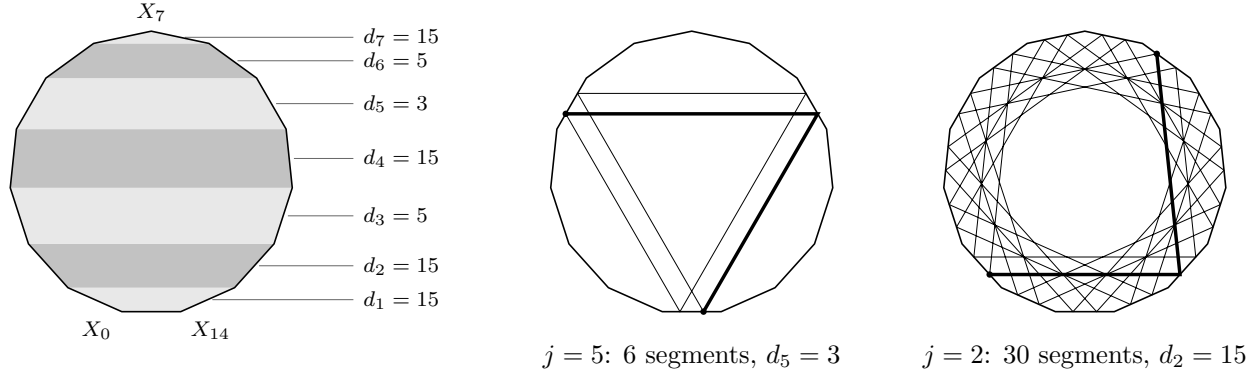


Figure 4: The regular 15-gon. Left: the strips $S_1, \dots, S_7$ parallel to the side X_0X_{14} , with the factors of their bands. Middle and right: the closed trajectories which start horizontally at the point which divides the side e_{j-1} in the ratio 3 : 7, for $j = 5$ and $j = 2$; the simple preclosed piece (two segments, between the two dots) is drawn thick.

8.3 Even n

Proof of Theorem H(a). Let n be even, and let a closed trajectory be not the middle trajectory of its band. By Lemma 8.1 its simple preclosed piece has an even number j_* of segments, and $g_{j_*} \circ r^{a_*}$ is a translation; so the linear part of g_{j_*} is the inverse of the linear part of r^{a_*} . Now g_{j_*} is a composition of an even number of reflections in lines which are parallel to sides of P_0 . For even n the lines of the sides have the directions $\frac{2\pi i}{n}$, $i \in \mathbb{Z}$, modulo π , and the composition of the reflections in two lines with the directions φ_1, φ_2 is a rotation by $2(\varphi_1 - \varphi_2)$ or a translation. Hence the linear part of g_{j_*} is a rotation by a multiple of $\frac{4\pi}{n}$, the residue a_* is even, and the factor $n/\gcd(n, a_*)$ divides $\frac{n}{2}$. In particular no factor is equal to n , before and after the division by a common divisor. $\square$

In Veech's description this is the fact that for even n the surface of the billiard covers the double n -gon with the degree $\frac{n}{2}$; see Section 8.5.

Remark 8.4 (middle trajectories). (a) For even n the regular n -gon through the midpoints of the sides of P_0 is a closed trajectory with $x_1 = r^{\pm 1}x_0$; its simple preclosed piece is one segment and its factor is n . It is the middle trajectory of the band of the trajectories near a corner which are parallel to a shortest diagonal; the other trajectories of this band have simple preclosed pieces of two segments and the factor $\frac{n}{2}$. If the middle trajectory were counted with its own factor, Conjecture 1.7 would fail for $n = 6$ and $n = 10$: there the class of this band contains a band with the factor 1 (Table 5), so that the greatest common divisor is 1 and the factor n remains. This would contradict the table of the source, which lists for these classes the factors 3, 1 and 5, 5, 1. So Conjecture 1.7 concerns the factors of the bands. For $n = 8$ and $n = 12$ the corresponding classes have the factors 4, 4 and 6, 2, 6, and the conjecture would hold in both readings.

(b) For odd n the middle trajectory has the factor of its band (Lemma 8.1); Proposition 8.2, Corollary 8.3 and parts (b) and (c) of Theorem H are not affected by the choice of the reading. A closed trajectory which passes through the midpoint M of a side is the middle trajectory of its band, because the point reflection in M maps the unfolded chain of polygons to itself (Lemma 2.1) and the unfolded trajectory to itself. Let n be an odd prime. Then the factor of such a trajectory is n if and only if the trajectory is mapped to itself by all rotations of P_0 : if the factor is n , the trajectory consists of the n rotated copies of its simple preclosed piece; and if a rotation r^b , $b \neq 0$, maps the trajectory to itself, then r^{2b} maps it to itself with its orientation, so $x_j = r^{2b}x_0$ for some j and the factor is not 1. This is the translation between Proposition 1.8 of [Fuc21] and the theorems of McMullen which was used in Section 1.5.

8.4 The rotation class of the double n -gon

We call the side e_s of P_0 and the side $-e_s$ of P'_0 , which are glued in D_n , the sides with the label s . Let x be a closed curve in D_n which avoids the vertices, is transversal to the sides and starts in P_0 , and let $s_1, s_2, \dots, s_{2t}$ be the labels of the sides which it crosses, in this order. We put

$$\text{rot}(x) := 2 \sum_{i=1}^t (s_{2i} - s_{2i-1}) \in \mathbb{Z}/n. \quad (16)$$

Lemma 8.5. (a) *Let a line pass through the polygons $Q_i = g_i(P_0)$ of an unfolding, as in Section 8.1, and let $s_1, \dots, s_{2t}$ be the labels of the sides of D_n which the corresponding geodesic crosses on its way from Q_0 to Q_{2t} . Then $g_{2t}(X_a) = c_{2t} + X_{a+\rho}$ for all a , with $\rho = 2 \sum_{i=1}^t (s_{2i} - s_{2i-1})$ and $Q_{2t} = c_{2t} + P_0$. That is, g_{2t} is r^ρ followed by the translation by c_{2t} .*

(b) *rot depends only on the class of x in $H := H_1(D_n; \mathbb{Z})$, and it is a homomorphism $H \rightarrow \mathbb{Z}/n$.*

Proof. (a) By Lemma 2.1, $Q_i = c_i + (-1)^i P_0$. Let $\pi_i(a)$ be defined by $g_i(X_a) = c_i + (-1)^i X_{\pi_i(a)}$. If the line passes from Q_i to Q_{i+1} through the side $c_i + (-1)^i e_s$, which has the label s , then $g_{i+1} = \nu \circ g_i$ for the reflection ν in the line of this side. As in the proof of Lemma 2.1, ν is the composition of the reflection in the perpendicular bisector of the side, which maps $c_i + (-1)^i X_b$ to $c_i + (-1)^i X_{2s+1-b}$, and of the point reflection in its midpoint, which maps $c_i + (-1)^i X_b$ to $c_{i+1} - (-1)^i X_b$. Hence $\pi_{i+1}(a) = 2s + 1 - \pi_i(a)$, and after two steps $\pi_{i+2}(a) = \pi_i(a) + 2(s' - s)$ if the labels are s and then s' .

