

Gasull’s Finite Moment Problem for Polynomials with Few Monomials

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September 30, 2026

Abstract

Let $f \in \mathbb{C}[x]$ have at most k monomials and put $M_n = \int_0^1 f(x)^n dx$. A. Gasull asked whether there is a number $N(k)$ such that $M_1 = \dots = M_{N(k)} = 0$ forces $f = 0$, and, if so, for its value or a good upper bound. We show that $N(k)$ exists for every k . The proof treats the exponents as parameters, applies Hilbert’s basis theorem to the moments with their denominators cleared, and uses the known fact, proved by Pakovich and by Françoise, Pakovich, Yomdin and Zhao, that the moments $\int_0^1 f^n dx$, $n \geq 1$, of a nonzero polynomial cannot all vanish. It gives no explicit bound. We also show that $N(k) \geq k$ for every k , that $N(1) = 1$ and $N(2) = 2$, and, by an exact computer-assisted certificate, that $N(3) = 3$: after a linear change of variables, a resultant that controls the case $k = 3$ has only nonnegative coefficients. The statement for $k = 3$ holds for all real exponents greater than $-1/3$. For every k , the first k moments force $f = 0$ when the exponents are sufficiently lacunary. For $k = 4$ and $k = 5$ we report exact certificates for all sets of integer exponents with largest exponent at most 50 and at most 16, respectively; these are finite-range computations. Numerically, the real-exponent version of the case $k = 4$ fails for some exponents with $e_1 < 0$. The value of $N(k)$, and any explicit upper bound for it, remain open for $k \geq 4$. This is an unrefereed note.

1 Introduction

In his survey of open problems in low dimensional dynamical systems, Gasull [Gas21] records a question of van den Essen (Problem 26 there): if $f \in \mathbb{C}[x_1, \dots, x_n]$ and $\int_{[0,1]^n} f^m = 0$ for all $m \geq 1$, must f vanish? The question is related to the Jacobian conjecture [vdE00, DvdEZ17, Fra14]. For $n = 1$ the answer is yes. Gasull refers to a theorem of Pakovich [Pak13] (see Theorem 3.1 below); the same statement is [FPYZ11, Corollary 4.1], proved there by an argument of M. Boyarchenko, and it is a special case of a theorem of van den Essen and Zhao [vdEZ13, Theorem 6.8] (see Section 3). Gasull then asks for a finite version [Gas21, Problem 27].

Problem 1.1 (Gasull). *(i) Let $f \in \mathbb{C}[x]$ have k monomials. Is there a number $N(k)$ such that*

$$M_n := \int_0^1 f(x)^n dx = 0 \quad (1 \leq n \leq N(k))$$

implies $f = 0$? (ii) If so, find $N(k)$ or a good upper bound for it.

Gasull points out that questions of this type arise in the center problem for Abel differential equations [CGM13]. The problem is record AMR-046-0027 of the UnsolvedMath dataset [ula26], where it is listed as open. Since the conclusion is $f = 0$, we read “ k monomials” as “at most k monomials”.

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Definition 1.2. For $k \geq 1$ let $N(k)$ be the least integer $N \geq 1$ such that every $f \in \mathbb{C}[x]$ with at most k monomials and $M_1 = \dots = M_N = 0$ is the zero polynomial, and $N(k) = \infty$ if there is no such N . Let $N^=(k)$ be the least $N \geq 1$ such that no polynomial with exactly k monomials satisfies $M_1 = \dots = M_N = 0$.

Clearly $N(k)$ is nondecreasing, and $N(k) = \max_{j \leq k} N^=(j)$.

Theorem 1.3. (a) $N(k) < \infty$ for every $k \geq 1$. More precisely, for each k there is an integer $N_0(k)$ with the following property: if $e_1, \dots, e_k$ are distinct nonnegative rational numbers, $c \in \mathbb{C}^k$, and $\int_0^1 (\sum_i c_i x^{e_i})^n dx = 0$ for $1 \leq n \leq N_0(k)$, then $c = 0$. The proof gives no explicit value of $N_0(k)$.

(b) $N(k) \geq k$ for every k . For $k \leq 4$ also $N^=(k) \geq k$.

(c) $N(1) = N^=(1) = 1$, $N(2) = N^=(2) = 2$ and $N(3) = N^=(3) = 3$.

Part (a) answers Problem 1.1(i) affirmatively, and part (c) answers (ii) for $k \leq 3$. The value $N(3) = 3$ rests on an exact computer calculation. We prove it in the following stronger form, for real exponents.

Theorem 1.4. Let $-1/3 < e_1 < e_2 < e_3$ be real numbers, let $c \in \mathbb{C}^3$, and put $f(x) = c_1 x^{e_1} + c_2 x^{e_2} + c_3 x^{e_3}$ for $0 < x \leq 1$. If $\int_0^1 f(x)^n dx = 0$ for $n = 1, 2, 3$, then $c = 0$.

We also show that for every k the first k moments force $f = 0$ when the exponents are sufficiently lacunary (Lemma 6.1 and Corollary 6.2). For $k = 4$ and $k = 5$ we report exact certificates on finite ranges of exponents, and numerical evidence that the analogue of Theorem 1.4 for $k = 4$ fails for some real exponents with $e_1 < 0$ (Section 7). The value of $N(k)$ for $k \geq 4$, and any explicit upper bound, remain open.

