

A Sharp Three-Cycle Bound for Two-Zone Affine Systems with a Virtual Focus

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10 October 2026

Abstract

We prove that a planar two-zone piecewise-affine system with a straight switching line, two strict off-line foci, and at least one virtual focus has at most three isolated crossing limit cycles. The estimate includes nonhyperbolic cycles and all values of the switching offset. Known examples with three cycles show that the bound is sharp. We use the Poincaré half-map and Khovanskiĭ framework of Carmona, Fernández-Sánchez and Novaes. The additional argument rationally parametrizes contacts of their conic, locates them according to two normalized equilibrium parameters, and proves a trace-sign monotonicity inequality. An adaptive choice between the full opposite-sign half-plane and an energy half-plane then gives the required intersection bound. Degenerate conics are removed by a one-parameter perturbation. This addresses the stated focus family, not the general non-focus problem or all entries of Gasull's comparison table. This manuscript is unrefereed.

1 Statement and relation to the half-map method

Consider the piecewise-affine system

$$\begin{aligned} \dot{x} &= tx - y, & \dot{y} &= dx - a, & x < 0, \\ \dot{x} &= zx - y + b, & \dot{y} &= ex - c, & x > 0. \end{aligned} \tag{1}$$

The switching line is $x = 0$. A crossing periodic orbit meets this line transversely and does not contain a sliding segment. A crossing limit cycle is an isolated crossing periodic orbit. We assume

$$d, e > 0, \quad t^2 < 4d, \quad z^2 < 4e, \quad actz \neq 0. \tag{2}$$

Thus the two affine extensions have strict foci, rather than centers, and neither equilibrium lies on the switching line. The left equilibrium has $x = a/d$ and is virtual if $a > 0$; the right one has $x = c/e$ and is virtual if $c < 0$. No sign condition is imposed on b .

Theorem 1.1. *Suppose (2) holds and $a > 0$ or $c < 0$. System (1) has at most three crossing limit cycles, including nonhyperbolic ones. This bound is sharp within this class.*

The canonical form (1) is the usual two-zone Lienard form. The theorem may equivalently be applied after the line-preserving canonical reduction described in [4, Section 2.1]. The broader question of the maximum number of crossing cycles occurs in Gasull's comparison between quadratic systems and two-zone affine systems [1, Problem 17]. Its question-marked numbers are lower bounds,

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not established upper bounds. Theorem 1.1 treats a specified subclass of the affine entry; it does not settle the combined table, non-focus configurations, sliding cycles, or the quadratic-system entries.

The method is based on prior work. Carmona, Fernández-Sánchez and Novaes developed a uniform-bound argument using half-maps and an extended Khovanskii intersection theorem [2], and subsequently organized it into an algebraic method with worked examples [4]. We use their conic and tangent fields. The endpoint-sum sign used below is also known [3, Proposition 4.1]. The additional ingredients here are an explicit parametrization of the conic contacts, a sign-restricted monotonicity certificate, and a choice of containing half-plane adapted to the trace. We do not claim the half-map machinery or the known three-cycle examples as new. No absolute priority claim is made.

2 The conic and its contacts

Put

$$A = \frac{a^2}{d}, \quad L = \frac{at}{d}, \quad C = \frac{c^2}{e}, \quad R = \frac{cz}{e}. \quad (3)$$

Then $A, C > 0$, $L^2 < 4A$, $R^2 < 4C$, and

$$W_L(y) = y^2 - Ly + A, \quad W_R(y) = y^2 - Ry + C$$

are positive for every real y . These are the half-map boundary quadratics divided by d, e , respectively.

Write $u > 0 > v$ for left boundary heights, and set

$$s = u + v, \quad p = uv < 0, \quad q = (u - b)(v - b) = p - bs + b^2. \quad (4)$$

The map $(u, v) \mapsto (s, p)$ is a diffeomorphism from the open fourth quadrant onto $H = \{p < 0\}$. Its inverse is

$$u = \frac{s + \sqrt{s^2 - 4p}}{2}, \quad v = \frac{s - \sqrt{s^2 - 4p}}{2}.$$

The normalized conic from [4, Section 2.4] is

$$\begin{aligned} F(s, p) = & A(Cb + b^3 + b^2R) - Ab(2b + R)s + Abs^2 \\ & + [-C(b + L) + b(A - b^2 + bL) + (A - b^2)R]p \\ & + (C - A + b^2 - bL + bR)sp + (L - R - b)p^2. \end{aligned} \quad (5)$$

