

On the Billiard Problem of Dragović and Radnović for Two Concentric Half-Circles: A Periodicity Criterion and the Periods in Special Cases

Alper Ferudun*

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Abstract

Dragović and Radnović posed the following problem (Arnold Math. J. 1 (2015)). A billiard table is bounded by two concentric half-circles of radii $R_1 > R_2$, lying on opposite sides of a common diameter, and by the two segments of this diameter between them. Consider the trajectories tangent to a fixed concentric circle of radius $r < R_2$, and let $\rho_i = \frac{1}{\pi} \arccos(r/R_i)$ be the two rotation numbers. Given ρ_1 and ρ_2 , are these trajectories periodic, how many periodic and non-periodic regions do they form on the boundary, and what are the periods? We give a partial answer. An unfolding reduces the billiard to an explicit map of two circles, which is one of McMullen's coupled rotations; its first-return map is an exchange of three arcs of a circle, and the period of a trajectory, counted as the number of reflections off the arcs and the segments, is given by a formula in terms of the reduced orbit. Through this reduction published theorems apply; the next two statements are such applications and are not claimed as new. By Boshernitzan's theorem that minimal interval exchange transformations of rank two are uniquely ergodic (alternatively, by McMullen's theorems on measured foliations of genus two), all these trajectories are periodic if and only if $\rho_1 + \rho_2$ is rational. By the decomposition of interval exchange transformations into periodic and minimal components, there are at most two regions; if $\rho_1 + \rho_2$ is irrational, there is exactly one non-periodic region and at most one periodic region. Each region is symmetric under the mirror symmetry of the table. For irrational $\rho_1 + \rho_2$ we decide whether the periodic region exists, and find its period, when ρ_1 is rational, when $\rho_1 + q\rho_2 \in \frac{1}{2}\mathbb{Z}$ and when $p\rho_1 + \rho_2 \in \frac{1}{2}\mathbb{Z}$ for integers $p, q \geq 2$. For $\rho_1 + \rho_2 = \frac{1}{2}$ the periods are $3s - 2$ (s even) or $6s - 4$ (s odd), where s is $\lfloor 1/(2\rho_2) \rfloor$ or $\lfloor 1/(2\rho_2) \rfloor + 1$. All examples of Dragović and Radnović are recovered. A closed formula for the periods when $\rho_1 + \rho_2$ is rational and different from $\frac{1}{2}$, and the existence of the periodic region when the only relations are $p\rho_1 + q\rho_2 \in \frac{1}{2}\mathbb{Z}$ with $p, q \geq 2$, remain open; tables with more arcs are not treated. This is an unrefereed note.

1 Introduction

1.1 The problem

A billiard trajectory in a disc of radius R is tangent to a concentric circle, its caustic. If the caustic has radius r , the trajectory is periodic if and only if the *rotation number* $\rho = \frac{1}{\pi} \arccos \frac{r}{R}$ is rational, and its period is then the denominator of ρ [DR15, §1]. In their problem contribution [DR15], Dragović and Radnović ask what happens in a table bounded by two concentric half-circles and two segments lying on one diameter [DR15, §2, Fig. 1]. The figure there, the remark that the table has two reflex

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: <https://github.com/AlperTheKing>.
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angles [DR15, Remark 4], and the example with radii $2R$ and $R\sqrt{2}$ in the companion paper [DR14, §4.1] show that the two half-discs lie on opposite sides of the common diameter. Up to a rigid motion the table is therefore

$$D = \{z \in \mathbb{C} : |z| \leq R_1, \operatorname{Im} z \geq 0\} \cup \{z \in \mathbb{C} : |z| \leq R_2, \operatorname{Im} z \leq 0\}, \quad R_1 > R_2 > 0$$

(Figure 1). Its boundary consists of the *large arc* $A_1 = \{R_1 e^{i\varphi} : 0 \leq \varphi \leq \pi\}$, the *small arc* $A_2 = \{R_2 e^{i\varphi} : \pi \leq \varphi \leq 2\pi\}$ and the two *segments* $[R_2, R_1]$ and $[-R_1, -R_2]$ of the real axis. The corners $\pm R_1$ are right angles, and the corners $\pm R_2$ are reflex. The interval $(-R_2, R_2)$ of the real axis lies in the interior of D ; we call it the *gap*.

All links of a billiard trajectory in D are tangent to one circle $|z| = r$ (see Section 2). We fix r and assume $r < R_2$, so that both rotation numbers

$$\rho_i = \frac{1}{\pi} \arccos \frac{r}{R_i} \quad (i = 1, 2), \quad 0 < \rho_2 < \rho_1 < \frac{1}{2},$$

are defined. The source asks [DR15, Question 1]: “Given ρ_1, ρ_2 , determine if the billiard trajectories are periodic”, and to find the periods if they are. Its Question 2 asks for an arithmetic criterion, in terms of ρ_1 and ρ_2 , for the number of non-periodic regions on the boundary, the number of periodic regions, and the corresponding periods. Question 3 asks the same for tables bounded by finitely many concentric arcs and radial segments; we do not treat it. The problem is record AMR-044-0001 of the UnsolvedMath collection [ula26], where it is listed as open.

Three conventions of [DR15, DR14] are needed to read these questions.

Period. The source gives examples, but no definition. In [DR14, §4.1] the 12-periodic trajectories for $(\rho_1, \rho_2) = (\frac{1}{3}, \frac{1}{4})$ are described as having four bounces on the smaller circle, six on the bigger one and one on each segment. So the period is the number of reflections off the whole boundary, segments included, during one period of the motion.

Corners. At a right-angled corner the reflection is defined by continuity (the particle returns along the same line) and counts as two bounces. At a reflex corner it is not defined. A trajectory is *regular* if it never reaches a reflex corner [DR14, §1].

Regions. By [DR14, Theorem 6.1] the boundary is the closure of the union of finitely many pairwise disjoint invariant sets, each a finite union of open arcs, such that the trajectories with reflection points in one of these sets are all periodic with the same period, or all non-periodic with reflection points dense in the set. For $(\frac{1}{3}, \frac{1}{4})$, for instance, there are two regions, with periods 12 and 7 [DR15, Example 1]; see Figure 1.

1.2 Results

Throughout the paper we put

$$\alpha := 2\rho_1, \quad \lambda := 2\rho_2, \quad N := \lfloor 1/\lambda \rfloor, \quad \delta := 1 - N\lambda,$$

so that $0 < \lambda < \alpha < 1$, $N \geq 1$ and $0 \leq \delta < \lambda$. Let $s_1 := \alpha$, $s_2 := \lambda$ and $\mathbb{T} := \mathbb{R}/\mathbb{Z}$. (In [DR15] the letter α denotes a central angle; we do not use that notation.)

Theorem 1.1 (reduction). *Let γ be a regular trajectory in D whose links are tangent to $|z| = r$.*

- (a) *Unfolding. Reflect in the real axis those links of γ which run clockwise around 0; the sense changes exactly at the reflections off the segments (Section 2.2). The result is a broken line $\tilde{\gamma}$ whose links, the chords of γ , are tangent to $|z| = r$ and run counterclockwise, and whose vertices are the reflections of γ off the two arcs. Let the k -th chord ($k \in \mathbb{Z}$) touch the caustic at*

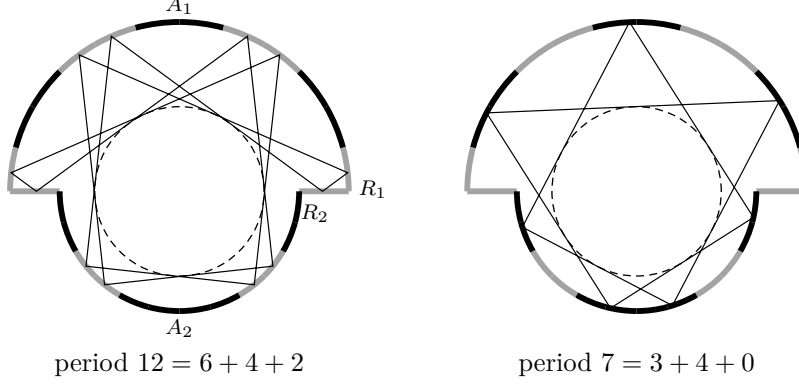


Figure 1: The table D for $R_1 = 2$, $R_2 = \sqrt{2}$ with the caustic $r = 1$ (dashed), so that $(\rho_1, \rho_2) = (\frac{1}{3}, \frac{1}{4})$ as in [DR15, Example 1] and [DR14, §4.1], and its two regions (grey and black). Left: a trajectory of period 12 in the grey region, with 6 reflections off the large arc A_1 , 4 off the small arc A_2 and 2 off the segments. Right: a trajectory of period 7 in the black region.

$re^{i\theta_k}$, with $0 < \theta_{k+1} - \theta_k < \pi$, let it start at a reflection off the arc A_{b_k} , $b_k \in \{1, 2\}$, and put $u_k := \theta_k/\pi + \rho_2$. Then $u_k \bmod 1 \notin \{0, \lambda\}$ and

$$(u_{k+1}, b_{k+1}) = \Phi(u_k, b_k), \quad \Phi(u, b) := (u + s_f, f), \quad f := \begin{cases} 3 - b & \text{if } u \bmod 1 \in (0, \lambda), \\ b & \text{if } u \bmod 1 \in (\lambda, 1). \end{cases} \quad (1)$$

The pair $(u_k \bmod 2, b_k)$ determines the point and the direction of γ after its k -th reflection off an arc, and conversely.