Related work. For polynomials of bounded degree the analogous number is finite by a standard Noetherianity argument (the “moment Bautin index”), and Batenkov and Binyamini gave an explicit bound [BB15, Theorem 1]. Combined with Pakovich’s theorem it shows that $M_1 = \dots = M_{3d^2-5d+5} = 0$ forces $f = 0$ when $\deg f \leq d$ (Remark 3.5). Such bounds grow with the degree, whereas Gasull asks for a bound in terms of the number of monomials. Our proof of Theorem 1.3(a) is the same Noetherianity argument, in a polynomial ring in which the exponents are variables. The polynomial moment problem, which describes the pairs of polynomials P, q with $\int_0^1 P^i q dx = 0$ for all $i \geq 0$, was solved by Pakovich and Muzychuk [PM09]. The vanishing of all moments, which we use in the form of Corollary 3.2, was treated by Pakovich [Pak13], whose theorem also answers a question of Zhao [Zha10], and, for the Lebesgue measure on a segment, by Françoise, Pakovich, Yomdin and Zhao [FPYZ11, Section 4]. Corollary 3.2 itself is a special case of [vdEZ13, Theorem 6.8(i)]; see the remarks after its proof. We found no earlier statement of Theorem 1.3(a) or of the value $N(3) = 3$.

2 Notation and elementary facts

For real $e_1 < \dots < e_k$ and $c \in \mathbb{C}^k$ put $f_{e,c}(x) = \sum_{i=1}^k c_i x^{e_i}$ for $0 < x \leq 1$. For a multi-index $\alpha \in \mathbb{Z}_{\geq 0}^k$ write $|\alpha| = \sum \alpha_i$, $\binom{n}{\alpha} = n! / \prod \alpha_i!$, $c^\alpha = \prod c_i^{\alpha_i}$ and $\alpha \cdot e = \sum \alpha_i e_i$. If $ne_1 > -1$, then $1 + \alpha \cdot e \geq 1 + ne_1 > 0$ for $|\alpha| = n$, so $f_{e,c}^n$ is integrable and

$$M_n(e; c) := \int_0^1 f_{e,c}(x)^n dx = \sum_{|\alpha|=n} \binom{n}{\alpha} \frac{c^\alpha}{1 + \alpha \cdot e}. \quad (1)$$

Thus $M_n(e; \cdot)$ is a form of degree n on $\mathbb{C}^k$, and $M_n(e; \lambda c) = \lambda^n M_n(e; c)$. We clear denominators with

$$D_n(e) = \prod_{|\alpha|=n} (1 + \alpha \cdot e), \quad P_n(e, c) = \sum_{|\alpha|=n} \binom{n}{\alpha} c^\alpha \prod_{|\beta|=n, \beta \neq \alpha} (1 + \beta \cdot e). \quad (2)$$

So D_n and P_n are polynomials with integer coefficients in e and in (e, c) , respectively. If $ne_1 > -1$, then $D_n(e) > 0$ and $P_n(e, c) = D_n(e)M_n(e; c)$.

Definition 2.1. Let $e_1 < \dots < e_k$ be real with $ke_1 > -1$. We say that e has *property* (H_k) if $c \in \mathbb{C}^k$ and $M_1(e; c) = \dots = M_k(e; c) = 0$ imply $c = 0$.

A polynomial with at most k monomials can be written as $f_{e,c}$ with k distinct exponents $e_i \in \mathbb{Z}_{\geq 0}$ (some c_i may vanish). Hence $N(k) \leq k$ if and only if every integer vector $0 \leq e_1 < \dots < e_k$ has property (H_k) .

Lemma 2.2. Let $e_1 < \dots < e_k$ be real with $e_1 > -1/2$.

- (a) The functions $x^{e_1}, \dots, x^{e_k}$ are linearly independent on $(0, 1]$. Hence $f_{e,c} = 0$ only if $c = 0$.
- (b) If $c_i = (1 + e_i)y_i$ for all i , then $M_1(e; c) = y_1 + \dots + y_k$.
- (c) If $c = \lambda c'$ with $\lambda \in \mathbb{C}$ and $c' \in \mathbb{R}^k$, then $M_2(e; c) = \lambda^2 \int_0^1 f_{e,c'}^2 dx$, and this is nonzero unless $c = 0$.

Proof. (a) If $f_{e,c} = 0$ and $c \neq 0$, let j be the least index with $c_j \neq 0$. Then $x^{-e_j} f_{e,c}(x) \rightarrow c_j$ as $x \rightarrow 0^+$, a contradiction. (b) is (1) with $n = 1$. (c) The function $f_{e,c'}$ is real-valued, so $\int_0^1 f_{e,c'}^2 dx > 0$ unless $f_{e,c'} = 0$, that is, unless $c' = 0$ by (a). $\square$

3 Existence of $N(k)$

We use the following result of Pakovich. We quote it from the preprint version of [Pak13]; we could not consult the journal version, whose numbering may differ.

Theorem 3.1 (Pakovich [Pak13, Corollary 3.5]). Let P, q be rational functions with $\deg P > 0$ and $q \neq 0$, and let $\gamma \subset \mathbb{C}$ be an oriented piecewise-smooth curve, with only transversal self-intersections, that contains no poles of P or q . Assume that $q^{-1}\{\infty\} \subseteq P^{-1}\{\infty\}$ and $P(\infty) = \infty$. If $\int_\gamma P(z)^i q(z) dz = 0$ for all $i \geq i_0$, for some $i_0 \geq 0$, then this holds for all $i \geq 0$.

In [Pak13] this corollary is deduced from Theorem 3.4 there: under the same assumptions, the function $I_\infty(t) = \frac{1}{2\pi\sqrt{-1}} \int_\gamma \frac{q(z) dz}{P(z)-t}$ is rational only if it vanishes identically. Pakovich remarks that the assumptions hold whenever P and q are polynomials.

Corollary 3.2. Let $e_1 < \dots < e_k$ be nonnegative rational numbers and $c \in \mathbb{C}^k$. If $M_n(e; c) = 0$ for every $n \geq 1$, then $c = 0$. In particular, a polynomial $f \in \mathbb{C}[x]$ with $\int_0^1 f^n dx = 0$ for all $n \geq 1$ is zero.