Equivalently,

$$(u - v)F = v(u - b)W_L(u)W_R(v - b) - u(v - b)W_L(v)W_R(u - b). \quad (6)$$

The left cubic tangent field in height coordinates is $X_L = (-vW_L(u), -uW_L(v))$. Its pushforward has components

$$V_s = -ps + 2Lp - As, \quad V_p = -2p^2 + Lps - A(s^2 - 2p). \quad (7)$$

Define $G = F_sV_s + F_pV_p$. At a regular point of $F = 0$, a contact with the left tangent field is exactly a common zero of F, G . Normalization rescales the original F, G by nonzero constants, so does not change their common zeros.

For the algebraic arguments until Section 4, assume $b \neq 0$.

Lemma 2.1. *Every common zero of F, G in H satisfies $q < 0$. Define*

$$D = Ab^2s + 2Abp - Abs^2 - 2bp^2 + p^2s. \quad (8)$$

Then $D \neq 0$ at such a zero, and

$$C = \mathcal{C}(s, p) := \frac{Asq^2}{D}, \quad R = \mathcal{R}(s, p; L) := -\frac{(s - 2b)[Ab(s^2 - bs - p) + (b - L)p^2]}{D}. \quad (9)$$

Conversely, substituting these values with $D \neq 0$ annihilates F, G .

Proof. On $F = 0$, equation (6) and positivity of the quadratics imply $pq > 0$: neither $u = b$ nor $v = b$ can make that equation zero, and its two nonzero terms must have the same sign. Hence $q < 0$.

The equations $F = G = 0$ are linear in C, R . Put $J = W_L(u)W_L(v) > 0$. Their coefficient determinant and Cramer numerators are respectively

$$DJ, \quad Asq^2J, \quad -(s - 2b)[Ab(s^2 - bs - p) + (b - L)p^2]J.$$

These identities follow by expanding (5) and (7). If a common zero had $D = 0$, consistency would force $s = 0$. But $D(0, p) = 2bp(A - p) \neq 0$ for $p < 0$, a contradiction. Cramer's rule proves (9); direct substitution gives the converse. In particular no admissible singular fiber was discarded. $\square$

Lemma 2.2 (Localization). *Suppose $b > 0$. Contacts in H satisfy $s < 0$ if $C < A$, and $s > 2b$ if $C > A$. If $A = C$, there are no contacts in H .*

Proof. Put $r = q/p > 0$ and $x = -p > 0$, so $s = b - (1 - r)x/b$. The equation $CD - Asq^2 = 0$ is equivalent to

$$P(r, x) := (C - Ar^2)(1 - r)x^2 + [b^2(C + Ar^2) + AC(1 - r)^2]x + ACb^2(1 + r) = 0. \quad (10)$$

Indeed, substitution into $CD - Asq^2$ gives $-xP(r, x)/b$. The linear and constant coefficients in x are strictly positive. Thus a positive root exists only when $(C - Ar^2)(1 - r) < 0$; in that case there is exactly one, since the two roots have opposite signs. It follows that r lies strictly between 1 and $\sqrt{C/A}$. For $A = C$ this is impossible.

If $C < A$, then $\sqrt{C/A} < r < 1$, and

$$P\left(r, \frac{b^2}{1 - r}\right) = \frac{2Cb^2(Ar - A - b^2)}{r - 1} > 0.$$

The unique positive root is larger than this comparison value, whence $s < 0$. If $C > A$, then $1 < r < \sqrt{C/A}$ and

$$P\left(r, \frac{b^2}{r - 1}\right) = \frac{2Ab^2r(Cr - C + b^2r)}{r - 1} > 0.$$

Now the larger root gives $s = b + (r - 1)x/b > 2b$. $\square$

3 Monotonicity of a contact curve

Proposition 3.1. *Let $b > 0$. If $C < A$ and either $L \geq 0$ or $R \geq 0$, the conic has at most one contact in H . If $C > A$ and either $L \leq 0$ or $R \leq 0$, the same conclusion holds.*

Proof. First suppose $C < A$ and $L \geq 0$. By Lemma 2.2, all contacts have $s < 0, q < 0$. In this domain each term of D is negative. For fixed $s < 0$, direct differentiation gives

$$\mathcal{C}_p = \frac{2Absq}{D^2}(-Ab^2 + 2Abs + Ap - As^2 + 2b^2p - 3bps + ps^2) < 0. \quad (11)$$