(b) The same computation applies to the unfolding along any path which avoids the vertices and is transversal to the sides, and by (a) the residue $\text{rot}(x)$ is the shift of the labels after the unfolding along the closed curve x . It does not change under a homotopy which avoids the vertices: if two consecutive crossings of the same side are inserted or removed, the sum in (16) does not change. For a small loop around a vertex the labels of the crossed sides increase by one from each crossing to the next, by (4), and there are $2n$ crossings if n is odd and n crossings if n is even; so rot of this loop is $2n$ or n , which is 0 in $\mathbb{Z}/n$. Therefore rot is defined on the fundamental group of D_n with a base point in P_0 . It is additive for the composition of loops, because the shifts of the labels add up, and its values lie in an abelian group; so it factors through H . $\square$

Lemma 8.6 (bands and cylinders). *Let Z be a cylinder of D_n of the direction θ , and let x be a closed geodesic in the interior of Z which runs once around Z in the direction θ . For a point p of x in the interior of P_0 , the billiard trajectory in P_0 through p in the direction θ is closed, and the number j_* and the residue a_* of its band (Lemma 8.1) are the number of the sides which x crosses and $a_* = -\text{rot}(x)$. So the simple preclosed pieces of the band have the length of x , which is the circumference of Z , and the factor of the band is the order of $\text{rot}(x)$ in $\mathbb{Z}/n$.*

Proof. Unfold the trajectory along the line L through p in the direction θ ; it is convenient to let $g_0 = \text{id}$ belong to the piece of the trajectory through p . The image of L in D_n is the geodesic

through p in the direction θ , which is x , traversed repeatedly. Let x cross $2t$ sides. After $2t$ crossings the geodesic is again at p ; so c_{2t} is a positive multiple of the direction vector, of length $|x|$, and by Lemma 8.5(a), $g_{2t} \circ r^{-\text{rot}(x)}$ is the translation by c_{2t} along L . By (15) the piece of the trajectory which corresponds to x is preclosed, with the rotation $r^{-\text{rot}(x)}$. If $g_{2t'} \circ r^b$ is a translation along L for some $0 < t' < t$, then, again by Lemma 8.5(a), this translation is the translation by $c_{2t'}$, and the geodesic of D_n is at p after $2t'$ crossings, which is impossible, because x runs once around Z . Hence $j_* = 2t$ and $a_* = -\text{rot}(x)$. $\square$

Corollary 8.7. *Let n be odd and θ the direction of a cylinder of D_n . The bands of the class of the closed trajectories which are parallel to the direction θ are, up to the symmetries of P_0 and the reversal of the trajectories, the bands which arise from the cylinders of D_n of the direction θ as in Lemma 8.6.*

Proof. For odd n the symmetries of P_0 act on the directions modulo π by all maps $\vartheta \mapsto \pm\vartheta + \frac{i\pi}{n}$. So every trajectory of the class is the image under a symmetry of P_0 of a trajectory which has a segment of the direction θ or $-\theta$. Such a trajectory, or the reversed one, passes through a point p in the interior of P_0 in the direction θ . If it is closed, the geodesic of D_n through p in the direction θ is closed, because its unfolding is the unfolding of the trajectory; it lies in a cylinder of the direction θ , and Lemma 8.6 applies. Conversely the closed geodesics of one cylinder cross the same sides in the same order, so the trajectories which arise from them are strongly parallel, up to rotations of P_0 . $\square$

8.5 Veech's formula

Parts (a) and (b) of Theorem H are contained in Veech's paper [Vee92]. We describe the relevant statements in the notation of that paper and translate them. Throughout [Vee92], $n > 4$; we put $N := n$ if n is odd and $N := \frac{n}{2}$ if n is even.

The two surfaces. Veech's surface $X(n)$ is obtained from the regular n -gon $P(n)$ whose vertices are the n -th roots of unity and from $Q(n) = -P(n)$ by gluing parallel sides by translations; up to a similarity it is the double n -gon D_n . His surface $Y(n)$ is the translation surface of the billiard in the regular n -gon, built from $2N$ copies of the n -gon, and it is an unbranched cyclic covering of $X(n)$ of degree N . The bands of closed billiard trajectories correspond to the cylinders of $Y(n)$, with the convention that a trajectory which closes after an odd number of reflections is counted with twice its length [Vee92, Remark 1.11]; this is the reading of Section 8.1, in which a band is described by its trajectories other than the middle one. The covering is given by a character $\chi: H_1(X(n)) \rightarrow \mathbb{Z}/N$; for odd n , χ is the sum of the coefficients with respect to the classes of the oriented sides of $P(n)$ [Vee92, (5.3)–(5.5)].

The formula. Let ν_i , $1 \leq i \leq [\frac{n}{2}]$, be the classes of the closed geodesics of the cylinders of the vertical direction of $X(n)$, which for odd n is the direction of a side, and for even n let μ_j , $1 \leq j \leq \frac{n-2}{2}$, be those of the cylinders in the direction of a side. In every periodic direction the cylinders of $X(n)$ are the images of the cylinders of one of these two directions under one affine automorphism σ , which multiplies their circumferences by one factor [Vee92, (7.5), (7.6)]; that the periodic directions arise in this way is taken there from [Vee89] [Vee92, pp. 349 and 363]. One may take σ in the group $E(n)$ which is generated by the twist in the cylinders of the vertical direction, by the rotation by $\frac{2\pi}{n}$ and, for even n , by the twist in the cylinders of the direction of a side. Veech shows that the action of $E(n)$ on the homology of $X(n)$ relative to the vertices, reduced modulo N , yields a homomorphism $\rho: E(n) \rightarrow \text{SL}(2, \mathbb{Z}/N)$, and that its image consists of the matrices whose upper right entry is divisible by $\gcd(N, 4)$; in particular ρ maps onto $\text{SL}(2, \mathbb{Z}/n)$ if n is odd [Vee92, Lemmas 4.3 and 4.10]. If (a, b) is the first row of $\rho(\sigma)$, then for odd n

$$\chi(\sigma_* \nu_i) \equiv -4i(a - b) \pmod{n}, \quad (17)$$

and for even n there are analogous formulas for $\chi(\sigma_*\nu_i)$ and $\chi(\sigma_*\mu_j)$ [Vee92, Lemmas 8.3, 8.8 and Proposition 8.11]. Finally, if the closed geodesics of a cylinder of $X(n)$ have the class x , then the preimage of the cylinder in $Y(n)$ consists of $d = \gcd(\chi(x), N)$ cylinders, each N/d times as long as the cylinder itself [Vee92, (7.7), (8.16)].