Proof. Write $e_i = p_i/s$ with integers $p_i \geq 0$ and $s \geq 1$, and put $P(z) = \sum_i c_i z^{p_i} = f_{e,c}(z^s)$ and $q(z) = sz^{s-1}$. The substitution $x = z^s$ gives $M_n(e; c) = \int_0^1 P(z)^n q(z) dz$ for $n \geq 0$ (for $n = 0$ both sides equal 1). If P is constant, then $f_{e,c}$ is a constant function, equal to $M_1(e; c) = 0$, and $c = 0$ by Lemma 2.2(a). If P is not constant, then P and q are polynomials, $P(\infty) = \infty$ and $q^{-1}\{\infty\} \subseteq \{\infty\} = P^{-1}\{\infty\}$. Theorem 3.1 with $\gamma = [0, 1]$ and $i_0 = 1$ then gives $\int_0^1 q(z) dz = 0$, which is false, since this integral equals 1. $\square$

Corollary 3.2 is not new. The substitution $z = (1+t)/2$ turns $\int_0^1 P(z)^n s z^{s-1} dz$ into a constant multiple of $\int_{-1}^1 P((1+t)/2)^n (1+t)^{s-1} dt$. So Corollary 3.2 is the special case $\alpha = 0$, $\beta = s - 1$ of a theorem of van den Essen and Zhao [vdEZ13, Theorem 6.8(i)]. That theorem concerns the Jacobi weights $(1-t)^\alpha(1+t)^\beta$ on $(-1, 1)$ with integers $\alpha, \beta \geq 0$, needs the vanishing only for all large n , and is deduced there from the same Theorem 3.4 and Corollary 3.5 of [Pak13]. For the Lebesgue measure on an interval (the case $s = 1$) the statement also follows from [FPYZ11, Proposition 4.2] with $\lambda = 1/2$, as noted in [vdEZ13, Theorem 6.8(ii)]; that proposition treats the weights $(1-t^2)^{\lambda-1/2}$ with $\lambda \in \frac{1}{2}\mathbb{Z}_{\geq 0}$. The case $s = 1$ is also [FPYZ11, Corollary 4.1], which is deduced there from [FPYZ11, Theorem 4.1], a generalised form of an argument of M. Boyarchenko; [vdEZ13] mentions his unpublished proof for $[0, 1]$ with the Lebesgue measure. We include the short deduction from Theorem 3.1 for completeness.

Theorem 3.3. *For every $k \geq 1$ there is an integer $N_0(k)$ such that the following holds. If $e_1 < \dots < e_k$ are nonnegative rational numbers, $c \in \mathbb{C}^k$ and $M_n(e; c) = 0$ for $1 \leq n \leq N_0(k)$, then $c = 0$. In particular $N(k) \leq N_0(k) < \infty$.*

Proof. Let $A_k = \mathbb{Q}[\epsilon_1, \dots, \epsilon_k, \gamma_1, \dots, \gamma_k]$ be the polynomial ring in $2k$ indeterminates, and regard the polynomials P_n of (2) as elements $P_n(\epsilon, \gamma)$ of A_k . For real e with $e_1 \geq 0$ and $c \in \mathbb{C}^k$ we have $P_n(e, c) = D_n(e)M_n(e; c)$ and $D_n(e) \geq 1$. The ideals $I_N = (P_1, \dots, P_N) \subseteq A_k$ form an ascending chain. Since A_k is Noetherian (Hilbert's basis theorem), there is $N_0 = N_0(k)$ with $I_N = I_{N_0}$ for all $N \geq N_0$. Hence for every N there are $h_{N,1}, \dots, h_{N,N_0} \in A_k$ with $P_N = \sum_{j \leq N_0} h_{N,j} P_j$.

Now let e and c be as in the statement. Then $P_n(e, c) = D_n(e)M_n(e; c) = 0$ for $n \leq N_0$, so $P_N(e, c) = 0$ for every N , and $M_N(e; c) = P_N(e, c)/D_N(e) = 0$ for every $N \geq 1$. By Corollary 3.2, $c = 0$. Every polynomial with at most k monomials is of the form $f_{e,c}$ with distinct $e_i \in \mathbb{Z}_{\geq 0}$, so $N(k) \leq N_0(k)$. $\square$

Remark 3.4 (Effectivity). The number $N_0(k)$ is the index at which the chain (I_N) becomes stationary, and Hilbert's basis theorem gives no bound for it. So Theorem 3.3 is not effective. We know no explicit upper bound for $N(k)$ for any $k \geq 4$; for $k \leq 3$ see Theorem 1.3(c).

Remark 3.5 (Bounded degree). For $P, q \in \mathbb{C}[z]$ let $\tilde{m}_i = \int_0^1 P^i q dz$, $i \geq 0$. By [BB15, Theorem 1], if $\deg P \leq d_P$, $\deg q \leq d_q$ and $\tilde{m}_0 = \dots = \tilde{m}_{K-1} = 0$ with $K = 2 + d_q + 3(d_P - 1)^2$, then $\tilde{m}_i = 0$ for all i . Taking $P = q = f$ with $\deg f \leq d$, so that $\tilde{m}_i = M_{i+1}$, and using Corollary 3.2, we get: if $\deg f \leq d$ and $M_1 = \dots = M_{3d^2-5d+5} = 0$, then $f = 0$. This bound grows with the degree and gives no bound in terms of the number of monomials. The finiteness of the corresponding index for bounded degree also follows from Noetherianity, as remarked in [BB15]; in Theorem 3.3 the exponents are treated as variables instead.