To check the sign, put $s = -y, p = -x$ with $x, y > 0$; the factor in parentheses is $-A((b+y)^2 + x) - x(b+y)(2b+y) < 0$. The endpoint values, on $p < bs - b^2$, are

$$\lim_{p \rightarrow -\infty} \mathcal{C}(s, p) = \frac{As}{s - 2b}, \quad \mathcal{C}(s, bs - b^2) = 0.$$

Consequently the level set $\mathcal{C} = C$ is a single smooth graph $p = p(s)$ on exactly the interval

$$-\infty < s < -\frac{2bC}{A - C}. \quad (12)$$

The Jacobian of the two contact parameters factors as

$$\mathcal{C}_s \mathcal{R}_p - \mathcal{C}_p \mathcal{R}_s = \frac{A^2 b^2 q}{D^3} (AU + LV + W), \quad (13)$$

where

$$\begin{aligned} U &= 6b^2ps - 3b^2s^3 - 14bps^2 + 5bs^4 - 2p^2s + 7ps^3 - 2s^5, \\ V &= -2p(2b^2p - 2b^2s^2 - 6bps + 3bs^3 + 2p^2 + 2ps^2 - s^4), \\ W &= 2p^2s(-3b^2 + 3bs + p - s^2). \end{aligned} \quad (14)$$

In the variables $s = -y, p = -x$,

$$\begin{aligned} U &= y(6b^2x + 3b^2y^2 + 14bxy + 5by^3 + 2x^2 + 7xy^2 + 2y^4) > 0, \\ W &= 2x^2y(3b^2 + 3by + x + y^2) > 0. \end{aligned}$$

If $V \geq 0$, positivity of $AU + LV + W$ follows immediately. Suppose $V < 0$ and put $V_0 = V/(-2p) < 0$. Define

$$Z = 2b^2p + b^2s^2 - 10bps + bs^3 + 2p^2 + 5ps^2 - s^4.$$

The identity

$$UW - V^2 = -2p^2(4p - s^2)qZ \quad (15)$$

has a useful strict sign certificate. Write $p = -(by + b^2 + k)$, $s = -y$, where $k = -q > 0$. Then

$$Z - V_0 = -y(4b^3 + 4b^2y + 4bk + by^2 + 3ky) < 0.$$

Thus $Z < 0$, and (15) is positive. Since $0 \leq L < 2\sqrt{A}$, we obtain

$$\begin{aligned} AU + LV + W &\geq AU - 2|V|\sqrt{A} + W \\ &= U \left(\sqrt{A} - \frac{|V|}{U} \right)^2 + W - \frac{V^2}{U} > 0. \end{aligned}$$

Both q and D are negative. Equations (11) and (13) therefore give

$$\frac{d}{ds}\mathcal{R}(s, p(s); L) = -\frac{\mathcal{C}_s\mathcal{R}_p - \mathcal{C}_p\mathcal{R}_s}{\mathcal{C}_p} > 0.$$

Because (12) is one interval, a fixed value of R can occur at most once.

For the alternative $C < A, R \geq 0$, the bracket in (9) is strictly positive whenever $L \leq b$. As $s - 2b < 0$ and $D < 0$, this would force $R < 0$. Hence every contact in this alternative has $L > b > 0$, and the preceding argument applies.

Finally use the reflection

$$(A, L, C, R, s, p) \mapsto (C, -R, A, -L, 2b - s, q). \quad (16)$$

It preserves the conic equation. To justify preservation of contacts, introduce the right shifted field

$$X_R = (-(v - b)W_R(u - b), -(u - b)W_R(v - b)).$$

The determinant of X_L, X_R is $(u - v)F$. On $F = 0$ in H , both fields are nonzero and collinear, since $q < 0$ and the quadratics are positive. Thus contact with one is equivalent to contact with the other; this is also [4, Remark 4]. The height reflection $(u, v) \mapsto (b - v, b - u)$ exchanges the fields up to nonzero scaling and induces (16). Applying the first part to the reflected parameters proves the $C > A$ assertion. $\square$

4 Containing domains and cycle counts

We first recall the exact intersection principle needed from [2, Theorem 4], also stated in [4, Theorem 2]. On a simply connected open planar domain, a smooth curve with N noncompact components and k contacts with a smooth vector field has at most $N + k$ isolated intersections with a separating orbit. Here a noncompact orbit is separating if its finite limiting endpoints lie on the boundary of the domain. Compact components of the curve are allowed.