- (b) Period. γ is periodic if and only if the sequence $(u_k \bmod 1, b_k)_{k \in \mathbb{Z}}$ is periodic. Let L be its minimal period, let m_j be the number of $k \in \{0, \dots, L-1\}$ with $b_{k+1} = j$, let v be the number of these k with $(b_k, b_{k+1}) = (1, 2)$, and let $n := u_L - u_0 = m_1\alpha + m_2\lambda$, an integer. Then during one period γ is reflected

$$\begin{aligned} & m_1 \text{ times by } A_1, \quad m_2 \text{ times by } A_2, \quad n - 2v \text{ times by the segments,} & \text{if } n \text{ is even,} \\ & 2m_1 \text{ times by } A_1, \quad 2m_2 \text{ times by } A_2, \quad 2(n - 2v) \text{ times by the segments,} & \text{if } n \text{ is odd.} \end{aligned} \quad (2)$$

So its period is $m_1 + m_2 + n - 2v$ if n is even and $2(m_1 + m_2 + n - 2v)$ if n is odd.

- (c) First return. The first-return map of Φ to the states with $b = 1$ is

$$G : \mathbb{T} \rightarrow \mathbb{T}, \quad G(u) = u + \alpha + \mu(u)\lambda, \quad \mu(u) := \begin{cases} N + 1 & \text{on } [0, \delta), \\ N & \text{on } [\delta, \lambda), \\ 0 & \text{on } [\lambda, 1). \end{cases} \quad (3)$$

G is a bijection of the circle and a translation on each of these three arcs. If $\delta \neq 0$ it is discontinuous at the three points of $\Sigma := \{0, \delta, \lambda\}$; if $\delta = 0$ it is the rotation by α . The trajectory γ is periodic if and only if the G -orbit O of $u_k \bmod 1$ (for any k with $b_k = 1$) is periodic, and then $m_1 = |O|$, $m_2 = \sum_{u \in O} \mu(u)$ and $v = |O \cap (0, \lambda)|$ in (2).

- (d) Models. Φ is the coupled rotation $F_t = R \circ S_t$ of McMullen [McM15, §9] with parameters

$$(L_1, \tau_1, L_2, \tau_2; t) = (1, 2\rho_1, 1, 2\rho_2; 2\rho_2). \quad (4)$$

It is also the first-return map, to the horizontal sides, of the flow of slope 1 on the translation surface of genus two obtained from the L-shaped polygon $[0, 1] \times [0, \alpha] \cup [1, 2] \times [0, \lambda]$ by identifying opposite sides; this surface has a single cone point.

Theorem 1.2 (periodicity criterion). *Every regular trajectory with the caustic $|z| = r$ is periodic if and only if $\rho_1 + \rho_2$ is rational.*

To state the result on regions we use the following facts (Sections 3 and 5). The regular trajectories with the given caustic correspond to the regular orbits of an exchange of arcs of a circle, which is a double cover of G . The circle decomposes, up to finitely many points, into finitely many invariant sets, each a finite union of arcs, of two kinds: *periodic components*, in which all orbits are periodic with the same data (m_1, m_2, v, n) , and *minimal components*, in which no orbit is periodic and every orbit is dense. The *region* of a component is the set of all points of ∂D at which its trajectories are reflected; it is called periodic or non-periodic accordingly. Up to end points of arcs these are the regions of [DR14, Theorem 6.1].

Theorem 1.3 (regions). (a) *There are at most two regions. Different regions are disjoint, and each region is symmetric with respect to the imaginary axis, the axis of symmetry of D . A regular trajectory which is reflected at a point of a region has all its reflection points in this region. All trajectories of a periodic region have the same period, given by (2).*

- (b) *If $\rho_1 + \rho_2 \in \mathbb{Q}$, there are one or two regions, and they are periodic; there are two if $\rho_2 \notin \mathbb{Q}$.*
- (c) *If $\rho_1 + \rho_2 \notin \mathbb{Q}$, there is exactly one non-periodic region, in which the reflection points of every regular trajectory are dense, and at most one periodic region.*

For comparison, the source records a topological upper bound for the number of regions, which is 6 for a table with two reflex angles [DR15, Remark 1]. For $\rho_1 + \rho_2 \notin \mathbb{Q}$ the remaining question is whether the periodic region exists, and what its period is. We write $\{x\} := x - \lfloor x \rfloor$.

Theorem 1.4 (the periodic region when $\rho_1 + \rho_2$ is irrational). *Let $\rho_1 + \rho_2 \notin \mathbb{Q}$ and $N = \lfloor 1/(2\rho_2) \rfloor$.*

- (a) *If $1, \rho_1, \rho_2$ are linearly independent over $\mathbb{Q}$, or if $\rho_2 \in \mathbb{Q}$, there is no periodic region.*
- (b) *Let $\rho_1 = s/t$ in lowest terms. A periodic region exists if and only if $\rho_2 < \frac{1}{2t}$ (t odd), respectively $\rho_2 < \frac{1}{t}$ (t even). Its trajectories have period $t + 2s$: they are reflected t times by A_1 , $2s$ times by the segments and never by A_2 .*
- (c) *Let $\rho_1 + q\rho_2 = \frac{n}{2}$ with integers $q \geq 2$ and n . A periodic region exists if and only if q divides N or $N + 1$. If q divides $s \in \{N, N + 1\}$, its trajectories are reflected s/q times by A_1 , s times by A_2 and $sn/q - 2$ times by the segments, with period $P_0 := s/q + s + sn/q - 2$, if sn/q is even; if sn/q is odd, all these numbers are doubled.*
- (d) *Let $p\rho_1 + \rho_2 = \frac{n}{2}$ with integers $p \geq 2$ and n . A periodic region exists if and only if*

$$n = p \quad \text{and} \quad \left\{ \frac{1}{2\rho_2} \right\} > 1 - \frac{2}{p}.$$

Its trajectories are reflected Np times by A_1 , N times by A_2 and $Np - 2$ times by the segments, with period $(2p + 1)N - 2$, if Np is even; if Np is odd, all these numbers are doubled. In particular there is no periodic region if $2\rho_1 + \rho_2 = \frac{1}{2}$, and there is always one if $2\rho_1 + \rho_2 = 1$.

- (e) *In general, if a periodic region exists, then the integer relations $i\rho_1 + j\rho_2 = \frac{k}{2}$ are the multiples of one relation $p\rho_1 + q\rho_2 = \frac{n}{2}$ with $p \geq 1$, $q \geq 0$, $n \geq 1$, and $q \leq (N + 1)p$; moreover the data of the region satisfy $(m_1, m_2) = (cp, cq)$ for some integer $c \geq 1$.*

Cases (b), (c) and (d) cover all relations with $p = 1$ or $q \leq 1$. For $p, q \geq 2$ the condition in (e) is not sufficient, and we have no criterion (Section 9).

Theorem 1.5 ($\rho_1 + \rho_2 = \frac{1}{2}$). *Let $\rho_1 + \rho_2 = \frac{1}{2}$ and $N = \lfloor 1/(2\rho_2) \rfloor$; then $N \geq 2$. Every regular trajectory is periodic. For $s = N$, and also for $s = N + 1$ unless $1/(2\rho_2)$ is an integer, there is exactly one periodic region whose trajectories are reflected*

$$\begin{cases} s \text{ times by each arc and } s - 2 \text{ times by the segments (period } 3s - 2) & \text{if } s \text{ is even,} \\ 2s \text{ times by each arc and } 2s - 4 \text{ times by the segments (period } 6s - 4) & \text{if } s \text{ is odd,} \end{cases}$$

and there are no other regions.

For rational pairs (ρ_1, ρ_2) the regions and periods are the cycle data of an explicit permutation (Proposition 8.1; Table 2 lists them for denominators up to 7), and Proposition 8.2 gives a finite certificate for the absence of further periodic trajectories. Table 1 compares the results with all examples of [DR15, DR14].

(ρ_1, ρ_2)	stated in the sources	here: period = $A_1 + A_2 + \text{segments}$
1 $(\frac{1}{3}, \frac{1}{4})$	periods 12 and 7 [DR15, Ex. 1]	$12 = 6 + 4 + 2$, $7 = 3 + 4 + 0$
2 $(\frac{1}{4}, \frac{1}{6})$	periods 5 and 6 [DR15, Ex. 2]	$5 = 2 + 3 + 0$, $6 = 4 + 0 + 2$
3 $(\frac{1}{3}, \frac{1}{5})$	periods 13 and 21 [DR15, Ex. 2]	$13 = 6 + 5 + 2$, $21 = 9 + 10 + 2$
4 $(\frac{1}{4}, \frac{1}{\sqrt{30}})$	6, and non-periodic [DR15, Ex. 3]	$6 = 4 + 0 + 2$; one non-periodic region
5 $(\frac{5-\sqrt{5}}{10}, \frac{\sqrt{5}}{10})$	periods 14 and 4 [DR14, §4.3]	$14 = 6 + 6 + 2$, $4 = 2 + 2 + 0$
6 $(\frac{5}{11} + \frac{1}{22\pi}, \frac{5}{11} - \frac{1}{220\pi})$	all dense [DR14, §8.2]	no periodic region

Table 1: The examples of the source and of the companion paper. In the last column a period is written as the sum of the numbers of reflections off the large arc, the small arc and the segments. Rows 1–3 follow from Proposition 8.1, row 4 from Theorem 1.4(b) and Theorem 1.3(c), row 5 from Theorem 1.5 ($N = 2$), and row 6 from Theorem 1.4(c) ($q = n = 10$, $N = 1$).