4 The lower bound and one or two monomials

Proposition 4.1. (a) *Let $k \geq 2$ and let $e_1 < \dots < e_k$ be real with $e_1 \geq 0$. There is $c \in \mathbb{C}^k \setminus \{0\}$ with $M_1(e; c) = \dots = M_{k-1}(e; c) = 0$. Hence $N(k) \geq k$ for every $k \geq 1$.*

(b) *If the e_i are integers and $N(k-1) \leq k-1$, then every c as in (a) has all coordinates nonzero. So $N^=(k) \geq k$ whenever $N(k-1) \leq k-1$.*

(c) *Every real $e_1 > -1$ has property (H₁), and every pair $-1/2 < e_1 < e_2$ has property (H₂). Hence $N(1) = N^=(1) = 1$ and $N(2) = N^=(2) = 2$.*

Proof. (a) The forms $M_1(e; \cdot), \dots, M_{k-1}(e; \cdot)$ are $k-1$ homogeneous polynomials in k variables. By the projective dimension theorem [Har77, Chapter I, Theorem 7.2] they have a common zero in $\mathbb{P}^{k-1}(\mathbb{C})$, that is, a common zero $c \neq 0$. If the e_i are integers, then $f_{e,c}$ is a nonzero polynomial (Lemma 2.2(a))

with at most k monomials and $M_1 = \dots = M_{k-1} = 0$. So $N(k) > k - 1$. For $k = 1$ the bound $N(1) \geq 1$ holds by definition.

(b) If $c_i = 0$ for some i , then $f_{e,c}$ is a nonzero polynomial with at most $k - 1$ monomials and $M_1 = \dots = M_{k-1} = 0$, contrary to $N(k - 1) \leq k - 1$.

(c) $M_1(e; c) = c_1/(1 + e_1)$. For $k = 2$ write $c_i = (1 + e_i)y_i$. If $M_1 = 0$, then $y_2 = -y_1$ by Lemma 2.2(b), so $c = y_1(1 + e_1, -(1 + e_2))$ is a multiple of a real vector, and $M_2 = 0$ forces $c = 0$ by Lemma 2.2(c). With (a) and (b) this gives the values of $N(1), N(2), N^=(1)$ and $N^=(2)$. $\square$

5 Three monomials

Let $k = 3$ and $e = (e_1, e_2, e_3)$. Put $c(r) = (1 + e_1, (1 + e_2)r, -(1 + e_3)(1 + r))$, the vector with $c_i = (1 + e_i)y_i$ and $(y_1, y_2, y_3) = (1, r, -1 - r)$. For $n \in \{2, 3\}$, by (2),

$$P_n(e, c(r)) = \sum_{|\alpha|=n} \binom{n}{\alpha} (1 + e_1)^{\alpha_1} (1 + e_2)^{\alpha_2} (1 + e_3)^{\alpha_3} r^{\alpha_2} (-1 - r)^{\alpha_3} \prod_{|\beta|=n, \beta \neq \alpha} (1 + \beta \cdot e), \quad (3)$$

a polynomial in r with coefficients in $\mathbb{Z}[e_1, e_2, e_3]$, and $P_n(e, c(r)) = D_n(e)M_n(e; c(r))$ when $ne_1 > -1$. We write $P_n(r)$ for $P_n(e, c(r))$. Write $P_2 = Ar^2 + Br + C$ and $P_3 = pr^3 + qr^2 + sr + t$, and let

$$R(e) = \det \begin{pmatrix} A & B & C & 0 & 0 \\ 0 & A & B & C & 0 \\ 0 & 0 & A & B & C \\ p & q & s & t & 0 \\ 0 & p & q & s & t \end{pmatrix} \in \mathbb{Z}[e_1, e_2, e_3] \quad (4)$$

be the Sylvester resultant of P_2 and P_3 with formal degrees 2 and 3. If $A \neq 0$ and ρ_1, ρ_2 are the roots of P_2 , then $R = A^3 P_3(\rho_1) P_3(\rho_2)$. All constants below refer to this normalisation of R .

Proposition 5.1 (exact computation). *The polynomial R has 10426 terms, of total degrees 18 to 45, and content 4, i.e. the greatest common divisor of its coefficients is 4. Moreover the following identities hold in $\mathbb{Q}[u, v, w]$.*

- (i) $R(u, u + v, u + v + w) = 4(vw(v + w))^6 R_1(u, v, w)$, where R_1 has 3290 terms, all with positive integer coefficients, total degrees 0 to 27, and $R_1(0, 0, 0) = 29$.
- (ii) $R(-\frac{1}{3} + u, -\frac{1}{3} + u + v, -\frac{1}{3} + u + v + w) = 4(vw(v + w))^6 R_2(u, v, w)$, where R_2 has 3115 terms, all with positive rational coefficients, total degrees 6 to 27, and the term of lowest degree of $R_2(u, 0, 0)$ is $\frac{200000}{177147} u^6$.
- (iii) $R(u, u + 1 + v, u + 2 + v + w)$ has 13400 terms, all with positive integer coefficients, and its constant term is $R(0, 1, 2) = 82556485632000 = 4 \cdot 20639121408000$.