Lemma 4.1. *For $b \neq 0$, a nondegenerate conic $F = 0$ has at most one noncompact component in $H = \{p < 0\}$. In $H \cap \{s < h\}$ or $H \cap \{s > h\}$ it has at most two noncompact components, for any real h .*

Proof. We have

$$F(s, 0) = Ab((s - b)^2 - R(s - b) + C) \neq 0. \quad (17)$$

Moreover the coefficient of s^2 is $Ab \neq 0$. Thus no asymptotic direction of a hyperbola can have zero p component. A hyperbola has two ends tending to $p = +\infty$ and two tending to $p = -\infty$. No branch can join an upward end to a downward end without crossing $p = 0$. Exactly one branch is therefore below that line. A parabola is connected and cannot cross the line; an ellipse contributes no noncompact component before clipping.

On a lower hyperbola branch or a parabola, the line $s = h$ has at most two intersections counted with multiplicity. A transverse intersection changes the retained side; a tangency that splits a retained interval uses multiplicity two. Hence at most two open intervals remain on either side. A clipped ellipse gives at most one open arc or a compact component. This proves the second bound. It is important not to keep $N = 1$ after clipping, which can split a branch into two components. $\square$

Let $y_L(u)$ be the forward left half-map and $y_R(w)$ the backward right half-map when $b = 0$. Their shifted displacement is

$$\delta_b(u) = y_R(u - b) + b - y_L(u). \quad (18)$$

These maps and the following facts are supplied by [4, Sections 2.2–2.3]. They are analytic in the interiors of their domains, their derivatives are negative, and their graphs are orbits of the corresponding cubic tangent fields. Interior zeros of δ_b correspond to crossing periodic orbits; simple and isolated zeros correspond to hyperbolic and isolated crossing cycles, respectively. At a zero with $v = y_L(u) < \min(0, b)$,

$$\delta'_b(u) = \frac{(u - v)F(s, p)}{v(v - b)W_L(u)W_R(u - b)}. \quad (19)$$

The factor multiplying F is positive.

Lemma 4.2. *If $a > 0$ and (2) holds, the full graph of y_L is a separating orbit in H and in $H \cap \{ts < 0\}$, after the sum-product change of coordinates.*

Proof. For a virtual left focus, the half-map starts at $(u, v) = (0, 0)$, is defined for every $u > 0$, and tends to infinity as $u \rightarrow \infty$. This is the endpoint characterization in [4, Remark 2]: $a > 0$ gives the zero endpoint and positivity of W_L excludes a finite other endpoint. In particular the graph has no finite limiting endpoint inside the fourth quadrant.

For completeness, its endpoint-sum sign has an elementary energy proof. Along the left linear system put

$$E = (tx - y)^2 + dx^2 - 2ax, \quad \dot{E} = 2t(tx - y)^2.$$

At the line $x = 0$, $E = y^2$. Integration over a left passage from u to v gives $\text{sgn}(v^2 - u^2) = \text{sgn } t$. The integrated square is strictly positive, since otherwise x would be constant, contradicting transverse endpoints. As $v - u < 0$, this gives $ts < 0$; the same sign relation is [3, Proposition 4.1].

The sum-product graph thus lies wholly in $H \cap \{ts < 0\}$, a convex open domain. Its only finite limiting endpoint is $(s, p) = (0, 0)$, on the boundary of both domains. This is the required separating property. $\square$

Proof of Theorem 1.1. First assume $a > 0$ and $b > 0$. Then $\text{sgn } L = \text{sgn } t$. For a nondegenerate conic, Lemmas 2.2, 4.1, 4.2 and Proposition 3.1 give the following choices. Here k bounds contacts and N bounds noncompact components.

Parameters	Trace	Domain	k	N
$C < A$	$t > 0$	H	≤ 1	≤ 1
$C < A$	$t < 0$	$H \cap \{s > 0\}$	0	≤ 2
$C > A$	$t < 0$	H	≤ 1	≤ 1
$C > A$	$t > 0$	$H \cap \{s < 0\}$	0	≤ 2
$C = A$	either	H	0	≤ 1

Every row has $N + k + 1 \leq 3$. The zero-contact rows exclude contacts by localization; they do not assert that the conic itself is absent. These choices work uniformly for b in a positive neighborhood, with the other six physical coefficients fixed.