Translation. By Lemma 8.5(a) the unfolding of the billiard along a closed curve x of D_n ends with the rotation $r^{\text{rot}(x)}$ of the polygon; so x lifts to a closed curve of the surface of the billiard if and only if $\text{rot}(x) = 0$. Let $n = 2m + 1$ be odd, and identify D_n with $X(n)$. The cylinder with the class ν_j is the horizontal cylinder C_j of Section 2.4, with $\nu_j = \pm c_j$ in the notation of Lemma 8.8, and $\chi(\nu_j) = -4j$ by (17), while $\text{rot}(c_j) = 4j$ by Lemma 8.10. Now rot and χ are homomorphisms of the first homology group to $\mathbb{Z}/n$ with the same kernel, the classes of the curves which lift to closed curves, and both are onto, because 4 is a unit; so $\text{rot} = u\chi$ for a unit u of $\mathbb{Z}/n$, and $u = \pm 1$ by the values on c_1 . The sign depends on the orientations and on the choice of a generator of the group of the covering; only the orders of the values matter. By Lemma 8.6 the factor of the band which belongs to a cylinder is the order of rot on its closed geodesics, that is, the number N/d . Hence, for odd $n > 4$, (17) says: *in the class of the direction $\sigma(\text{vertical})$ the bands have the factors*

$$\frac{n}{\gcd(n, 4j(a-b))} = \frac{n'}{\gcd(n', j)}, \quad n' = \frac{n}{\gcd(n, a-b)}, \quad 1 \leq j \leq m. \quad (18)$$

As ρ is onto $\text{SL}(2, \mathbb{Z}/n)$, the difference $a - b$ takes all values in $\mathbb{Z}/n$, so every divisor n' of n occurs; and $\sigma = \text{id}$ gives $n' = n$ for the direction of a side. Together with the statement on the circumferences this is Theorem H(b). For even n the covering has the degree $\frac{n}{2}$, so every factor N/d divides $\frac{n}{2}$, which is Theorem H(a); the formulas of Veech give the factors themselves, and they reproduce the factors of Table 5 (Section 9).

Thus the description of the factors is due to Veech. In [Vee92] the numbers d enter the residue of a zeta function, and through it the constant of the asymptotic formula for the number of bands of length at most t [Vee92, Theorems 1.3 and 9.25]; the factors of the cylinders of one direction are not compared with each other there. The following deduction is not in [Vee92].

Proof of Theorem H(c) from (b). Let $n = 3$. The vertex of D_3 has the angle 2π , so D_3 is a flat torus with one marked point, and in every direction of a closed geodesic it has one cylinder. By Lemma 8.6 and Corollary 8.7 all bands of a class then have the same factor; so all quotients are 1, and Conjecture 1.7 holds. Let now $n \geq 5$. Let $n = q^e$ be a power of a prime q , and consider a class with the divisor $n' = q^b$, $0 \leq b \leq e$. If $b < e$, all factors d_j divide q^b and are smaller than n . If $b = e$, then $d_j = n/\gcd(n, j)$ is divisible by q for every $j \leq m < n$; so the greatest common divisor of the factors is divisible by q , and all quotients are smaller than n . Hence Conjecture 1.7 holds for n . If $n = q$ is prime, the factors are all 1 (for $n' = 1$) or all q (for $n' = q$). If n is not a power of a prime, Conjecture 1.7 fails by Corollary 8.3, or by (18) with $n' = n$ and the arithmetic in the proof of Corollary 8.3; in a class with $n' < n$ all factors are smaller than n , so it fails exactly in the classes with $n' = n$. $\square$

For $n = 15$ the classes of directions have, according to the divisor $n' = 15, 5, 3, 1$, the factors 15, 15, 5, 15, 3, 5, 15, then 5, 5, 5, 5, 1, 5, 5, then 3, 3, 1, 3, 3, 1, 3, and 1, $\dots$, 1; see Table 4.

8.6 A second derivation of the formula for odd n

We now prove Theorem H(b) without the use of [Vee92]. The idea is the same, the action of the affine group on the homology modulo n ; in place of Veech's matrices in an explicit basis we use intersection numbers. Lemma 8.10 below corresponds to Lemmas 4.3, 4.10 and 8.3 of [Vee92], and Lemma 8.11 to the fact that the group $E(n)$ represents all affine automorphisms up to those with the derivative $\pm \text{id}$.

Let $n = 2m + 1 \geq 5$. Then D_n has one vertex and the genus m . For $s \in \mathbb{Z}/n$ let $\hat{e}_s \in H$ be the class of the side e_s , oriented from X_s to X_{s+1} ; it is a closed curve. We write $\langle \cdot, \cdot \rangle$ for the intersection form on H .

Lemma 8.8. (a) H is generated by $\hat{e}_0, \dots, \hat{e}_{n-1}$, with the single relation $\sum_s \hat{e}_s = 0$.

(b) There is a sign $\varepsilon_0 = \pm 1$ such that $\langle \hat{e}_s, \hat{e}_t \rangle = \varepsilon_0 (-1)^{(t-s) \bmod n}$ for all $s \neq t$, where $(t-s) \bmod n$ is the representative in $\{1, \dots, n-1\}$.

(c) The horizontal chord L_j , oriented from X_j to X_{n-1-j} , has the class $D_j := \hat{e}_j + \hat{e}_{j+1} + \dots + \hat{e}_{n-2-j}$ for $0 \leq j \leq m-1$, and so has the chord $-L_j$ of P'_0 , oriented in the same direction. With $D_m := 0$, the closed geodesics of the cylinder C_j , oriented in the direction of L_j , have the class $c_j := D_{j-1} + D_j$ for $1 \leq j \leq m$.

(d) $\varrho_* \hat{e}_s = -\hat{e}_{s+m}$.

Proof. (a) D_n is a cell complex with one vertex, the n edges $\hat{e}_s$ and the two faces P_0, P'_0 . The boundary of P_0 is $\sum_s \hat{e}_s$. The side $-e_s$ of P'_0 , traversed from $-X_{s+1}$ to $-X_s$, is glued to e_s , traversed from X_s to X_{s+1} ; so the boundary of P'_0 is $-\sum_s \hat{e}_s$.

(b) Let $r_0, \dots, r_{2n-1}$ be the rays along the sides at the vertex, in counterclockwise order, where r_i is the end ray of the i -th corner $((-1)^i, i \bmod n)$; see (4). The loop $\hat{e}_s$ leaves the vertex along the end ray of the corner $(+, s)$ and arrives along the end ray of the corner $(-, s)$, traversed backwards. These are r_i and r_{i+n} , where i is even and $i \equiv s$ modulo n . So the two ends of every $\hat{e}_s$ are opposite among the $2n$ rays, two different loops $\hat{e}_s, \hat{e}_t$ cross at the vertex exactly once, and the sign of the crossing is determined by whether, going counterclockwise from the beginning of $\hat{e}_s$, one meets the beginning of $\hat{e}_t$ before its end. If i' is the even index of the beginning of $\hat{e}_t$, this happens if and only if $(i' - i) \bmod 2n < n$, that is, if and only if $(t - s) \bmod n$ is even.

(c) In P_0 the chord L_j is homotopic, with fixed ends, to the path $X_j X_{j+1} \dots X_{n-1-j}$ along the boundary; in P'_0 the chord $-L_j$, oriented from $-X_{n-1-j}$ to $-X_j$, is homotopic to the path through $-X_{n-2-j}, \dots, -X_{j+1}$, whose pieces are the classes $\hat{e}_s$ by the gluing. A boundary component of C_j consists of L_{j-1} and $-L_j$ (Lemma 2.8), and it is homologous to the closed geodesics of C_j .