The computation is exact. The coefficients of r^0, r^1, r^2 in P_2 have 64, 59, 64 terms, and those of $r^0, \dots, r^3$ in P_3 have 278, 296, 296, 278 terms. The determinant (4) expands into 13 monomials in A, B, C, p, q, s, t ; this expansion was checked symbolically against the identity $A^2 R = \lambda^2 C - \lambda \mu B + A \mu^2$, where $A^2 P_3 \equiv \lambda r + \mu \pmod{P_2}$, which also proves $R = A^3 P_3(\rho_1) P_3(\rho_2)$. For (i)–(iii) the substitutions are made in P_2 and P_3 before R is formed, the division by $(vw(v + w))^6$ is exact, and each identity is confirmed by multiplying back. In (ii) we put $E = 3e$, so that each factor $1 + \beta \cdot e$ in (3) becomes $(3 + \beta \cdot E)/3$ and each factor $1 + e_i$ becomes $(3 + E_i)/3$. A term of P_n has $n + \binom{n+2}{2} - 1$ such factors, so clearing these denominators multiplies P_2 by 3^7 and P_3 by 3^{12} . Since R is homogeneous of degree 3 in the coefficients of P_2 and of degree 2 in those of P_3 , it is multiplied by $3^{3 \cdot 7 + 2 \cdot 12} = 3^{45}$. The three versions agree with $R(e)$ at random points, and $R(e)$ agrees with $D_2^3 D_3^2 \text{Res}(M_2, M_3)$ computed from (1) in rational arithmetic. The verification is described at the end of the paper.

Proof of Theorem 1.4. Let $-1/3 < e_1 < e_2 < e_3$ and suppose that $M_1 = M_2 = M_3 = 0$ for $f = f_{e,c}$. All three integrals converge, since $3e_1 > -1$, and $1 + \beta \cdot e \geq 1 + |\beta|e_1 > 0$ for $|\beta| \leq 3$. Write $c_i = (1 + e_i)y_i$; then $y_1 + y_2 + y_3 = 0$ by Lemma 2.2(b).

If $y_1 = 0$, then $c = y_2(0, 1 + e_2, -(1 + e_3))$ is a multiple of a real vector, and $M_2 = 0$ forces $c = 0$ by Lemma 2.2(c).

Let $y_1 \neq 0$. By homogeneity we may assume $y_1 = 1$, so $(y_1, y_2, y_3) = (1, r, -1 - r)$ for some $r \in \mathbb{C}$, $c = c(r)$, and $P_2(r) = P_3(r) = 0$. Here $f_{e,c(r)} = g_0 + rg$ with $g_0 = (1 + e_1)x^{e_1} - (1 + e_3)x^{e_3}$ and $g = (1 + e_2)x^{e_2} - (1 + e_3)x^{e_3}$, so the leading coefficient of P_2 is $A = D_2(e) \int_0^1 g^2 dx > 0$. Thus r is a root of P_2 at which P_3 vanishes, and $R(e) = A^3 P_3(\rho_1) P_3(\rho_2) = 0$.

On the other hand, put $u = e_1 + \frac{1}{3}$, $v = e_2 - e_1$ and $w = e_3 - e_2$, which are all positive. All coefficients of R_2 are positive, so by Proposition 5.1(ii)

$$R(e) = 4(vw(v+w))^6 R_2(u, v, w) \geq 4(vw(v+w))^6 \cdot \frac{200000}{177147} u^6 > 0.$$

This contradiction shows $c = 0$. □

Proof of Theorem 1.3. (a) is Theorem 3.3. (b) is Proposition 4.1(a), together with Proposition 4.1(b) and the values $N(1) = 1$, $N(2) = 2$, $N(3) = 3$ from (c). (c) The values for $k \leq 2$ are Proposition 4.1(c). By Theorem 1.4, every integer vector $0 \leq e_1 < e_2 < e_3$ has property (H_3) , so $N(3) \leq 3$, and $N(3) \geq 3$ by Proposition 4.1(a). Finally $N^-(3) \leq N(3) = 3$, and $N^-(3) \geq 3$ by Proposition 4.1(b). □

Remark 5.2. (1) If $e_1 \geq 0$ and $e_2 - e_1 \geq 1$, $e_3 - e_2 \geq 1$, in particular for integer exponents, part (iii) of Proposition 5.1 alone suffices: then $u = e_1$, $v = e_2 - e_1 - 1$ and $w = e_3 - e_2 - 1$ are nonnegative, so $R(e)$ is at least the constant term 82556485632000. Part (i) gives the quantitative bound $R(e) \geq 116((e_2 - e_1)(e_3 - e_2)(e_3 - e_1))^6$ for all real $0 \leq e_1 < e_2 < e_3$.

(2) The bound $-1/3$ in Theorem 1.4 cannot be lowered in the same form: if $e_1 \leq -1/3$ and $c_1 \neq 0$, then $\int_0^1 |f|^3 dx = \infty$.

(3) With g_0, g as in the proof of Theorem 1.4, for real r we have $P_2(r) = D_2(e) \int_0^1 (g_0 + rg)^2 dx > 0$. So P_2 has no real root, $\rho_2 = \bar{\rho}_1$, and $R(e) = A^3 |P_3(\rho_1)|^2 \geq 0$. So a positivity certificate such as Proposition 5.1 is the natural form of a proof.

6 Lacunary exponents

Lemma 6.1. *Let $k \geq 2$, and let $0 \leq e_1 < \dots < e_{k-1}$ be real numbers such that $e' = (e_1, \dots, e_{k-1})$ has property (H_{k-1}) . Then there is $E_0 > e_{k-1}$ such that $(e_1, \dots, e_{k-1}, E)$ has property (H_k) for every real $E \geq E_0$.*

Proof. For $c' \in \mathbb{C}^{k-1}$ put $g(c') = \max_{1 \leq n \leq k-1} |M_n(e'; c')|^{1/n}$. This is a continuous function with $g(\lambda c') = |\lambda|g(c')$, and by (H_{k-1}) it is positive on the compact set $\{\|c'\|_1 = 1\}$. So there is $K > 0$ with $\|c'\|_1 \leq K g(c')$ for all c' .