1.3 What is new, what is not, and what stays open

No answer to the questions of [DR15] for this table was found in print; the searches are described in the paragraph “Scope and priority” at the end. The companion paper [DR14] contains an interval exchange map of six or eight intervals for the full table, the examples in rows 1, 2, 5 and 6 of Table 1 and a sufficient condition for minimality, but neither the criterion of Theorem 1.2, nor the bound of two regions, nor period formulas.

The two structural statements are applications of published theorems once Theorem 1.1 is available, and we do not claim them as new mathematics. Theorem 1.2 follows from the classical decomposition of interval exchange transformations into periodic and minimal components, in the precise form given by Boshernitzan [Bos88, Proposition 2.9, Theorem 2.16], and from his theorem that a minimal interval exchange transformation of rank two is uniquely ergodic [Bos88, Theorem 1.1]; this is the proof given in Section 4. Alternatively it follows from McMullen’s theorems on measured foliations of rank two in genus two [McM15, Corollaries 3.2 and 7.4, Proposition 9.3]; see Remark 4.3, where the case of quadratic irrationals is also derived from [McM03]. The bound on the number of components (at most two, and at most one periodic component next to a minimal one) rests on the same decomposition [Bos88, §2], in which the components are determined by the connections between the discontinuities; given this, the count for three discontinuities in Lemma 3.3 is short (compare the estimate of the

number of minimal components after [Bos88, Remark 2.18]). It corresponds to the count of saddle connections on a translation surface of genus two with one cone point (compare [McM15, §3]). What, to our knowledge, is not in print is the reduction itself, that is Theorem 1.1 with the dictionary (4) and the formula (2) for the period of the billiard trajectory; the statement on the regions on the boundary, Theorem 1.3(a); and the explicit results of Theorem 1.4(b)–(d) and Theorem 1.5.

The following questions remain open (Section 9): a closed formula for the periods when $\rho_1 + \rho_2$ is rational and different from $\frac{1}{2}$ (they are computed by a finite procedure, Proposition 8.1 and Section 10); the existence of the periodic region when the generating relation $p\rho_1 + q\rho_2 = \frac{n}{2}$ has $p, q \geq 2$; and Question 3 of the source.

2 The reduction

In this section we prove Theorem 1.1. Let $c(z) := \bar{z}$ and $\beta_i := \arccos(r/R_i) = \pi\rho_i$, so that $0 < \beta_2 < \beta_1 < \frac{\pi}{2}$, $\alpha = 2\beta_1/\pi$ and $\lambda = 2\beta_2/\pi$.

2.1 Tangent lines

For $\theta \in \mathbb{R}$ let ℓ_θ be the oriented line $t \mapsto e^{i\theta}(r + it)$, $t \in \mathbb{R}$. It touches the circle $|z| = r$ at $re^{i\theta}$ and runs counterclockwise around 0. Since $\ell_\theta(t) = \sqrt{r^2 + t^2} e^{i(\theta + \arctan(t/r))}$, the argument increases strictly along ℓ_θ , from $\theta - \frac{\pi}{2}$ to $\theta + \frac{\pi}{2}$, and

$$\text{the point of } \ell_\theta \text{ with argument } \psi \text{ has modulus } r/\cos(\psi - \theta). \quad (5)$$

So, for $R > r$ and $\beta = \arccos(r/R)$, the line ℓ_θ is inside the disc $|z| < R$ exactly at the arguments ψ with $|\psi - \theta| < \beta$, and it meets the circle $|z| = R$ at the arguments $\theta \pm \beta$.

Lemma 2.1. (a) *c maps ℓ_θ onto $\ell_{-\theta}$ and reverses the orientation.*

(b) *Let a point move along ℓ_θ in the positive direction inside the disc $|z| \leq R$, and let it be reflected by the circle $|z| = R$. Then it continues along $\ell_{\theta+2\beta}$ in the positive direction, where $\beta = \arccos(r/R)$.*

Proof. (a) $c(\ell_\theta(t)) = e^{-i\theta}(r - it) = \ell_{-\theta}(-t)$.

(b) The point reaches the circle at $P = Re^{i\varphi}$ with $\varphi = \theta + \beta$. Let s_P be the reflection of the plane in the line through 0 and P . The two tangents from P to the circle $|z| = r$ touch it at the arguments $\varphi \pm \beta$, and s_P exchanges them. The billiard reflection at P maps a velocity w to $-s_P(w)$. So the reflected link lies on the tangent which touches at the argument $\varphi + \beta = \theta + 2\beta$. Both s_P and $w \mapsto -w$ reverse the sense of rotation around 0; hence the reflected link runs counterclockwise, that is, along $\ell_{\theta+2\beta}$ in the positive direction. $\square$

In particular every link of a trajectory in D is tangent to the same circle $|z| = r$, because reflections in the real axis also preserve the distance of a line from 0.

2.2 Unfolding

Let $\gamma : \mathbb{R} \rightarrow D$ be a regular trajectory, parametrised by arc length, whose links are tangent to $|z| = r$. Each link runs counterclockwise or clockwise around 0. A reflection off an arc preserves this sense (Lemma 2.1(b), applied to γ or to $c \circ \gamma$), a reflection off a segment reverses it (Lemma 2.1(a)), and a passage through the gap does not change it. Let $\varepsilon(t) := 0$ if γ runs counterclockwise at time t and $\varepsilon(t) := 1$ otherwise, and let

$$\tilde{\gamma}(t) := c^{\varepsilon(t)}(\gamma(t)).$$

This is the *unfolding* of γ . At a reflection of γ off a segment, ε changes, and $\tilde{\gamma}$ passes straight across the real axis. Hence $\tilde{\gamma}$ is a broken line whose vertices are the reflections of γ off the arcs. Each of its links, the *chords* of γ , is a link of γ or the mirror image of one, runs counterclockwise, and is therefore a piece of a line ℓ_θ traversed in the positive direction (Figure 2).

We say that $\tilde{\gamma}$ is on *sheet* $\varepsilon(t)$ at time t : on sheet 0 it moves in D , on sheet 1 in the mirror table $c(D)$. The *type* of a point $\tilde{\gamma}(t)$ off the real axis is the index $j \in \{1, 2\}$ of the half-disc of radius R_j of the current table which contains it. Thus

$$\tilde{\gamma}(t) \text{ has type } 1 \iff (\operatorname{Im} \tilde{\gamma}(t) > 0 \text{ and } \varepsilon(t) = 0) \text{ or } (\operatorname{Im} \tilde{\gamma}(t) < 0 \text{ and } \varepsilon(t) = 1). \quad (6)$$

A chord at a point of type j stays in the disc $|z| < R_j$ until it meets the real axis or the circle $|z| = R_j$, where γ is reflected by the arc A_j .

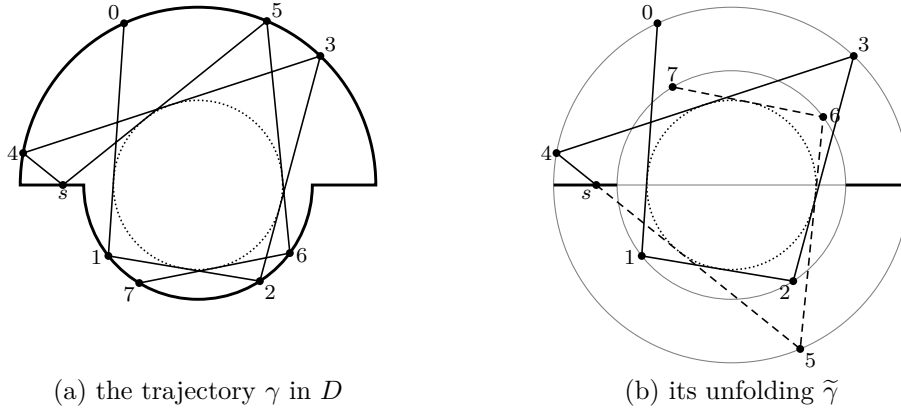


Figure 2: Unfolding, for $R_1 = 2.1$, $R_2 = 1.35$, $r = 1$. (a) A piece of a trajectory, reflected by A_1 at 0, 3, 4, 5, by A_2 at 1, 2, 6, 7 and by a segment at s . (b) Its unfolding: seven chords, tangent to the caustic (dotted) and counterclockwise; after the reflection at s it is the mirror image of γ (dashed; sheet 1). The chords $0 \rightarrow 1$, $2 \rightarrow 3$ and $5 \rightarrow 6$ cross the gap and change their type; the chord $4 \rightarrow 5$ crosses a segment and keeps the type 1.

Lemma 2.2. *Let $\tilde{\gamma}$ cross the real axis at a point of modulus ϱ . If $\varrho < R_2$, then γ passes through the gap and the type of $\tilde{\gamma}$ changes. If $\varrho > R_2$, then γ is reflected by a segment and the type of $\tilde{\gamma}$ is 1 before and after the crossing. The case $\varrho = R_2$ does not occur.*

Proof. If $\varrho < R_2$, then γ passes through the gap: ε does not change and $\operatorname{Im} \tilde{\gamma}$ changes sign, so the type changes by (6). If $\varrho > R_2$, then γ is on a segment and is reflected there. Before the reflection γ is in the upper half-disc of D , because the segments bound D from above only; so the type is 1. At the reflection both ε and the sign of $\operatorname{Im} \tilde{\gamma}$ change, and the type is 1 again by (6). If $\varrho = R_2$, then γ runs into a reflex corner, which is excluded. $\square$

Number the chords by $k \in \mathbb{Z}$ in the order in which they are traversed. Let chord k lie on ℓ_{θ_k} , and let b_k and f_k be the types at its beginning and at its end. So chord k starts at a reflection of γ off A_{b_k} , at the argument $\theta_k - \beta_{b_k}$, and ends on A_{f_k} at the argument $\theta_k + \beta_{f_k}$. We have $b_{k+1} = f_k$, and by Lemma 2.1(b) the numbers θ_k can be chosen such that

$$\theta_{k+1} = \theta_k + 2\beta_{f_k} \quad (k \in \mathbb{Z}). \quad (7)$$

Lemma 2.3. *For every k the distance d_k from θ_k to the set $\pi\mathbb{Z}$ is different from β_2 . If $d_k < \beta_2$, then $f_k = 3 - b_k$, and chord k meets the real axis exactly once, in the gap. If $d_k > \beta_2$, then $f_k = b_k$, and*

chord k contains a reflection of γ off a segment if and only if $b_k = 1$ and $d_k < \beta_1$; it does not meet the gap.