(d) By Lemma 2.6(b), ϱ maps e_s , traversed from X_s to X_{s+1} , to the side $-e_{s+m}$ of P'_0 , traversed from $-X_{s+m}$ to $-X_{s+m+1}$, which is $-\hat{e}_{s+m}$. $\square$

Let $c'_j := \varrho_* c_j$ for $1 \leq j \leq m$. These are the classes of the closed geodesics of the cylinders $\varrho(C_j)$ of the direction $\frac{\pi}{n}$.

Lemma 8.9. For $1 \leq i, j \leq m$ we have $\langle c_i, c_j \rangle = \langle c'_i, c'_j \rangle = 0$ and

$$\langle c'_j, c_i \rangle = \begin{cases} -\varepsilon_0 & \text{if } i + j \in \{m, m+1\}, \\ 0 & \text{otherwise.} \end{cases}$$

Consequently $c_1, \dots, c_m, c'_1, \dots, c'_m$ is a basis of H , and for $1 \leq j \leq m$

$$\sum_{i=1}^m i \langle c'_j, c_i \rangle \equiv 2\varepsilon_0 j, \quad \sum_{i=1}^m i \langle c_j, c'_i \rangle \equiv -2\varepsilon_0 j \pmod{n}. \quad (19)$$

Proof. The c_i are classes of pairwise disjoint curves, and so are the c'_i . Let $0 \leq i \leq m-1$. By Lemma 8.8(b), for $i \leq t \leq n-2-i$,

$$\langle \hat{e}_t, D_i \rangle = \varepsilon_0 \left(\sum_{u=1}^{n-2-i-t} (-1)^u - \sum_{u=1}^{t-i} (-1)^u \right) = -\varepsilon_0 (-1)^{t-i},$$

because n is odd and exactly one of the numbers $n - 2 - i - t$ and $t - i$ is odd; and $\langle \hat{e}_t, D_i \rangle = 0$ for the other t , since then the sum has an even number of terms with alternating signs. By Lemma 8.8(a),(d),

$$\varrho_* D_j = - \sum_{s=j}^{n-2-j} \hat{e}_{s+m} = \sum_{t=m-j-1}^{m+j-1} \hat{e}_t \quad (0 \leq j \leq m-1).$$

The interval $[m-j-1, m+j-1]$ lies in $[i, n-2-i]$ if $i+j \leq m-1$; then $\langle \varrho_* D_j, D_i \rangle$ is an alternating sum of $2j+1$ terms, equal to its first term $\varepsilon_0(-1)^{m-i-j}$. If $i+j \geq m$, the two intervals intersect in $[i, n-2-i]$, which has an even number of elements, and $\langle \varrho_* D_j, D_i \rangle = 0$. With $D_m = 0$ this gives, for $1 \leq i, j \leq m$ and $\sigma := i+j$,

$$\langle c'_j, c_i \rangle = \varepsilon_0(-1)^{m-\sigma} ([\sigma \leq m+1] - 2[\sigma \leq m] + [\sigma \leq m-1]),$$

where $[\cdot]$ is 1 if the condition holds and 0 otherwise. This is the stated value. The matrix $(\langle c'_j, c_i \rangle)_{i,j}$ has the entries $-\varepsilon_0$ on the antidiagonal $i+j = m+1$ and zeros below it, so its determinant is ± 1 . Hence the Gram matrix of the $2m$ classes c_i, c'_j has the determinant 1. As H has the rank $2m$ and its intersection form is unimodular, they form a basis of H . Finally $\sum_i i \langle c'_j, c_i \rangle = -\varepsilon_0((m-j) + (m+1-j)) = -\varepsilon_0(n-2j)$, and the second sum in (19) is obtained in the same way. $\square$

Lemma 8.10. (a) $\text{rot}(c_j) = 4j$ and $\text{rot}(c'_j) = -4j$ for $1 \leq j \leq m$.

(b) There is a sign $\varepsilon = \pm 1$ with the following property. Let U be the set of the homomorphisms $f: H \rightarrow \mathbb{Z}/n$ for which there are $\alpha, \beta \in \mathbb{Z}/n$ with $f(c_i) = \alpha i$ and $f(c'_j) = \beta j$ for all i, j ; we write $f = f_{\alpha, \beta}$. Then for $\tau := \tau_0$ and $\tau' := \tau_1$,

$$f_{\alpha, \beta} \circ \tau_* = f_{\alpha, \beta + 2\varepsilon\alpha}, \quad f_{\alpha, \beta} \circ \tau'_* = f_{\alpha - 2\varepsilon\beta, \beta}.$$

(c) The set $\{\text{rot} \circ \psi_* : \psi \in \langle \tau, \tau' \rangle\}$ is the set of all $f_{\alpha, \beta}$ with $\gcd(\alpha, \beta, n) = 1$.

Proof. (a) A closed geodesic of C_j , oriented in the direction of L_j and started in S_j , leaves P_0 through the side e_{n-1-j} and returns through the side with the label $j-1$. So $\text{rot}(c_j) = 2((j-1) - (n-1-j)) = 4j$. Let x be a closed curve which starts in P_0 and crosses the sides with the labels $s_1, \dots, s_{2t}$. By Lemma 2.6(b), $\varrho(x)$ starts in P'_0 and crosses the labels $s_1+m, \dots, s_{2t}+m$; started after its first crossing, it has the labels $s_2+m, \dots, s_{2t}+m, s_1+m$, and (16) gives $\text{rot}(\varrho(x)) = 2 \sum_i (s_{2i+1} - s_{2i}) = -\text{rot}(x)$.

(b) In each cylinder C_j the twist τ is a shear which fixes the boundary and moves one boundary component once around the cylinder relative to the other one (Lemma 2.10). If a closed curve crosses C_j , then its image under τ differs from it by one turn around C_j , in a sense which depends only on the direction of the crossing, because the shear is the same linear map T in all horizontal cylinders. So there is a sign ε_2 , the same for all j , with

$$\tau_* y = y + \varepsilon_2 \sum_{j=1}^m \langle y, c_j \rangle c_j \quad (y \in H),$$

and, as $\tau' = \varrho \tau \varrho^{-1}$ and ϱ preserves the intersection form, $\tau'_* y = y + \varepsilon_2 \sum_j \langle y, c'_j \rangle c'_j$. Therefore $\tau_* c_i = c_i$ and $\tau_* c'_j = c'_j + \varepsilon_2 \sum_i \langle c'_j, c_i \rangle c_i$, and by (19), with $\varepsilon := \varepsilon_0 \varepsilon_2$,

$$f_{\alpha, \beta}(\tau_* c_i) = \alpha i, \quad f_{\alpha, \beta}(\tau_* c'_j) = \beta j + \varepsilon_2 \alpha \sum_i i \langle c'_j, c_i \rangle = (\beta + 2\varepsilon\alpha)j.$$

The formula for τ' follows in the same way from the second sum in (19). Since the c_i, c'_j form a basis of H , every pair (α, β) defines an element of U .