Let $E > e_{k-1}$, and suppose that $c = (c', c_k) \neq 0$ satisfies $M_n((e', E); c) = 0$ for $1 \leq n \leq k$. If $c_k = 0$, then $c' \neq 0$ contradicts (H_{k-1}) ; so $c_k \neq 0$, and by homogeneity $c_k = 1$. Put $f' = f_{e', c'}$ and $\rho = \|c'\|_1$, so that $|f'| \leq \rho$ on $(0, 1]$ because all $e_i \geq 0$. Expanding $(f' + x^E)^n$ gives

$$0 = M_n((e', E); c) = M_n(e'; c') + \sum_{j=1}^n \binom{n}{j} I_{n,j}, \quad I_{n,j} = \int_0^1 x^{jE} f'(x)^{n-j} dx, \quad |I_{n,j}| \leq \frac{\rho^{n-j}}{1 + jE}.$$

Put $\varepsilon = (1 + E)^{-1/(k-1)} \in (0, 1)$.

For $1 \leq n \leq k-1$ the display and $1 + jE \geq 1 + E$ give $|M_n(e'; c')| \leq \sum_{j \geq 1} \binom{n}{j} \rho^{n-j} / (1 + E) \leq (1 + \rho)^n / (1 + E)$, hence $|M_n(e'; c')|^{1/n} \leq (1 + \rho)(1 + E)^{-1/n} \leq (1 + \rho)\varepsilon$. Thus $\rho \leq K(1 + \rho)\varepsilon$, and if $K\varepsilon \leq \frac{1}{2}$, then $\rho \leq 2K\varepsilon \leq 1$.

For $n = k$ we isolate the term $j = k$, which is $I_{k,k} = 1/(1 + kE)$. Using $|M_k(e'; c')| \leq \int_0^1 |f'|^k \leq \rho^k$, $\rho^{k-j} \leq \rho$ for $j < k$, and $1 + jE \geq 1 + E$, we get

$$\frac{1}{1 + kE} \leq \rho^k + \frac{(2^k - 2)\rho}{1 + E}.$$

Multiply by $1 + E$ and use $(1 + E)/(1 + kE) \geq 1/k$ and $(1 + E)\varepsilon^k = \varepsilon$:

$$\frac{1}{k} \leq (1 + E)(2K\varepsilon)^k + (2^k - 2)2K\varepsilon = ((2K)^k + (2^{k+1} - 4)K)\varepsilon.$$

This is false once ε is small enough, that is, for all sufficiently large E . For such E no nonzero c exists. $\square$

Corollary 6.2. *Let $k \geq 1$. There are functions $T_1, \dots, T_{k-1}$, where T_j is defined on the j -tuples $0 \leq e_1 < \dots < e_j$ with property (H_j) , such that every real vector $0 \leq e_1 < \dots < e_k$ with $e_{j+1} \geq T_j(e_1, \dots, e_j)$ for $j = 1, \dots, k-1$ (each value $T_j(e_1, \dots, e_j)$ is then defined, by induction) has property (H_k) . In particular, if $f \in \mathbb{C}[x]$ has at most k monomials with such exponents and $M_1 = \dots = M_k = 0$, then $f = 0$.*

Proof. Every $e_1 \geq 0$ has property (H_1) . If $(e_1, \dots, e_j)$ has property (H_j) , Lemma 6.1 gives $T_j(e_1, \dots, e_j)$ such that $(e_1, \dots, e_{j+1})$ has property (H_{j+1}) whenever $e_{j+1} \geq T_j(e_1, \dots, e_j)$. Induction on j proves the claim. $\square$

With Theorem 1.4, Lemma 6.1 shows that for every real triple $0 \leq e_1 < e_2 < e_3$ the vector (e_1, e_2, e_3, E) has property (H_4) for all large E . The thresholds are not explicit, since K is not.

7 Four and five monomials: finite-range computations

Exact certificates. For $k = 4$ and fixed exponents, put $c_i = (1 + e_i)y_i$ and $y_4 = -(y_1 + y_2 + y_3)$. Then M_2, M_3, M_4 are ternary forms in (y_1, y_2, y_3) with rational coefficients, and (H_4) says that they have no common zero in $\mathbb{P}^2(\mathbb{C})$. The coefficient of y_1^2 in M_2 is the constant $\int_0^1 ((1 + e_1)x^{e_1} - (1 + e_4)x^{e_4})^2 dx > 0$. On the chart $y_3 = 1$ we reduce M_3 and M_4 modulo M_2 as polynomials in y_1 , which gives $M_n \equiv \lambda_n y_1 + \mu_n$ with $\lambda_n, \mu_n \in \mathbb{Q}[y_2]$. At a common zero, y_2 is a common root of the norms $N_n(y_2) = \prod_{\rho} (\lambda_n \rho + \mu_n)$, $n = 3, 4$, where ρ runs over the two roots of $M_2(\cdot, y_2, 1)$; these are polynomials of degree at most 6 and 8 in y_2 . So $\gcd(N_3, N_4) = 1$ excludes this chart. On the line $y_3 = 0$ the same computation with $y_2 = 1$ gives two numbers that are not both zero, and at the point $(1 : 0 : 0)$ we have $M_2 \neq 0$. All arithmetic is in $\mathbb{Q}$.