Proof. Write $\theta = \theta_k$ and $b = b_k$. By (5), at an argument $\psi \in \pi\mathbb{Z}$ the line ℓ_θ has modulus less than, equal to, or greater than R_2 according as $|\psi - \theta|$ is less than, equal to, or greater than β_2 . Whatever its types are, chord k covers the arguments in $[\theta - \beta_2, \theta + \beta_2]$. So if $d_k = \beta_2$, the chord contains a reflex corner, which is excluded.

Let $d_k < \beta_2$, say $|\psi - \theta| < \beta_2$ with $\psi \in \pi\mathbb{Z}$. The chord starts at the argument $\theta - \beta_b \leq \theta - \beta_2 < \psi$, and no other multiple of π lies between its start and ψ , because $\psi - \pi < \theta + \beta_2 - \pi < \theta - \beta_1$. So the chord reaches the argument ψ with type b and crosses the real axis there at a modulus less than R_2 . By Lemma 2.2 its type becomes $3 - b$. It then ends at the argument $\theta + \beta_{3-b}$, before the next multiple of π , because $\psi + \pi > \theta - \beta_2 + \pi > \theta + \beta_1$. Hence $f_k = 3 - b$.

Let $d_k > \beta_2$. If $b = 2$, the chord runs through the arguments in $(\theta - \beta_2, \theta + \beta_2)$, which contain no multiple of π ; so $f_k = 2$ and the chord does not meet the real axis. If $b = 1$, the chord runs, as long as its type is 1, through arguments in $(\theta - \beta_1, \theta + \beta_1)$. This interval is shorter than π , so the chord meets the real axis at most once, at a modulus greater than R_2 , and its type remains 1 by Lemma 2.2. Hence $f_k = 1$, and the chord meets the axis if and only if $d_k < \beta_1$. $\square$

Proof of Theorem 1.1(a). Put $u_k := \theta_k/\pi + \lambda/2$; note that $\lambda/2 = \rho_2$. Then $d_k < \beta_2$ if and only if $u_k \bmod 1 \in (0, \lambda)$, and $d_k > \beta_2$ if and only if $u_k \bmod 1 \in (\lambda, 1)$. With $b_{k+1} = f_k$ and (7), Lemma 2.3 gives (1).

Chord k starts at the vertex $V_k = R_{b_k} e^{i(\theta_k - \beta_{b_k})}$ of $\tilde{\gamma}$, which has type b_k . By (6) the sign of $\text{Im } V_k$ and b_k determine the sheet ε , and γ leaves the point $c^\varepsilon(V_k)$ of the arc A_{b_k} in the direction $c^\varepsilon(i e^{i\theta_k})$. Conversely, a point P of an arc A_b which is not a corner, together with a direction at P which is tangent to the caustic, determines ε (the sense of rotation), the vertex $V = c^\varepsilon(P)$ and θ modulo 2π , hence $(u \bmod 2, b)$. $\square$

For later use we note what happens when u is replaced by $u + 1$: then θ is replaced by $\theta + \pi$, the vertex V by $-V$ and, by (6), ε by $1 - \varepsilon$. The point $P = c^\varepsilon(V)$ and the direction $w = c^\varepsilon(i e^{i\theta})$ become $-\bar{P}$ and $-\bar{w}$. So

$$(u, b) \mapsto (u + 1, b) \text{ corresponds to the mirror symmetry } z \mapsto -\bar{z} \text{ of } D. \quad (8)$$

Proof of Theorem 1.1(b). A trajectory is determined by one of its points together with the direction there. So γ is periodic if and only if the sequence $(u_k \bmod 2, b_k)$ is periodic. This holds if and only if $(u_k \bmod 1, b_k)$ is periodic: if the latter has minimal period L , then by (1) the number $n = u_L - u_0 = m_1\alpha + m_2\lambda$ is an integer and $(u_{k+2L}, b_{k+2L}) = (u_k + 2n, b_k)$. The minimal period L' of $(u_k \bmod 2, b_k)$ is a multiple of L and divides $2L$; so $L' = L$ if n is even and $L' = 2L$ if n is odd.

During one period of length L' the trajectory γ is reflected L' times by the arcs, namely $(L'/L) m_j$ times by A_j . The argument of $\tilde{\gamma}(t)$ is continuous and strictly increasing. From the start of chord 0 to the start of chord L' it increases by $\theta_{L'} - \theta_0 = \pi n L'/L$, because chord k starts at the argument $\theta_k - \beta_{b_k}$ and $b_{L'} = b_0$. So $\tilde{\gamma}$ crosses the real axis exactly $n L'/L$ times in a period. By Lemma 2.2 each crossing is a passage of γ through the gap, which changes the type, or a reflection of γ off a segment, which does not. The type changes from 1 to 2 and from 2 to 1 alternately, and by Lemma 2.3 it changes on chord k exactly when $b_{k+1} \neq b_k$. So there are $2v L'/L$ passages through the gap and $(n - 2v) L'/L$ reflections off the segments in a period. This is (2). $\square$

Remark 2.4. A trajectory may pass through a right-angled corner $\pm R_1$. With the convention of the introduction this counts as one reflection off A_1 and one off a segment, and the unfolded path has a vertex on the real axis there. Everything above remains true for such trajectories, with the obvious changes in the wording of Lemmas 2.2 and 2.3 (the case $d_k = \beta_1$).

2.3 The first-return map

Proof of Theorem 1.1(c). Let $b_k = 1$ and $u := u_k \bmod 1$. If $u \in (\lambda, 1)$, then $b_{k+1} = 1$ and $u_{k+1} = u_k + \alpha$. Let $u \in (0, \lambda)$. Then $b_{k+1} = 2$, and by (1) we have $b_{k+j} = 2$ and $u_{k+j} = u_k + j\lambda$ for $1 \leq j \leq J$, and $b_{k+J+1} = 1$, $u_{k+J+1} = u_k + J\lambda + \alpha$, where $J \geq 1$ is the least integer with $(u + J\lambda) \bmod 1 \in (0, \lambda)$. To find J , note that for $1 \leq j \leq N - 1$ the number $u + j\lambda$ lies in $(j\lambda, (j+1)\lambda) \subset (\lambda, 1)$, and that

$$u + N\lambda = 1 + (u - \delta), \quad u + (N+1)\lambda = 1 + (u + \lambda - \delta).$$

If $u > \delta$, then $J = N$. If $u < \delta$, then $u + N\lambda \in (1 - \delta, 1) \subset (\lambda, 1)$ and $J = N + 1$. If $u = \delta$, chord $k + N$ passes through a reflex corner. In all cases $J = \mu(u)$. Hence the next state with $b = 1$ after $(u_k, 1)$ is $(u_k + \alpha + \mu(u)\lambda, 1)$; this proves (3) on $\mathbb{T} \setminus \Sigma$, and we extend G to $\mathbb{T}$ by (3). A run of consecutive states with $b = 2$ has at most $N + 1$ members: if $b_j = 2$ for $a \leq j < a + m$, then by (1) the numbers $u_a + i\lambda$, $0 \leq i \leq m - 2$, lie in $(\lambda, 1)$ modulo 1, and passing from one to the next does not cross an integer; so $(m - 2)\lambda < 1 - \lambda$ and $m \leq N + 1$. Hence every regular trajectory has infinitely many k with $b_k = 1$ in both directions.

Let R_α be the rotation $u \mapsto u + \alpha$ of $\mathbb{T}$, and let S be the map which is the identity on $[\lambda, 1)$ and exchanges the two arcs $[0, \delta)$ and $[\delta, \lambda)$:

$$S(u) = u + \lambda - \delta \text{ on } [0, \delta), \quad S(u) = u - \delta \text{ on } [\delta, \lambda). \quad (9)$$

Since $(N + 1)\lambda = 1 + \lambda - \delta$ and $N\lambda = 1 - \delta$, we have $G = R_\alpha \circ S$. Thus G is a right-continuous bijection of $\mathbb{T}$ and a translation on each of the arcs $[0, \delta)$, $[\delta, \lambda)$, $[\lambda, 1)$. The jumps of G at 0 , δ and λ are $\lambda - \delta$, $-\lambda$ and δ . So G is discontinuous at the three points of Σ if $\delta \neq 0$, and $G = R_\alpha$ if $\delta = 0$.