(c) By (a), $\text{rot} = f_{4,-4}$, and $\gcd(4, n) = 1$. By (b) the set in question is the orbit of $(4, -4)$ under the group generated by the maps $(\alpha, \beta) \mapsto (\alpha, \beta + 2\varepsilon\alpha)$ and $(\alpha, \beta) \mapsto (\alpha - 2\varepsilon\beta, \beta)$ of $(\mathbb{Z}/n)^2$. As 2 is invertible modulo n , this group contains $E_1: (\alpha, \beta) \mapsto (\alpha, \beta + \alpha)$ and $E_2: (\alpha, \beta) \mapsto (\alpha + \beta, \beta)$. These maps preserve $\gcd(\alpha, \beta, n)$. Let $\gcd(\alpha, \beta, n) = 1$. For every prime q which divides n at most one residue u modulo q has $\alpha + u\beta \equiv 0$ modulo q , unless q divides β , in which case q does not divide α . So there is $u \in \mathbb{Z}$ such that $\alpha + u\beta$ is invertible modulo n . Then E_2^u maps (α, β) to a pair (α_1, β) with α_1 invertible, a power of E_1 maps it to $(\alpha_1, 1)$, and a power of E_2 to $(1, 1)$. Hence all pairs with $\gcd(\alpha, \beta, n) = 1$ lie in one orbit. $\square$

Lemma 8.11. *For odd n an affine automorphism of D_n is determined by its derivative, $\tau_1\tau_0 = \varrho^{n+2}$, and G_0 is generated by τ_0 and τ_1 .*

Proof. Let ψ be an affine automorphism with $D\psi = \text{id}$. The horizontal saddle connections are the chords L_j and $-L_j$ (Lemma 2.7), of the lengths λ_{n-2-2j} , and only L_0 , which is the same saddle connection as $-L_0$, has the length 1. So ψ maps the oriented saddle connection L_0 to itself. Then ψ is the identity near L_0 , hence everywhere.

The matrix $TR_{\pi/n}^{-1}$ has the determinant 1 and the trace $2 \cos \frac{\pi}{n} - 2 \cot \frac{\pi}{n} \sin \frac{\pi}{n} = 0$; so its square is $-\text{id}$, and

$$T_1T = R_{\pi/n} (TR_{\pi/n}^{-1})^2 R_{\pi/n} = -R_{2\pi/n}.$$

This is the derivative of $\mu\varrho^2 = \varrho^{n+2}$ (Lemma 2.6); hence $\tau_1\tau_0 = \varrho^{n+2}$. The rotation ϱ has the order $2n$, and $\gcd(n+2, 2n) = 1$; so ϱ is a power of $\tau_1\tau_0$. Finally $\mu = \varrho^n$ and $\tau_k = \varrho^k\tau_0\varrho^{-k}$, because the two sides have the same derivative. $\square$

Second proof of Theorem H(b). Let θ be the direction of a closed trajectory. The geodesic of D_n which is the image of its unfolding is closed, so θ is the direction of a cylinder and of a saddle connection γ in its boundary. By Lemma 2.12, an element of G_0 maps γ onto some δ_k , which has a base direction, and a power of ϱ maps this direction to the horizontal one. So there is $\psi \in G_0$ which maps the horizontal direction to $\pm\theta$. It maps the horizontal cylinders $C_1, \dots, C_m$ onto the cylinders of the direction θ , multiplies their circumferences ℓ_j of (8) by one factor $c > 0$, and the closed geodesics of $\psi(C_j)$ have the class $\pm\psi_*c_j$. By Lemma 8.11, $\psi \in \langle \tau, \tau' \rangle$, and by Lemma 8.10(c), $\text{rot} \circ \psi_* = f_{\alpha, \beta}$ for some α, β . Hence $\text{rot}(\psi_*c_j) = \alpha j$, and by Lemma 8.6 and Corollary 8.7 the bands of the class of θ are, up to symmetries and reversal, m bands with the simple preclosed lengths $c\ell_j$ and the factors

$$d_j = \frac{n}{\gcd(n, \alpha j)} = \frac{n'}{\gcd(n', j)}, \quad n' := \frac{n}{\gcd(n, \alpha)}.$$

The lengths $c\ell_j$ are pairwise distinct, so n' is determined by the class. Conversely, by Lemma 8.10(c) every pair $(\alpha, 1)$ is $\text{rot} \circ \psi_*$ for some $\psi \in \langle \tau, \tau' \rangle$, and then the class of the direction $D\psi(1, 0)$ has the divisor $n' = n/\gcd(n, \alpha)$; so every divisor n' of n occurs. For the trajectories parallel to a side we may take $\psi = \text{id}$; then $c = 1$, and $\alpha = 4$ gives $n' = n$, in accordance with Proposition 8.2. $\square$

8.7 Tables

Table 4 lists the factors of Theorem H(b) for some odd n , together with the number of classes with each divisor n' which were found by unfolding billiard trajectories directly, without the use of D_n (Section 9). In all 2407 classes which were computed for 14 odd values of n , the bands and their factors are as stated in Theorem H(b).

Table 5 compares the table of experiments of the source [Fuc21, §1.7.3.2] with the computation. Every line of the source is reproduced. Three remarks are necessary. For $n = 8$ and the class which the source calls odd, the factors of the two bands are 2 and 1, the larger factor belonging to the

n	n'	factors $d_1, \dots, d_m$	gcd	a quotient is n	classes
9	9	9, 9, 3, 9	3	no	338
	3	3, 3, 1, 3	1	no	75
	1	1, 1, 1, 1	1	no	41
15	15	15, 15, 5, 15, 3, 5, 15	1	yes	116
	5	5, 5, 5, 5, 1, 5, 5	1	no	38
	3	3, 3, 1, 3, 3, 1, 3	1	no	22
	1	1, ..., 1	1	no	5
21	21	21, 21, 7, 21, 21, 7, 3, 21, 7, 21	1	yes	74
	7	7, 7, 7, 7, 7, 7, 1, 7, 7, 7	1	no	18
	3	3, 3, 1, 3, 3, 1, 3, 3, 1, 3	1	no	4
	1	1, ..., 1	1	no	6
25	25	25, 25, 25, 25, 5, 25, 25, 25, 25, 5, 25, 25	5	no	54
	5	5, 5, 5, 5, 1, 5, 5, 5, 5, 1, 5, 5	1	no	6
	1	1, ..., 1	1	no	4
27	27	27, 27, 9, 27, 27, 9, 27, 27, 3, 27, 27, 9, 27	3	no	49
	9	9, 9, 3, 9, 9, 3, 9, 9, 1, 9, 9, 3, 9	1	no	11
	3	3, 3, 1, 3, 3, 1, 3, 3, 1, 3, 3, 1, 3	1	no	3
	1	1, ..., 1	1	no	1
45	45	45, 45, 15, 45, 9, 15, 45, 45, 5, 9, 45, 15, 45, 45, 3, 45, 45, 5, 45, 9, 15, 45	1	yes	15
	15	$d_j = 15 / \gcd(15, j)$, $1 \leq j \leq 22$	1	no	1
	9	$d_j = 9 / \gcd(9, j)$	1	no	2
	5	$d_j = 5 / \gcd(5, j)$	1	no	3
	3	$d_j = 3 / \gcd(3, j)$	1	no	4