For $k = 5$, an exact rational change of coordinates $c = Cz$ gives $M_1 \equiv 0$ and $M_2 = \sum_{j=1}^4 d_j z_j^2$ with rational $d_j > 0$. On the chart $z_4 = 1$ we write $M_n \equiv \lambda_n z_1 + \mu_n$ modulo M_2 , that is, after replacing z_1^2 by $-(d_2 z_2^2 + d_3 z_3^2 + d_4)/d_1$, with $\lambda_n, \mu_n \in \mathbb{Q}[z_2, z_3]$. A common zero of $M_2, \dots, M_5$ is a common zero of the norms $N_n = \mu_n^2 - \lambda_n^2 z_1^2$ (with z_1^2 replaced in the same way) and of $W_{mn} = \lambda_m \mu_n - \lambda_n \mu_m$. The polynomials $N_3, W_{34}, W_{35}, N_4, W_{45}$ have total degrees at most 6, 6, 7, 8, 8, so the resultants $\text{Res}_{z_2}(N_3, W_{34})$, $\text{Res}_{z_2}(N_3, W_{35})$ and $\text{Res}_{z_2}(N_4, W_{45})$, taken with these formal degrees, are polynomials in z_3 of degree at most 36, 42 and 64. We check that $\text{Res}_{z_3}(\text{Res}_{z_2}(N_3, W_{34}), \text{Res}_{z_2}(N_3, W_{35}) + \sigma \text{Res}_{z_2}(N_4, W_{45})) \neq 0$ for a fixed integer σ , with formal degrees 36 and 64. On the chart $z_4 = 0, z_3 = 1$ the same reduction gives polynomials N_3, W_{34}, W_{35} in z_2 alone, and we check that $\text{Res}_{z_2}(N_3, W_{34})$ or $\text{Res}_{z_2}(N_3, W_{35})$ is

nonzero. On the chart $z_4 = z_3 = 0$, $z_2 = 1$ the three numbers N_3, W_{34}, W_{35} are not all zero, and at the point $(1 : 0 : 0 : 0)$ we have $M_2 = d_1 \neq 0$.

After the exact rational set-up, these eliminations are carried out in $\mathbb{F}_p$ with $p = 2^{61} - 1$. This is sound for the following reasons. The coefficients of the forms M_n in the coordinates z and the numbers d_j are checked to be p -integral; otherwise the exponent set is reported as not certified. Our coordinates have $d_1 = M_2(e; (1+e_1, 0, 0, 0, -(1+e_5)))$, a positive rational number whose numerator is smaller than p for integer exponents with $e_5 \leq 1000$. So $1/d_1$ is p -integral as well, and every rational coefficient that occurs is p -integral. Reduction modulo p is a ring homomorphism on p -integral rationals, and a determinant is a polynomial in its entries. The resultants in z_2 are Sylvester determinants, evaluated at 37, 43 and 65 distinct integer values of z_3 and interpolated by Lagrange interpolation over $\mathbb{F}_p$. By the degree bounds above, this gives the reductions modulo p of the rational resultants, and the final resultant in z_3 is again a Sylvester determinant. So a nonzero residue certifies that the corresponding rational resultant is nonzero.

k	exponent sets certified to have property (H_k)	count
4	all $0 \leq e_1 < e_2 < e_3 < e_4 \leq 50$	249,900
4	random sets with $e_4 \leq 10^4$ (largest 9825)	2000
5	all $0 \leq e_1 < \dots < e_5 \leq 16$	6188
5	random sets with $e_5 \leq 1000$ (largest 972)	300

Table 1: Finite-range computations. No exponent set in these ranges failed.

As negative controls, replacing M_k by a combination of lower moments that vanishes at their common zeros makes the certificates fail, as it must. These computations prove (H_4) and (H_5) only for the exponent sets in Table 1. They do not prove $N(4) = 4$ or $N(5) = 5$.

Numerical experiments. The following observations are numerical (floating point after an exact rational reduction) and are not proofs. Let $\rho_k(e)$ be the minimum, over the nonzero solutions c of $M_1 = \dots = M_{k-1} = 0$, of $|M_k(e; c)|$ divided by $(\int_0^1 |f_{e,c}|^{2a} \int_0^1 |f_{e,c}|^{2b})^{1/2}$, where $a = \lfloor k/2 \rfloor$ and $b = \lceil k/2 \rceil$. In the cases we checked there were $(k-1)!$ solutions up to scaling, the Bézout number. When there are finitely many, $\rho_k(e) > 0$ if and only if e has property (H_k) .

- $k = 4$. Over all integer sets with $e_4 \leq 30$ the minimum is $\rho_4 \approx 0.0984$, at $(1, 2, 3, 4)$. Local minimisation over real $e_1 \geq 0$ with all gaps at least 1 gives 0.0976, near $(0.305, 1.305, 2.305, 3.305)$; with arbitrary gaps it gives 0.045, as the exponents approach the confluent point $e_1 = \dots = e_4 \approx 0.856$.
- $k = 5$ and $k = 6$. $\rho_5(0, 1, 2, 3, 4) \approx 0.0092$, the minimum over integer sets with $e_5 \leq 11$ is ≈ 0.0065 , and $\rho_6(0, 1, \dots, 5) \approx 6 \cdot 10^{-4}$ (all 120 solutions found by homotopy continuation).
- Real exponents with $e_1 < 0$. For $k = 4$ and real $-1/4 < e_1 < \dots < e_4$, local minimisation in double precision reaches values of ρ_4 of order 10^{-14} to 10^{-13} , which is the floor of double-precision arithmetic. For example, $\rho_4 \approx 2.0 \cdot 10^{-14}$ at the point e^* with

$$\begin{aligned} (e_1^*, e_2^*) &= (-0.18271900153639156, -0.1809796744363755), \\ (e_3^*, e_4^*) &= (769.0851161513428, 825.7277931734245), \end{aligned}$$

where $1 + 4e_1^* \approx 0.269$. This value needs all the digits shown: the rounded point $(-0.18272, -0.18098, 769.085, 825.728)$ gives $\rho_4 \approx 9.4 \cdot 10^{-6}$. With $e_1 = e_1^*$ and $e_2 = e_2^*$

fixed, Newton's method in (e_3, e_4) , in 60-digit and in 80-digit arithmetic, converges quadratically to the point

$$(e_3, e_4) = (769.08511614717282823869\dots, 825.72779317804078768512\dots),$$

where ρ_4 drops to the level of the working precision (below $3 \cdot 10^{-44}$ and below 10^{-64} , respectively). Along a continuous branch of solutions of $M_1 = M_2 = M_3 = 0$, normalised by $c_1 = 1$, the value M_4 has winding number 1 around 0 when (e_3, e_4) runs over circles of radius 0.5 and 2 about (e_3^*, e_4^*) , and over the circle of radius 10^{-6} about this point. This indicates that M_4 vanishes on the branch at this point, so that (H_4) fails there, unlike the case $k = 3$ (Theorem 1.4). We have not proved this. If it is correct, a proof of $N(4) = 4$ must use $e_1 \geq 0$ or the integrality of the exponents.