Call a point of $\mathbb{T}$ *regular* if its G -orbit avoids Σ . By the above, the regular trajectories correspond to the regular points, and a regular trajectory is periodic if and only if the orbit O of $u_k \bmod 1$ ($b_k = 1$) is periodic. In this case the sequence $(u_k \bmod 1, b_k)$ has minimal period $L = |O| + \sum_{u \in O} \mu(u)$, with $m_1 = |O|$ states which are followed by a state with $b = 1$ and $m_2 = \sum_O \mu$ states which are followed by a state with $b = 2$; and the pairs $(b_k, b_{k+1}) = (1, 2)$ correspond to the points of $O \cap (0, \lambda)$. $\square$

We shall also use the set

$$\Sigma' := G(\Sigma) = \{\alpha + \lambda - \delta, \alpha, \alpha + \lambda\} \pmod{1}, \quad (10)$$

where $G(0) = \alpha + \lambda - \delta$, $G(\delta) = \alpha$ and $G(\lambda) = \alpha + \lambda$. Geometrically, $u = 0$ and $u = \lambda$ are the chords from A_1 which pass through a reflex corner, and $u = \delta$ is the chord from A_1 whose continuation, after N reflections off A_2 , reaches a reflex corner from inside the small half-disc.

2.4 The coupled rotation and the L-shaped surface

Proof of Theorem 1.1(d). McMullen [McM15, §9] considers two circles $X_1 = \mathbb{R}/L_1\mathbb{Z}$ and $X_2 = \mathbb{R}/L_2\mathbb{Z}$ with $0 < L_1 \leq L_2$, the map R which rotates X_i by $\tau_i \in [0, L_i)$, and, for $t \in [0, L_1)$, the map S_t which exchanges the arc $[0, t)$ of X_1 with the arc $[0, t)$ of X_2 and fixes all other points; the coupled rotation is $F_t = R \circ S_t$. For $L_1 = L_2 = 1$, $\tau_1 = \alpha$, $\tau_2 = \lambda$ and $t = \lambda$, the map S_t changes the label b of a point (u, b) exactly when $u \in [0, \lambda)$, and then R adds s_f to u , where f is the new label. By (1) this is Φ , extended to the points with $u \in \{0, \lambda\}$. For the L-shaped surface see Lemma 2.5. $\square$

Let $Q := [0, 1] \times [0, \alpha] \cup [1, 2] \times [0, \lambda] \subset \mathbb{R}^2 = \mathbb{C}$, and let (X_Q, ω) be the translation surface obtained from (Q, dz) by the identifications of Figure 3(b):

$$\begin{aligned} [0, 1] \times \{0\} &\sim [0, 1] \times \{\alpha\}, & [1, 2] \times \{0\} &\sim [1, 2] \times \{\lambda\}, \\ \{0\} \times [0, \lambda] &\sim \{2\} \times [0, \lambda], & \{0\} \times [\lambda, \alpha] &\sim \{1\} \times [\lambda, \alpha]. \end{aligned}$$

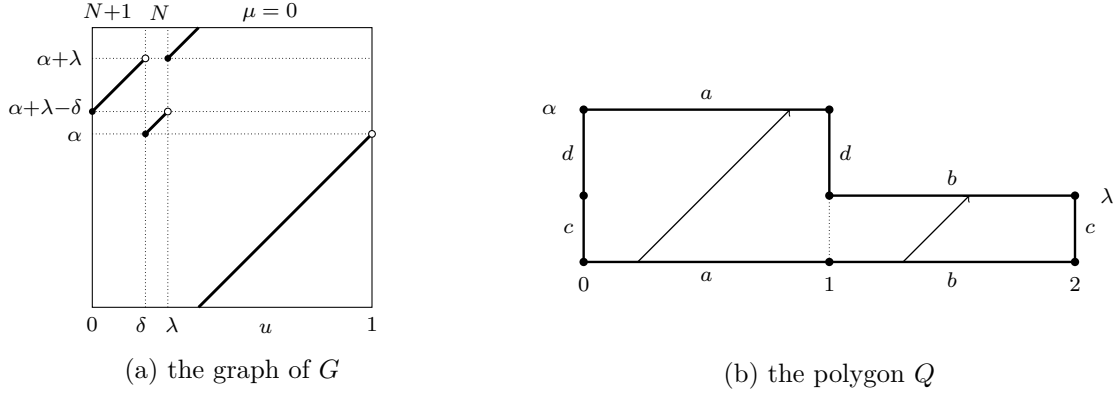


Figure 3: (a) The first-return map G of (3) for $\alpha = 0.62$, $\lambda = 0.27$ (so $N = 3$, $\delta = 0.19$): three branches of slope 1, with discontinuities at the points of $\Sigma = \{0, \delta, \lambda\}$ and values $\Sigma' = \{\alpha + \lambda - \delta, \alpha, \alpha + \lambda\}$ there. Above the graph: the number μ of reflections off the small arc. (b) The L-shaped polygon Q ; sides with the same letter are identified by translations, the eight marked points become one cone point, and the flow of slope 1 returns to the horizontal sides by the chord map Φ (Lemma 2.5).

The eight marked points of the boundary of Q are identified to a single point, at which the total angle is 6π . So X_Q has genus two, and ω has a single zero, of order two.

Lemma 2.5. *For $(u, b) \in \mathbb{T} \times \{1, 2\}$ let $\mathbf{p}(u, b) := ((b - 1) + \{u - \lambda\}, 0)$, a point of the lower side of Q . If $u \notin \{0, \lambda\}$, the leaf of slope 1 which starts upwards at $\mathbf{p}(u, b)$ returns to the lower side of Q for the first time at $\mathbf{p}(\Phi(u, b))$.*

Proof. Let $x := \{u - \lambda\}$; thus $u \in (\lambda, 1)$ means $x < 1 - \lambda$, and $u \in (0, \lambda)$ means $x > 1 - \lambda$ and $u = x + \lambda - 1$.

Let $b = 1$; the leaf starts at $(x, 0)$ in $[0, 1] \times [0, \alpha]$. If $x < 1 - \lambda$, it reaches the upper side at $(x + \alpha, \alpha)$ when $x + \alpha \leq 1$; otherwise it reaches the side $\{1\} \times [\lambda, \alpha]$ at the height $1 - x \in (\lambda, \alpha)$, continues from $(0, 1 - x)$ and reaches the upper side at $(x + \alpha - 1, \alpha)$. In both cases it returns at $(\{x + \alpha\}, 0) = \mathbf{p}(u + \alpha, 1)$. If $x > 1 - \lambda$, it reaches the line $\operatorname{Re} z = 1$ at the height $1 - x < \lambda$, enters $[1, 2] \times [0, \lambda]$ and reaches its upper side at $(x + \lambda, \lambda)$; it returns at $(x + \lambda, 0) = (1 + u, 0) = \mathbf{p}(u + \lambda, 2)$.

Let $b = 2$; the leaf starts at $(1 + x, 0)$. If $x < 1 - \lambda$, it reaches the upper side of $[1, 2] \times [0, \lambda]$ at $(1 + x + \lambda, \lambda)$ and returns at $(1 + x + \lambda, 0) = \mathbf{p}(u + \lambda, 2)$. If $x > 1 - \lambda$, it reaches the side $\{2\} \times [0, \lambda]$ at the height $1 - x$, continues from $(0, 1 - x)$ and reaches the upper side of $[0, 1] \times [0, \alpha]$ at $(x + \alpha - 1, \alpha)$; it returns at $(x + \alpha - 1, 0) = \mathbf{p}(u + \alpha, 1)$.

In all four cases the point of return is $\mathbf{p}(\Phi(u, b))$, by (1). $\square$

So Φ is the first-return map of the flow of slope 1 on (X_Q, ω) to the horizontal sides, the leaves through the cone point correspond to the trajectories through the reflex corners, and the closed leaves to the periodic regular orbits of Φ . In the notation of [McM15], this flow is the measured foliation $\mathcal{F}(\varrho)$ of the harmonic form $\varrho := \operatorname{Re}((1 + i)\omega) = dx - dy$. In the proof of [McM15, Proposition 9.1] (see Figure 7 there) the coupled rotation F_t is realised in the same way, as the first-return map of the diagonal foliation of a surface of genus two glued from two stacked rectangles with sides L_i and τ_i . We work with the polygon Q , on which it is evident that the form has a single zero. By (4) the billiards of the problem are, up to scaling, the coupled rotations with $L_1 = L_2$ and $t = \tau_2$. The invariant surface of the billiard itself has genus three [DR15, Remark 4]; in our description it corresponds to the states $(u \bmod 2, b)$, a double cover of X_Q on which the mirror symmetry (8) acts as the covering involution.

3 Exchanges of arcs of a circle

3.1 Families and minimal components

A *circle exchange* is a bijection T of $\mathbb{T}$ for which there are points $d_1 < d_2 < \dots < d_r < d_1 + 1$ ($r \geq 1$) such that T is a translation on each arc $[d_i, d_{i+1})$, where $d_{r+1} := d_1 + 1$. We fix such a set $\Sigma_T = \{d_1, \dots, d_r\}$ of *marked points*; T need not be discontinuous at all of them. T is right-continuous, and T^{-1} is a circle exchange with marked points $T(\Sigma_T)$. A point is *regular* if its orbit $\{T^k u : k \in \mathbb{Z}\}$ avoids Σ_T ; all but countably many points are regular. Let U_T be the set of regular periodic points. The map G of Theorem 1.1 is a circle exchange with $\Sigma_G = \Sigma$.