Table 4: The factors $d_j = n' / \gcd(n', j)$ of the $m = \frac{n-1}{2}$ bands of a class of parallel closed trajectories, for odd n (Theorem H(b)); the band with the index j has the simple preclosed length $c \lambda \sin \frac{2j\pi}{n} / \sin \frac{\pi}{n}$. Column “gcd”: the greatest common divisor of the factors. Last column: the number of classes with this divisor n' among the classes of the directions of the reachable points of length at most 60, 40, 30, 24, 24, 16 (for $n = 9, 15, 21, 25, 27, 45$), found by direct unfolding.

shorter preclosed piece; so the two closed lengths are in the ratio $\sqrt{2} : 1$, which the source prints as $1 : \sqrt{2}$. For $n = 10$ in the even class and for $n = 12$ in the odd class there are also classes in which all factors are 1; the corresponding ratios are not in the table of the source, which lists the ratios known at that time. None of this affects Conjecture 1.7.

9 Computations and reproducibility

All theorems, propositions, lemmas and corollaries of this note are proved without computation. Lemma 7.2 is a finite statement; its proof is the computation which is written out there, and a program repeats it. The computations do three things: they test the reading of the definitions against the explicit descriptions, the examples and the tables of the source; they test the statements on many exact instances; and they produce the tables and the numbers which are quoted in Remarks 4.2, 5.1, 6.3 and 7.4, Sections 7.4 and 8.7. The reproducibility package has five parts.

The programs with which the results were first obtained (original/). They compute in the ring $\mathbb{Z}[\zeta_n]$ or $\mathbb{Z}[\zeta_{2n}]$, with elements stored as integer vectors; equalities are tested exactly, and signs of real numbers by floating point with a rigorous error bound and, where this does not decide, with multiple precision. The program `reach.py` enumerates all reachable points up to a given length, with their types, by a search through the developments, and contains an exact tracer for a single point; with the

n	class	preclosed lengths [Fuc21]	closed lengths [Fuc21]	computed: factors of the bands [classes]
6	even	1 : 2	1 : 2 and 3 : 2	(1, 1) [44], (3, 1) [122]
6	odd	1	1	(3) [374], (1) [123]
8	even	1 : (1 + $\sqrt{2}$)	1 : (1 + $\sqrt{2}$)	(4, 4) [136]
8	odd	1 : $\sqrt{2}$	1 : $\sqrt{2}$	(2, 1) [460]
9		1 : $\lambda : \lambda^2 - 1 : \lambda + 1$	the same, and 3 : $3\lambda : \lambda^2 - 1 : 3(\lambda + 1)$	(1, 1, 1, 1) [41], (9, 9, 3, 9) [338], (3, 3, 1, 3) [75]
10	even	1 : 1 + $\tau : 2\tau$	5 : $5(1 + \tau) : 2\tau$	(5, 5, 1) [106], (1, 1, 1) [17]
10	odd	1 : τ	1 : τ	(5, 5) [373], (1, 1) [72]
12	even	1 : (1 + $\sqrt{3}) : (2 + \sqrt{3})$	the same, and 3 : (1 + $\sqrt{3}) : 3(2 + \sqrt{3})$	(2, 2, 2) [29], (6, 2, 6) [86]
12	odd	1 : $\sqrt{3} : 2$	3 : $3\sqrt{3} : 2$	(3, 3, 1) [319], (1, 1, 1) [108]

Table 5: The table of experiments of [Fuc21, §1.7.3.2] and the computation. The classes “even” and “odd” of directions are named as in Section 1 of the source: in an odd class the short trajectories have the types A_k with even k . Here $\lambda = 2 \cos \frac{\pi}{9}$ and τ is the golden ratio. Last column: the factors of the bands, in the order of increasing length of the simple preclosed piece, and in brackets the number of classes with these factors among the classes of the directions of the reachable points of length at most 60. In all cases the computed ratios of the simple preclosed lengths are those of the third column.

type of Definition 1.1 it reproduces the descriptions of the source for $n = 4$ and $n = 6$. `test_conj.py` examines all reachable points of length at most 80 for $n = 5, \dots, 12$ (between 3289 and 3399 points for each n): 30,654 instances of Theorems B and C; all pairs with $\det = S$, which have the three pairs of types of Remark 4.2; for the 8028 unitary pairs the 60,360 points of Theorem E inside the disc, none missing and no further reachable point on the lines; and the unitary pairs whose n -gon lies in the disc, 668 in all, which span reachable n -gons with the types of Theorem D. For $n = 7, 9, 11$ the printed point $u + (\lambda + 1)v$ is not reachable for any of the 493 unitary pairs which were tested. `verify_theory.py` carries out the descent of Lemma 2.12 for all reachable points of length at most 60 and tests the action of the twists; `certificates.py` searches all reachable n -gons through the points of Remark 6.3; `dodeca.py` produces Table 2 and finds that for all 2095 geodesics of type A_0 with an even number of segments the end vertex is the image of the first vertex under the half-turn of Theorem G; `closed_sim.py` unfolds closed trajectories exactly; `factors_structure.py` verifies in integer arithmetic, for 13 odd values of $n \leq 49$, the statements of Lemmas 8.9 and 8.10 and that ϱ acts on the homology, up to the sign, as a word in the twists τ_0 and τ_1 ; and `factors_orbit.py` computes the orbit of rot and the resulting factors for all odd n from 5 to 49.

The programs written with this note (writing_stage/). `check_paper.py` tests the statements as they are printed, in twelve parts: (A) the formulas (1), (2), Lemmas 2.4 and 2.8 and the identity $T_1 T = -R_{2\pi/n}$ for $n \leq 60$; (B) Definition 1.1, Lemma 2.3 with (5), (6), and Lemma 7.1 on the 1899 reachable points of length at most 20 for $n = 4, \dots, 12$; (C) Theorems B and C on 1526 instances, with the exact tracer; (D) Theorem D, Remark 4.2 and Theorem E on 377 pairs and 66 lines; (E) the formulas of Lemma 6.1, of Steps 1 and 2 of the proof of Theorem D and of the proof of Theorem F(c) for $n \leq 60$, in floating point, the point w_n for $n \leq 12$, and Theorem E(c) for $n \leq 12$; (F) Example 6.2, by an enumeration which uses only the formulas of the source; (G) Lemma 7.2; (H) Proposition 8.2, Table 3 and Remark 8.4(a) with a floating-point billiard in the n -gon which uses nothing of this note (504 trajectories for odd $n \leq 45$); (I) Lemmas 8.8 to 8.10 for odd $n \leq 61$, and Theorem H(a) on 708 trajectories of the floating-point billiard for $n = 6, 8, 10, 12$; (J) Theorem H(b) against this billiard, in the directions which are the images of the horizontal direction under ten words in the twists, for $n = 9$ and for $n = 15$; (K) the tables and the numbers of this note against the recorded outputs of the other programs; (L) the formula (17) of Veech, and his formulas for even n as they were transcribed, against

the factors which were found by direct unfolding (Tables 4 and 5). All checks pass; the run takes less than a minute. The program `dodecahedron_halfturns.py` repeats the proof of Lemma 7.2 in exact arithmetic in $\mathbb{Q}(\sqrt{5})$ for all 20 vertices and prints the 20×20 matrix of the distances and the 15 half-turns. `witness_check.py` contains the computations for Remark 7.4, made with the programs of `original/`, and `make_figures.py` produced the coordinates of the four figures.