We give the parameter argument to cover all exceptional cases and to avoid assuming alternation across omitted multiple zeros. Choose a compact interval inside the displacement domain containing

any proposed four cycle zeros. It lies in a common open displacement domain for sufficiently small changes in b , because the crossings are transverse. Throughout the resulting rectangle,

$$\partial_b \delta_b(u) = 1 - y'_R(u - b) > 0. \quad (20)$$

Thus the zero set is a smooth curve. By regular-value perturbation of its projection to b , arbitrarily close parameter values have only simple zeros in the chosen compact interval. These are finitely many and their derivative signs alternate. By (19), m such zeros force at least $m - 1$ isolated intersections between the conic and the left half-map graph. Analyticity gives isolation: otherwise F would vanish identically on this orbit, contradicting its nonzero values at simple cycle zeros. The extended Khovanskii bound therefore gives $m - 1 \leq N + k \leq 2$.

It remains to remove conic degeneracy. The determinant of the homogeneous 3×3 matrix of the quadratic polynomial (5), as a polynomial in b , has degree five and leading coefficient $-AC/4 \neq 0$. It has only finitely many zeros. Thus the preceding argument applies at dense nearby regular parameter values with nondegenerate conics. Four simple cycle zeros at any other positive b would persist at such a parameter, a contradiction. We have proved a uniform bound of three on simple zeros for every b in a positive neighborhood.

The monotone-parameter lemma [2, Lemma 5], or [4, Lemma 3], now applies: a smooth family analytic in u , with strictly positive derivative in b and a uniform upper bound on simple zeros, has the same upper bound on isolated zeros. All its hypotheses hold on the common rectangle by (20). In elementary terms, odd zeros persist under perturbation and even zeros split into pairs in one of the two directions; one direction produces at least the original total number of simple zeros. Consequently the bound three includes nonhyperbolic isolated cycles.

For $b < 0$, the transformation $(x, y, \tau) \mapsto (x, -y, -\tau)$, with τ denoting time, changes (t, z, b) to $(-t, -z, -b)$ and leaves a, c, d, e fixed. It preserves virtuality and isolated crossing cycles. At $b = 0$, four simple cycle zeros would persist for small nonzero b , which has just been excluded. The same monotone-parameter lemma then excludes four arbitrary isolated zeros at $b = 0$.

Finally, the spatial transformation $(x, y) \mapsto (-x, b - y)$ exchanges the subsystems, sends (a, c, t, z, d, e) to $(-c, -a, z, t, e, d)$, and keeps b unchanged. A virtual right focus becomes a virtual left focus. This establishes the upper bound under either virtual-focus alternative.

For sharpness, the known Huan–Yang example, whose analytic existence proof is credited to Llibre–Ponce and whose exact count is established in [4, Section 3.1], has canonical parameters

$$(a, c, t, z, d, e, b) = \left(-\frac{117}{200}, -\frac{2861}{5000}, -\frac{3}{10}, \frac{19}{250}, \frac{117}{200}, \frac{2861}{250000}, \frac{9}{10} \right).$$

It has a real left focus and a virtual right focus, satisfying all strict hypotheses of the theorem. Its three cycles provide the required lower bound. This construction is prior work. $\square$

5 Scope and reproducibility

The proof is uniform over the stated focus class and does not infer a general bound from a finite parameter scan. In particular, contacts of the algebraic conic need not lie on the actual half-map. They were counted on containing domains, with the separating-orbit and component hypotheses checked separately. Clipping can increase the number of components; this is why the two zero-contact rows allow $N = 2$.

Exact symbolic checks accompany the source. They verify the conic and Lie derivative identities, Cramer determinants and substitutions, localization polynomial, derivative and Jacobian

factorizations, mirror symmetry, and the degree-five degeneracy determinant. Saved rational examples provide regressions. These computations support the displayed algebra; they do not replace the analytic and topological proof or constitute proof-assistant formalization.

This work was developed with AI assistance in literature retrieval, algebraic exploration, drafting and consistency checks. It received an originating-researcher self-audit and has not received independent peer review or formal verification. The prior half-map theory, intersection theorem, endpoint-sum sign and three-cycle examples are explicitly attributed. Bounded literature searches do not certify absolute priority.

References

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