Lemma 3.1 (families). *U_T is open. Let $I = (c, c + w)$ be a connected component of U_T . Then all points of I have the same minimal period m ; T^i is a translation on I for $0 \leq i \leq m$, and T^m is the identity on I ; the arcs $I, TI, \dots, T^{m-1}I$ are pairwise disjoint components of U_T ; and $T^m c = c$, while the orbit of c meets Σ_T .*

We call $P := I \cup TI \cup \dots \cup T^{m-1}I$ a *family* of T , m its period and w its width, and we put $P^+ := \bigcup_{i < m} [T^i c, T^i c + w)$. The set U_T is the disjoint union of the families. The left end points of the arcs of different families form different periodic orbits, each containing a marked point; so there are at most $|\Sigma_T|$ families.

Proof. Let $u \in U_T$ have minimal period m . The points $T^i u$ are not marked, so T is a translation near each of them. Hence T^m is a translation near u which fixes u , that is, the identity near u , and the points near u are regular and periodic. So U_T is open; it is not $\mathbb{T}$, because it contains no marked point. Let I be the component of u . Since $T(U_T) = U_T$ and $U_T \cap \Sigma_T = \emptyset$, every T^i is continuous, hence a translation, on I , and $T^i I$ is a component of U_T . T^m is a translation on I fixing u , so $T^m = \text{id}$ on I . If $T^{m'} u' = u'$ for some $u' \in I$ and $0 < m' < m$, the translation $T^{m'}|_I$ would be the identity, contradicting the minimality of m . If $T^i I = T^j I$ with $0 \leq i < j < m$, then T^{j-i} would map the arc I onto itself by a translation and would be the identity on I : again a contradiction. Finally $T^m c = c$ by right-continuity, and $c \notin U_T$ shows that the orbit of c meets Σ_T . $\square$

The following decomposition is classical. We use it in the form given by Boshernitzan, which contains the statement on *forward* orbits of *all* points that is needed in Section 4.

Proposition 3.2 (decomposition [Bos88, Proposition 2.9, Theorems 2.16 and 2.17]). *Let $T \neq \text{id}$ be a circle exchange. There are pairwise disjoint sets $M_1, \dots, M_b \subset \mathbb{T}$ ($b \geq 0$), the minimal components of T , with the following properties.*

- (i) $\mathbb{T}$ is the disjoint union of the sets P^+ , where P runs through the families of T , and of $M_1, \dots, M_b$.
- (ii) Each M_j is a finite union of arcs $[c, c')$ whose end points lie on orbits of marked points, and $T(M_j) = M_j$.
- (iii) No point of M_j is periodic, and the forward orbit $\{T^k z : k \geq 0\}$ of every point $z \in M_j$ is dense in M_j .

The sets P^+ and M_j are the *components* of T .

Derivation from [Bos88]. Identify $\mathbb{T}$ with $X = [0, 1)$ so that $d_1 = 0$. Then T is an interval exchange transformation of X in the sense of [Bos88, §2]: a right-continuous bijection which is a translation on each of finitely many intervals $[\beta_{i-1}, \beta_i)$ partitioning X . Its discontinuities in $(0, 1)$ belong to the finite set $\mathcal{D} := (\Sigma_T \cup \{T^{-1}(0)\}) \setminus \{0\}$. As in [Bos88, Definition 2.7] we call an orbit segment $x, Tx, \dots, T^k x$

with $k \geq 1$ and $x, T^k x \in \mathcal{D}$ a T -connection, and let $\mathcal{C}$ be the finite set of all points on T -connections; by the remark after [Bos88, Theorem 2.16] the results of [Bos88, §2] hold for this choice of $\mathcal{D}$.

If $\mathcal{C} = \emptyset$, then T satisfies the infinite distinct orbit condition [Bos88, (2.5)] with respect to its discontinuities, and T is minimal by Keane's theorem, quoted as [Bos88, Theorem 2.6]: the forward orbit of every point is dense. Then there is no family, and the claims hold with $b = 1$ and $M_1 = \mathbb{T}$.

Let $\mathcal{C} \neq \emptyset$, let $0 = c_0 < c_1 < \dots < c_t$ be the points of $\mathcal{C} \cup \{0\}$, $c_{t+1} := 1$ and $Y_j := [c_j, c_{j+1})$. By [Bos88, Theorem 2.16] the sets

$$\text{comp}(x) := \bigcup \{Y_j : Y_j \text{ meets the forward orbit of } x\} \quad (x \in X)$$

are the classes of an equivalence relation on X , and for each x one of the following holds: (1) x is periodic, with period p say, and $\text{comp}(x)$ is the union of p intervals Y_j which T permutes cyclically; (2) x is not periodic, and the forward orbit of every $z \in \text{comp}(x)$ is dense in $\text{comp}(x)$.

Every class K satisfies $T(K) \subset K$: the forward orbit of Tz is contained in that of z , so the class of Tz is contained in the class of z and equals it. Hence $T(K) = K$, because the classes partition X and T is bijective. In case (2) put $M := \text{comp}(x)$. No point $z \in M$ is periodic, since otherwise $M = \text{comp}(z)$ would be of the kind (1) and all its points would be periodic, as we show next. The end points of the intervals Y_j lie on orbits of marked points. This gives (ii) and (iii).

In case (1) let $\text{comp}(x) = Y \cup TY \cup \dots \cup T^{p-1}Y$ with $Y = [c, c')$. The first-return map of T to Y is $T^p|_Y$. By [Bos88, Proposition 2.9] it is the identity or it has no periodic point; as the orbit of the periodic point x meets Y , it is the identity. No marked point lies in the interior of an interval $T^i Y$: such a point d would be periodic, so $d \in \mathcal{D}$ would lie on the T -connection $d, Td, \dots, T^p d = d$ and hence in $\mathcal{C}$, and $d = 0$ is not interior to any Y_j . Hence the interior of Y consists of regular periodic points, while its end points lie on orbits of marked points and are not regular. So the interior of Y is a component of U_T , and $\text{comp}(x) = P^+$ for the family P of this component. Every family arises in this way, because each $u \in U_T$ lies in $\text{comp}(u)$, which is of the kind (1). This proves (i). $\square$

3.2 Three marked points

Lemma 3.3 (counting). *Let T be a circle exchange with three marked points which is discontinuous at each of them. Let a be the number of its families and b the number of its minimal components. Call connection an orbit segment $(e, Te, \dots, T^j e)$, $j \geq 0$, with $e \in \Sigma' := T(\Sigma_T)$, $T^j e \in \Sigma_T$ and $T^i e \notin \Sigma_T$ for $i < j$, and let S be the number of connections.*

- (a) $b \leq 1$ and $a \leq 2$; if $b = 1$ then $a \leq 1$.
- (b) $S \leq 3 - b$.
- (c) If $a = b = 1$, with family P and minimal component M , there are exactly two connections: the set κ_P of the left end points of the arcs of P , and the set κ_M of their right end points.

Proof. Write $\Sigma = \Sigma_T$. For $d \in \Sigma$ let $T_-(d) := \lim_{u \uparrow d} T(u) \neq T(d)$. The three arcs on which T is a translation are mapped onto arcs which tile $\mathbb{T}$; so $\Sigma' = \{T_-(d) : d \in \Sigma\}$, $|\Sigma'| = 3$, and T^{-1} is discontinuous at each point of Σ' . Consequently

$$\text{if } u \notin \Sigma, T \text{ is a translation near } u \text{ and } Tu \notin \Sigma'; \quad \text{if } u \notin \Sigma', T^{-1} \text{ is a translation near } u. \quad (11)$$

By (11) the points $T^i e$, $0 < i \leq j$, of a connection are not in Σ' . Hence two connections with a common point are equal: if $T^i e = T^{i'} e'$ with $i \leq i'$, then $e = T^{i'-i} e' \in \Sigma'$ forces $i = i'$ and $e = e'$. So at most one connection starts at each point of Σ' , different connections end at different points of Σ , and $S \leq 3$.

A *left end* of a component K is a point $c \in K$ with $(c - \eta, c) \cap K = \emptyset$ for some $\eta > 0$.

Step 1. Let $K \neq \mathbb{T}$ be a component and c a left end of K . Then c lies on a connection all of whose points are left ends of K . Since $T(K) = K$, (11) shows: if $c \notin \Sigma$ then Tc is a left end of K , and if $c \notin \Sigma'$ then $T^{-1}c$ is a left end of K . The orbit of c meets Σ (Lemma 3.1 and Proposition 3.2(ii)). If no $T^i c$ with $i \geq 0$ were in Σ , all these points would be left ends of K ; as there are finitely many left ends, c would be periodic and its orbit would avoid Σ . So there is a least $i \geq 0$ with $T^i c \in \Sigma$, and $c, \dots, T^i c$ are left ends of K . In the same way there is a least $i' \geq 0$ with $T^{-i'} c \in \Sigma'$ (otherwise the orbit of c would avoid Σ' , hence Σ), and $T^{-i'} c, \dots, c$ are left ends of K . By the minimality of i and i' none of the points $T^k c$, $-i' \leq k < i$, lies in Σ . So $(T^{-i'} c, \dots, T^i c)$ is a connection.

Step 2. Every minimal component M contains a point of Σ' at which no connection starts. As $T(M) = M$, the map T sends $\Sigma \cap M$ bijectively onto $\Sigma' \cap M$. Suppose that a connection starts at every point of $\Sigma' \cap M$. These connections lie in M and end at different points of $\Sigma \cap M$; so every point of $\Sigma \cap M$ is the end of a connection. If $\Sigma \cap M \neq \emptyset$, take $d \in \Sigma \cap M$. Its forward orbit passes to $Td \in \Sigma' \cap M$, follows a connection to a point of $\Sigma \cap M$, and so on; it stays in the finite set of points on connections. So d is periodic, contradicting Proposition 3.2(iii). If $\Sigma \cap M = \emptyset$, then $M \neq \mathbb{T}$, and T translates each maximal arc $[c, c')$ of M into M ; it therefore permutes the longest ones, and a power of T is the identity on such an arc. This is again a contradiction.