8 Concluding remarks

The margins ρ_k above shrink by a factor of about 10 to 15 from one k to the next, but they stay positive in all cases computed for integer exponents. Two questions remain.

Problem 8.1. *Is $N(k) = k$ for every k ? Equivalently, does every integer vector $0 \leq e_1 < \dots < e_k$ have property (H_k) ? This holds for $k \leq 3$, for the ranges of Table 1, and for sufficiently lacunary exponents (Corollary 6.2).*

Problem 8.2. *Find an explicit upper bound for $N(k)$ for $k \geq 4$.*

For $k = 4$ an analogue of Proposition 5.1 would need the resultant of three ternary forms whose coefficients are polynomials in four exponent variables. This is a very large polynomial, and we did not attempt the computation. If the numerics of Section 7 are correct, such a certificate cannot be positive on the whole region $e_1 > -1/4$.

Verification. The claims were checked as follows.

1. A self-contained program by the author, `check_theorem_C.py` (standard library only, about one minute), recomputes R from (3) and (4) and checks every statement of Proposition 5.1: the number of terms, degree range and content of R ; the three identities, by exact division and multiplication back; the positivity of all coefficients; and the constants 29 , $\frac{200000}{177147}$ and 82556485632000 . It checks the 13-term expansion of the Sylvester determinant symbolically, compares R with $D_2^3 D_3^2 \text{Res}(M_2, M_3)$ at 60 random rational points, and checks $\gcd(M_2(r), M_3(r)) = 1$ for all 2300 integer triples with $e_3 \leq 24$.
2. The first computation of R and of the certificates used separate code, which also compared R with Sylvester determinants at 300 random integer triples and solved $M_1 = M_2 = M_3 = 0$ exactly in quadratic fields for all integer triples with $e_3 \leq 60$.
3. A first independent verification run, which was AI-assisted, re-read the source problem and re-derived step by step the proofs of Theorem 3.3 (including the use of Theorem 3.1), Proposition 4.1 and Lemma 6.1. It recomputed R with separate code (memoized Laplace expansion, validated against direct rational determinants at 80 points) and reproduced the coefficient counts, the content 4 and the constants of Proposition 5.1. It also reproduced numerically the value $\rho_4(1, 2, 3, 4) \approx 0.0984$.

4. For the $k = 4$ entries of Table 1 there are two programs: the original one, which uses the reduction described above, and a separate program by the author, which uses resultants and interpolation instead. Both certify all 249,900 exponent sets with $e_4 \leq 50$. The $k = 5$ entries were re-run with the original program; the full range $e_5 \leq 16$ was also certified by the different method of item 5.
5. A second independent verification run, also AI-assisted, re-derived all the proofs step by step and checked the cited results again in the arXiv sources of [Gas21, Pak13, BB15, vdEZ13]. With new code it recomputed R and the identities (i)–(iii) of Proposition 5.1 from (3) and (4). It tested (H_3) directly, in exact arithmetic, on 17,660 exponent triples: all integer triples with $e_3 \leq 40$, random integer triples with $e_3 \leq 10^6$, and rational triples with $e_1 > -1/3$, including triples close to $-1/3$, close to confluence and with very large gaps. It certified all sets with $e_4 \leq 50$ for $k = 4$ and all sets with $e_5 \leq 16$ for $k = 5$ by a different criterion: after one coordinate is eliminated with $M_1 = 0$, the forms $M_2, \dots, M_k$ in the remaining $k - 1$ variables have no common projective zero if and only if their Macaulay matrix in degree $k(k - 1)/2 + 1$ has full rank, and this rank was computed modulo $2^{31} - 1$. Finally, it confirmed the numbers of the real example for $k = 4$ in Section 7 in 60-digit arithmetic. A separate program by the author, `check_k4_real_zero.py` (80-digit arithmetic, with a different elimination), gives the same values.

Scope and priority. Theorem 1.3(a) answers part (i) of Gasull’s problem [Gas21, Problem 27] affirmatively, but not effectively. Part (ii) is answered for $k \leq 3$, and $N(k) \geq k$ holds for all k . For $k \geq 4$ the value of $N(k)$, and any explicit upper bound for it, remain open. The results for $k = 4$ and $k = 5$ in Section 7 are finite-range computations, and the failure of the real-exponent statement for $k = 4$ is a numerical observation. We read the source problem in the arXiv version of [Gas21], and [Pak13] and [vdEZ13] in their arXiv versions; we did not consult the journal versions of these three papers. We read [FPYZ11] in its published version. We searched arXiv, Crossref, OpenAlex and zbMATH, including the works citing [Gas21] and [Pak13], and made three web searches. We found no earlier statement of Theorem 1.3(a) and no earlier value of $N(k)$ for $k \geq 3$. The argument for Theorem 1.3(a) is a routine Noetherianity argument, and experts may regard part (i) as easy. This negative search is not a proof of priority.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments and the certificates, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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