Two independent verification runs, each with its own programs and their recorded outputs (`verification_run_A/`, `verification_run_B/`); see the next paragraph.

The verification run on the final text (`independent_run_2/`), again with its own programs and their recorded outputs; see the next paragraph.

All programs of the first four parts which have a recorded output were run again when this note was written, one process at a time. The three programs of `writing_stage/` passed, and every other output is identical to the recorded one, or identical up to the fields which record running times; the details are in the file `RERUN_LOG.txt` of the package. The run on the final text repeated these re-runs from the archive of the package, with the same result.

Verification. The results were first obtained, with proofs and with the programs in `original/`. Three independent verification runs, all AI-assisted, followed; each wrote its own programs. The first two examined the first written version of the results, the third one the final text of this note.

The first run examined the conventions (the reading of [Fuc21, Fig. 8, Definition 2.1] against the examples and tables of the source), Section 2 and Theorems A to D. It re-derived the proofs. It found that the existence of the parallel short trajectory in Theorem B was used and not proved, and gave the lemma which proves it (now Lemma 2.13); it found no other gap. It required that the hypothesis $n \geq 4$ be stated in Lemma 2.11, and editorial corrections, among them the two misprints of the source in Section 1.4(3) and the precise form (3) of the characterization of the type A_0 . Its programs unfold by reflections of the polygon, in $\mathbb{Z}[\zeta_{2n}]$, and include a billiard in the fixed n -gon with 60 digits which does not unfold. They confirm Lemma 2.3, the descent and Theorems B and C without exception for $n = 4, \dots, 12$ up to the length 150 (between 10,743 and 12,029 reachable points for each n), for $n = 13, 14, 15, 16, 18, 20$ up to the length 60 and for $n = 17, 24, 25, 30$ up to the length 40, and Theorem D on the pairs of about half these lengths. In particular, for $n = 5, \dots, 12$ all 8028 unitary pairs of length at most 80 span reachable n -gons with the types of Theorem D, the vertices outside the disc being decided by exact tracing, and no other pair with $\det = S$ does.

The second run examined Theorems E to H and the literature. It re-derived the proofs and found no mathematical gap. It gave the elementary proof of Proposition 8.2; it found the record [Dan26] and made the analysis of Remark 7.4; and it required the corrections which are now Section 1.4(4) and (5), Remark 7.3, Remarks 5.1 and 8.4, the discussion of Table 5, and the facts on the census in Section 7.4. It also pointed out that Veech's paper [Vee92] was the closest earlier source for Theorem H and had not been read. Its programs compute in $\mathbb{Z}[\zeta_n]$ and unfold closed trajectories by reflections, without the double n -gon; they confirm Theorem E on 160 lines, the statements of Theorem F for $n = 5, \dots, 12$, the mechanism of Theorem G on 2901 geodesics, and Theorem H on 4776 classes of directions for 18 values of n ; and they contain the computation of Remark 5.1 ($|m| \leq 4$, 23 unitary pairs with $n = 7, 9, 11$). All corrections required by the two runs were made.

After the two runs, the paper [Vee92] was read in full. It contains the description of the factors, and the presentation of Theorem H was changed accordingly: parts (a) and (b) are attributed to Veech, Section 8.5 was added, and the argument of Section 8.6 is presented as a second derivation. When this note was written, the proofs were written out again. The following parts then had a form which the two runs had not seen: Lemmas 2.9 and 4.1; Lemma 6.1 as a separate lemma; the proof of Lemma 7.2 with coordinates, which replaces a shorter argument; in Section 8, Lemmas 8.1, 8.5 and 8.6 and Corollary 8.7, which make precise the reading of Conjecture 1.7 that the second run had

examined in its first form; and the dictionary of Section 8.5.

The third run examined the final text, and these parts first. It read the source again, with the pages which carry Fig. 8, Definition 2.1, the conjectures and the tables as page images, and compared Sections 1.1 and 1.2 with them. It checked all proofs line by line. It read the scan of [Vee92], all 39 pages as page images, compared Section 8.5 and the other statements on [Vee92] with the pages which carry the statements cited there, and looked on every page for a comparison of the factors of the cylinders of one direction; it did not check Veech's proofs. It compared the statements quoted from [McM26, AAH22, DFT11, DL18] with these texts; it read the metadata of the record [Dan26] and its file with the data of the geodesic; and it repeated the literature search. It found no gap which could not be closed, and no wrong statement apart from this: (3) and Lemma 7.1 were stated for $n \geq 3$ and are false for $n = 3$, a case which is not used. It required the following corrections, which were made. Mathematical ones: the hypothesis $n \geq 4$ in (3) and in Lemma 7.1; the hypothesis $n \geq 5$ in Theorem H(b), with the case $n = 3$ of part (c) treated separately; a more detailed proof of Lemma 2.9; and the precise form of the translation in Section 8.5. On earlier work: the paper [Tro23], which this run found, is now cited for the argument with the rotation about the midpoint of the geodesic; the cylinders and twists of Section 2.4 are attributed to [Vee89]; and the statement in Section 8.5 on what [Vee92] quotes from [Vee89] was corrected. The same run checked these corrections on the rebuilt text; no further run followed.

The programs of the third run use only the definitions of the source: the development by reflections of the labelled polygon, the angles of [Fuc21, Fig. 8] measured in the billiard table, the four lines of Definition 2.1 as printed, a billiard in the fixed n -gon in exact arithmetic in $\mathbb{Q}(\zeta_{2n})$, and the unfolding of closed trajectories without the double n -gon. They confirm the following. The rule on the parity of N , Lemma 2.3, (3) and Lemma 7.1 for $n = 4, \dots, 12$ up to the length 150 (the same 10,743 to 12,029 reachable points for each n as in the first run) and for ten further values of $n \leq 30$; the descriptions of the source for $n = 4$ and $n = 6$ on the 6875 and 7605 points of length at most 120. Theorems B and C on 147,446 instances for $n = 4, \dots, 12$ up to the length 60. Theorem D, Remark 4.2 and Theorem E on 2798 pairs with $\det = S$, among them 938 unitary pairs, and on 1876 lines with 13,780 reachable points, for ten values of $n \leq 15$. Theorem F(b), (c) and the numbers of Remark 6.3 for $n = 5, \dots, 12$, by a search which is exhaustive for the vertices $v_2, \dots, v_{n-2}$. Proposition 8.2 and Table 3, with the billiard in exact arithmetic, on 972 trajectories for 17 odd values of $n \leq 63$, among them $n = 15, 21, 33, 35, 45$. Tables 4 and 5 with all numbers of classes (4776 classes; in 114 of them the bands were compared with a floating-point billiard). Lemma 7.2, Table 2 and the numbers of Section 7.4; the half-turn of Theorem G on 2484 geodesics; the facts of Remark 7.4(a), (b) and the numbers of (c), (d). Lemma 8.1 on 867 pairs of trajectories of the floating-point billiard, Lemmas 8.5 and 8.6 on 2557 unfolded bands, and Lemma 8.9 for odd $n \leq 61$. The formula (17), with the matrices of [Vee92, Lemma 4.3], against the factors found by unfolding in the directions $D\sigma(1, 0)$, for 246 words σ in the two generators and ten odd values of $n \leq 45$, and the patterns which Veech's formulas give for $n = 6, 8, 10, 12$. The data of Figures 1, 3 and 4. The run also repeated the re-runs of the package from its archive; all recorded outputs were reproduced. No proof depends on a computation.