Step 3: proof of (b) and of (a) up to $a \leq 3$. By Step 2, $b \leq 3 - S$. If $a + b \geq 2$, every component is different from $\mathbb{T}$ and has a left end; by Step 1 its set of left ends contains a connection. As a point is a left end of at most one component, $a + b \leq S$. So $a + 2b \leq 3$ whenever $a + b \geq 2$. Hence $b \leq 1$, $a \leq 1$ if $b = 1$, and $a \leq 3$.

Step 4: $a = 3$ is impossible. Suppose $a = 3$. Then $b = 0$ and $S = 3$. Let P_1, P_2, P_3 be the families and O_i the periodic orbit formed by the left end points of the arcs of P_i . By Step 1, O_i contains a connection κ_i consisting of left ends of P_i^+ . The orbits O_i are disjoint, and each contains a point of Σ' and a point of Σ ; so $O_i \cap \Sigma' = \{e_i\}$, $O_i \cap \Sigma = \{d_i\}$, and κ_i runs from e_i to d_i . Since $T(\Sigma) = \Sigma'$ and $T(O_i) = O_i$, we get $Td_i = e_i$ and $O_i = \kappa_i$. So every arc of P_i is preceded by an arc of another family, and by (11), applied along κ_i , the arcs which end at the points of κ_i all belong to the same family $P_{\pi(i)}$. The right end point of an arc of P_k is the left end point of an arc of some P_i , and then $\pi(i) = k$; so π is a permutation of $\{1, 2, 3\}$. Now T translates a left neighbourhood of d_i , which lies in $P_{\pi(i)}$, onto a left neighbourhood of $T_-(d_i) \in \Sigma'$. If $T_-(d_i) = e_j$, this gives $\pi(j) = \pi(i)$, so $j = i$ and $T_-(d_i) = e_i = T(d_i)$: T would be continuous at d_i .

Step 5: proof of (c). Let $a = b = 1$. By Step 3, $2 \leq S \leq 3 - b = 2$. By Step 1 the left ends of P^+ contain a connection κ_P and the left ends of M contain a connection $\kappa_M \neq \kappa_P$; there are no other connections. Let O be the orbit formed by the left end points of the arcs of P , and let κ_P run from e to d . If $Td \neq e$, a third connection would start at $Td \in \Sigma' \cap O$, because the forward orbit of Td returns to $d \in \Sigma$. So $Td = e$ and $O = \kappa_P$: every arc of P is preceded by an arc of M . Hence the left ends of M are exactly the right end points of the arcs of P ; by Step 1 they form a union of connections contained in M , that is, the connection κ_M . $\square$

4 The periodicity criterion

By Theorem 1.1(c), Theorem 1.2 says that every regular point of G is periodic if and only if $\alpha + \lambda \in \mathbb{Q}$. We shall use that, by (3),

$$\int_{\mathbb{T}} \mu(u) du = (N + 1)\delta + N(\lambda - \delta) = N\lambda + \delta = 1. \quad (12)$$

Lemma 4.1 (relations). *Let O be a periodic orbit of G , $m_1 := |O|$ and $m_2 := \sum_{u \in O} \mu(u)$. Then $m_1\alpha + m_2\lambda \in \mathbb{Z}$. If $\alpha + \lambda \notin \mathbb{Q}$ and G has a periodic orbit, then the group of integer relations*

$\{(i, j, k) \in \mathbb{Z}^3 : i\alpha + j\lambda = k\}$ is generated by one element (p, q, n) with $p \geq 1$; moreover $q \geq 0$, $n \geq 1$, every periodic orbit has $(m_1, m_2) = (cp, cq)$ for some integer $c \geq 1$, and $\sum_{u \in O} (q - p\mu(u)) = 0$.

Proof. $G^{m_1}u = u + m_1\alpha + m_2\lambda$ in $\mathbb{T}$ by (3). If $\alpha + \lambda \notin \mathbb{Q}$, then α and λ are not both rational, so $1, \alpha, \lambda$ span a space of dimension at least 2 over $\mathbb{Q}$, and the group of relations has rank at most 1. It contains $(m_1, m_2, m_1\alpha + m_2\lambda)$ with $m_1 \geq 1$, and it contains an integer vector whenever it contains a non-zero multiple of it; so it is generated by a primitive vector (p, q, n) with $p \neq 0$, and we may take $p \geq 1$. Then $(m_1, m_2) = (cp, cq)$ with $c \geq 1$, $q \geq 0$ because $cq = m_2 \geq 0$, and $n \geq 1$ because $cn = m_1\alpha + m_2\lambda > 0$. Finally $\sum_{u \in O} (q - p\mu(u)) = qm_1 - pm_2 = 0$. $\square$

Proof of Theorem 1.2, “only if”. Suppose that every regular point of G is periodic and that $\alpha + \lambda \notin \mathbb{Q}$. Let (p, q, n) be as in Lemma 4.1 and $\kappa := q - p\mu$. By Lemma 3.1, $\mathbb{T}$ is the union of the families of G and a countable set. Let P be a family with period m_1 , width w and arc I . The arcs $G^i I$ contain no point of Σ , so μ is constant on each of them, and for $u \in I$

$$\int_P \kappa = w \sum_{i < m_1} \kappa(G^i u) = 0$$

by Lemma 4.1. Hence $\int_{\mathbb{T}} \kappa = 0$. But $\int_{\mathbb{T}} \kappa = q - p$ by (12). So $p = q$ and $\alpha + \lambda = n/p \in \mathbb{Q}$, a contradiction. $\square$

For the converse we use the following theorem. The definitions are those of [Bos88, §§2–3]: an *interval exchange transformation* is a right-continuous bijection T of an interval $X = [0, \ell)$ which is a translation on each of finitely many intervals $[\beta_{i-1}, \beta_i)$ partitioning X ; its *rank* is the dimension over $\mathbb{Q}$ of the span of the lengths $\beta_i - \beta_{i-1}$; T is *minimal* if the forward orbit of every point of X is dense in X ; and T is *uniquely ergodic* if normalised Lebesgue measure is its only invariant Borel probability measure. No irreducibility and no condition on the orbits of the discontinuities is assumed.

Theorem 4.2 (Boshernitzan [Bos88, Theorem 1.1]). *Every minimal interval exchange transformation of rank two is uniquely ergodic.*

Proof of Theorem 1.2, “if”. Let $\alpha + \lambda = n_0/g$ in lowest terms. If $\alpha \in \mathbb{Q}$, then $\lambda \in \mathbb{Q}$, and every regular point is periodic by Proposition 8.1. Let $\alpha \notin \mathbb{Q}$. Then $\lambda \notin \mathbb{Q}$ and $\delta \neq 0$, so G is a circle exchange which is discontinuous at its three marked points, and Lemma 3.3 applies. Suppose that some regular point is not periodic. By Proposition 3.2 and Lemma 3.3(a), G has exactly one minimal component M , and $\mathbb{T}$ is the disjoint union of M and of the sets P^+ .

Step 1. Let $\kappa := 1 - \mu$. For a periodic orbit, $m_1\alpha + m_2\lambda = (m_1 - m_2)\alpha + m_2n_0/g$ is an integer (Lemma 4.1), so $m_1 = m_2$ and $\sum_O \kappa = 0$. As in the “only if” part, $\int_P \kappa = 0$ for every family P . Since $\int_{\mathbb{T}} \kappa = 0$ by (12), we get $\int_M \kappa = 0$.

Step 2. Let $A'_1, \dots, A'_k$ be the maximal arcs of M in their cyclic order, ℓ the length of M , and $\iota : M \rightarrow X := [0, \ell)$ the bijection which translates $A'_1, \dots, A'_k$ onto consecutive intervals. Then $T_M := \iota \circ G \circ \iota^{-1}$ is an interval exchange transformation of X . It is minimal by Proposition 3.2(iii). The end points of the arcs A'_i lie on G -orbits of points of Σ (Proposition 3.2(ii)), hence in the subgroup $\Gamma := (\mathbb{Z} + \mathbb{Z}\alpha + \mathbb{Z}\lambda)/\mathbb{Z}$ of $\mathbb{T}$, which contains Σ and is invariant under G and G^{-1} . So all end points of the intervals exchanged by T_M are images under ι of points of Γ , and their lengths lie in $\mathbb{Z} + \mathbb{Z}\alpha + \mathbb{Z}\lambda \subset \mathbb{Q} + \mathbb{Q}\alpha$. Thus T_M has rank at most 2. If the rank were 1, the lengths would be integer multiples of one number $\eta > 0$, and T_M would permute the finite set $\eta\mathbb{Z} \cap X$; then 0 would be a periodic point, contradicting minimality. So T_M has rank two, and by Theorem 4.2 it is uniquely ergodic. In particular normalised Lebesgue measure on M is ergodic for $G|_M$, and by Birkhoff’s ergodic theorem, for almost every $u \in M$,

$$\frac{\sigma_t(u)}{t} \longrightarrow \frac{1}{\ell} \int_M \kappa = 0 \quad (t \rightarrow \infty), \quad \sigma_t(u) := \sum_{i=0}^{t-1} \kappa(G^i u). \quad (13)$$

Step 3. Fix $u \in M$ with (13); it is not periodic. By (3), for $t \geq 0$,

$$G^t u = u + t\alpha + (t - \sigma_t(u))\lambda = u + t \frac{n_0}{g} - \sigma_t(u)\lambda \quad \text{in } \mathbb{T}.$$

If $0 \leq t < t'$ satisfy $\sigma_t(u) = \sigma_{t'}(u)$ and $t \equiv t' \pmod{g}$, then $G^{t'} u = G^t u$, and u is periodic because G is bijective. So the pairs $(\sigma_t(u), t \bmod g)$, $0 \leq t \leq T$, are pairwise distinct. Hence $\sigma_t(u)$ takes at least $(T+1)/g$ different integer values for $0 \leq t \leq T$, and $\max_{t \leq T} |\sigma_t(u)| \geq \frac{T+1}{2g} - \frac{1}{2}$. This contradicts (13).