Scope and priority. The note decides the seven conjectures of [Fuc21] as listed in Table 1. It makes no statement on polyhedra other than the dodecahedron and none beyond the seven conjectures; the difference between the census of the source and the exact enumeration (Section 7.4) is reported and not explained. The problem and all definitions are due to Fuchs [Fuc21].

The closest earlier source for Conjecture 1.7 is Veech's paper [Vee92], which the source itself cites. For $n > 4$ Veech realises the translation surface $Y(n)$ of the billiard as an unbranched cyclic covering of degree N ($N = n$ for odd n , $N = \frac{n}{2}$ for even n) of the double n -gon $X(n)$, given by a character $\chi: H_1(X(n)) \rightarrow \mathbb{Z}/N$, and he computes χ on the closed geodesics of every cylinder in every periodic

direction: for odd n , $\chi(\sigma_*\nu_i) \equiv -4i(a-b)$ modulo n , where ν_i belongs to the i -th cylinder in the direction of a side, σ is an affine automorphism and (a, b) is the first row of its image under a surjection onto $\mathrm{SL}(2, \mathbb{Z}/n)$ [Vee92, Lemmas 4.3, 4.10, 8.3, 8.8, Proposition 8.11]; and the cylinder is covered by $d = \gcd(\chi(\sigma_*\nu_i), N)$ cylinders of $Y(n)$, each N/d times as long [Vee92, (7.7), (8.16)]. Analogous formulas are given there for even n . Since N/d is the factor of the corresponding band of billiard trajectories, the description of the factors in Theorem H(b) is contained in [Vee92, Proposition 8.11] and is due to Veech, and Theorem H(a) is the remark that the covering has the degree $\frac{n}{2}$ for even n . Our proof in Section 8.6 through intersection numbers is a second derivation of his formula, and not a new result. Veech uses these numbers to evaluate the constant of his main theorem, the asymptotic formula $c_n t^2 / \mathrm{area}(P_n)$ for the number of bands of periodic trajectories of length at most t in the regular n -gon P_n , with $c_n = (c(n) - n^3) / (48 \varepsilon(n)(n-2)\pi)$, $\varepsilon(n) = n/N$, for an explicit multiplicative function c [Vee92, Theorems 1.3 and 9.25]. He does not consider how the factors of the cylinders of one direction divide each other, and Conjecture 1.7, which is of 2021, is not mentioned. What this note adds is the observation that Veech's formula decides that conjecture: it fails for every odd n with at least two distinct prime factors (for $n = 15$ and the direction of a side the factors are 15, 15, 5, 15, 3, 5, 15), and it holds for odd prime powers and, for the trajectories other than the middle trajectories of bands, for even n . The deduction is a few lines of arithmetic. In addition we give a proof of the counterexample which uses neither the surface nor homology (Proposition 8.2). The prime case, in which all factors are n or all are 1, follows from theorems of McMullen [McM26], and for $n = 5$ from results of Davis, Fuchs and Tabachnikov [DFT11] and of Davis and Lelièvre [DL18] (Section 1.5); it is likewise an immediate consequence of Veech's formula, although it is not stated in [Vee92]. We claim no priority for any statement on the factors. Veech's paper contains nothing on the types of short trajectories, on reachable polygons, on unitary pairs or on polyhedra (Conjectures 2.3 to 2.6 and 3.2), and for Conjecture 2.7 only the analogous statement for closed geodesics [Vee92, (7.6), (7.15)] (Remark 3.2).

For Theorem G, the argument with the half-turn and the statement that no short geodesic of type A_0 is closed are due to Athreya, Aulicino and Hooper [AAH22], and the same argument for the geodesics between vertices of the other regular polyhedra is in [Tro23] (Remark 7.3); what is added here is Lemma 7.1 and the statement on the distance 2. The record [Dan26] concerns a different definition of the type (Remark 7.4). For Conjectures 2.3 to 2.7 and 3.2 no proof and no refutation was found in print, and for Conjecture 1.7 no statement of its failure for composite n . The proofs of this note for these conjectures are elementary: the flat geometry of the double n -gon and a descent. A search which finds nothing is not a proof of novelty, and we claim no priority.

What was read. The source [Fuc21] was read completely, in the file on the site of the journal, with its figures and tables. The paper [Vee92] was read in full, in the scan of the Göttingen digitisation centre (printed pages 341 to 379; the record of EuDML gives the pages 341 to 380); its statements were compared with those of this note, and its proofs were not checked step by step. Of [McM26], in the published version, Sections 1, 2 and 4 to 7 were read; of [DFT11], in the arXiv version 1, Sections 2.5 and 2.6 and the closing remarks, the remainder was searched for keywords; of [DL18], in the arXiv version 2, Sections 1.2 to 1.5, Theorems 3.30 and 3.31 and the conjectures of Section 4; of [AAH22], in the arXiv version 2, Sections 1 and 5 and Section 10.2.1; the paper [Tro23] completely, in the arXiv version 3. The journal versions of [DFT11], [AAH22] and [Tro23] were not compared. The paper [Vee89] was not read; the facts which [Vee92] quotes from it are taken on the word of [Vee92], and whether [Vee89] already contains a part of the description of the factors was not checked. Of [War98] only the abstract was read. Of the record [Dan26], three text files and the metadata were consulted, and not its paper or its programs (Remark 7.4). The bibliographic data were checked with Crossref, zbMATH and the arXiv; for [DL18] no journal publication was found.

The searches were made on 9 October 2026, and the third verification run repeated them: the

lists of works which cite [Fuc21] in OpenAlex and Crossref (one work, on another subject) and in Semantic Scholar (none); the review of [Fuc21] in zbMATH, which reports its statements as conjectures; queries to the arXiv on billiards in regular polygons, on the double pentagon and on geodesics on the dodecahedron and on the regular polyhedra; queries to Crossref, OpenAlex and zbMATH; and seven web searches. The only document found which claims to settle one of the seven conjectures is the record [Dan26].

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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