So every regular point of G is periodic. $\square$

Remark 4.3 (other derivations). (a) Step 3 is modelled on the argument by which McMullen shows that a uniquely ergodic interval exchange transformation defined over a real quadratic field has non-zero Galois flux [McM03, Theorem 2.1]; the sum σ_t plays the part of the Galois conjugate coordinate. If α and λ lie in a real quadratic field, the Galois flux of G in the sense of [McM03, §2] is $(\alpha + \lambda)' - (\alpha + \lambda)$, where the prime denotes Galois conjugation; it vanishes if and only if $\alpha + \lambda \in \mathbb{Q}$.

(b) *A second derivation, from McMullen's theorems on foliations.* Consider the form $\varrho = dx - dy$ on the surface X_Q of Section 2.4. It is a harmonic form with a single zero, and by Lemma 2.5 Φ is the first-return map of its foliation $\mathcal{F}(\varrho)$. In [McM15, §§2–3] the *rank* of a form is the rank of the group of its periods over $H_1(X, \mathbb{Z})$. If the rank is 2, say the periods are $\mathbb{Z}a_1 \oplus \mathbb{Z}a_2$ and $[\varrho] = a_1[\xi_1] + a_2[\xi_2]$ with integral classes ξ_i , the *degree* is $\int_X \xi_1 \wedge \xi_2$. The form is *periodic* if all components of its foliation are cylinders of closed leaves.

Let $\mathbf{a}_1$ and $\mathbf{b}_1$ be the classes of a horizontal and a vertical closed curve in the rectangle $[0, 1] \times [\lambda, \alpha]$ respectively $[0, 1] \times [0, \alpha]$, let $\mathbf{b}_2$ be the class of a vertical closed curve in $[1, 2] \times [0, \lambda]$, and let $\mathbf{a}_2$ be the class of a horizontal closed curve in $[0, 2] \times [0, \lambda]$ minus $\mathbf{a}_1$. They form a symplectic basis of $H_1(X_Q, \mathbb{Z})$, and the periods of ϱ on $\mathbf{a}_1, \mathbf{b}_1, \mathbf{a}_2, \mathbf{b}_2$ are $1, -\alpha, 1, -\lambda$. So the group of periods is $\mathbb{Z} + \mathbb{Z}\alpha + \mathbb{Z}\lambda$. Let $\alpha \notin \mathbb{Q}$ and $\alpha + \lambda = n_0/g$ in lowest terms. Then the periods are $\mathbb{Z}\frac{1}{g} \oplus \mathbb{Z}\alpha$, the rank is 2, and $\varrho = \frac{1}{g}\xi_1 + \alpha\xi_2$, where (ξ_1, ξ_2) takes the values $(g, 0), (0, -1), (g, 0), (-n_0, 1)$ on the basis. Hence $\int \xi_1 \wedge \xi_2 = \pm(g \cdot (-1) - 0 + g \cdot 1 - 0) = 0$: the degree is zero. By [McM15, Corollary 7.4], a form of genus two with degree zero and a single zero is periodic. So all leaves of slope 1 which avoid the cone point are closed, which is the “if” part of Theorem 1.2 for irrational ρ_i . (For forms with two zeros [McM15, Theorem 7.1] allows a second possibility, two minimal components of genus one.) Conversely, a periodic form of genus two has rank at most 2 and zero flux [McM15, Corollary 3.2], hence degree zero if the rank is 2 [McM15, (2.5)], and the same computation shows that the degree of ϱ is zero only if $\alpha + \lambda \in \mathbb{Q}$. In McMullen’s normal form the criterion is [McM15, Proposition 9.3]: a coupled rotation has degree zero if and only if both rotations are irrational and $\begin{pmatrix} L_1 \\ \tau_1 \end{pmatrix} = A \begin{pmatrix} L_2 \\ \tau_2 \end{pmatrix}$ for a rational matrix A with $\det A = -1$; with (4) this means $\alpha, \lambda \notin \mathbb{Q}$ and $\alpha + \lambda \in \mathbb{Q}$. This derivation does not use Lemma 3.3.

(c) If α and λ lie in a real quadratic field and $\alpha + \lambda \in \mathbb{Q}$, the conclusion also follows from [McM03]: a foliation whose relative periods lie in the field and whose flux is zero has a closed loop in its spine [McM03, Theorem 6.1], and in genus two such a foliation is periodic or a connected sum of two tori with irrational foliations [McM03, Theorem 7.1]. A connected sum of two tori has two simple zeros, at the ends of the slit, whereas ϱ has a double zero. By [McM03, Theorem 4.1] the flux of ϱ is the Galois flux of a first-return map, which is given in (a). (The proof of [McM03, Theorem 7.1] is written for two distinct zeros, and the case of a double zero, which is the present one, is said there to be similar.)

5 Regions

5.1 The full table

By Theorem 1.1(a) the states of the trajectories in the full table are the pairs (u, b) with $u \in \mathbb{R}/2\mathbb{Z}$, and the chord map is given by (1) with $u + s_f$ taken modulo 2. Its first-return map to the states with $b = 1$ is

$$\tilde{G} : \mathbb{R}/2\mathbb{Z} \rightarrow \mathbb{R}/2\mathbb{Z}, \quad \tilde{G}(u) = u + \alpha + \mu(u \bmod 1) \lambda.$$

Let $\pi : \mathbb{R}/2\mathbb{Z} \rightarrow \mathbb{T}$ be the projection and $\sigma(u) := u + 1$. Then $\pi \circ \tilde{G} = G \circ \pi$, σ commutes with $\tilde{G}$, and by (8) σ corresponds to the mirror symmetry $z \mapsto -\bar{z}$ of D . After rescaling, $\tilde{G}$ is a circle exchange with marked points $\pi^{-1}(\Sigma)$, and Lemma 3.1 and Proposition 3.2 apply to it.

Lemma 5.1. (a) *A point $\tilde{u}$ is regular, respectively regular and periodic, for $\tilde{G}$ if and only if $\pi(\tilde{u})$ is so for G . Let P be a family of G with period m_1 and $n = m_1\alpha + m_2\lambda$. If n is odd, $\pi^{-1}(P)$ is one family of $\tilde{G}$; if n is even, it is the union of two families exchanged by σ .*

(b) *If M is a minimal component of G , then $\pi^{-1}(M)$ is a minimal component of $\tilde{G}$ or the union of two minimal components exchanged by σ .*

(c) *σ permutes the components of $\tilde{G}$, and its orbits correspond to the components of G .*

Proof. (a) The first statement is clear, since $G^{m_1}u = u$ implies $\tilde{G}^{m_1}\tilde{u} = \tilde{u} + n$ and $\tilde{G}^{2m_1}\tilde{u} = \tilde{u}$. If $\tilde{I}$ is a lift of the arc of P , then $\tilde{G}^{m_1}\tilde{I} = \tilde{I} + n$, which gives the second statement.

(b) The set $\pi^{-1}(M)$ is invariant under $\tilde{G}$ and σ and contains no periodic point. By Proposition 3.2(i) each of its points lies in a minimal component of $\tilde{G}$. A minimal component $\tilde{M}$ of $\tilde{G}$ which meets $\pi^{-1}(M)$ is contained in it: the forward orbit of a common point stays in $\pi^{-1}(M)$ and is dense in $\tilde{M}$, and both sets are finite unions of arcs $[c, c')$. So $\pi^{-1}(M)$ is a union of minimal components of $\tilde{G}$. Let $\tilde{M}$ be one of them. Then $\pi(\tilde{M})$ and $M \setminus \pi(\tilde{M})$ are finite unions of arcs $[c, c')$ which are invariant under G . If the second set were not empty, the forward orbit of one of its points would be dense in M by Proposition 3.2(iii) and would enter $\pi(\tilde{M})$. So $\pi(\tilde{M}) = M$ and $\pi^{-1}(M) = \tilde{M} \cup \sigma(\tilde{M})$. Thus every minimal component of $\tilde{G}$ which is contained in $\pi^{-1}(M)$ is mapped onto M by π . Every point of M has exactly two preimages, and σ exchanges them. So there are at most two such components; and if there are two, each of them contains exactly one preimage of every point of M , so that σ exchanges them.

(c) By Proposition 3.2(i) the components of $\tilde{G}$ are the sets $\tilde{P}^+$, where $\tilde{P}$ runs through the families of $\tilde{G}$, and the minimal components of $\tilde{G}$. By (a), $\pi(\tilde{P})$ is a family P of G , and $\pi^{-1}(P^+)$ is $\tilde{P}^+$ or $\tilde{P}^+ \cup \sigma(\tilde{P}^+)$. A minimal component $\tilde{M}$ of $\tilde{G}$ contains no periodic point; so $\pi(\tilde{M})$ meets no set P^+ and therefore meets a minimal component M of G . By the first step of the proof of (b), $\tilde{M} \subset \pi^{-1}(M)$, and (b) applies. $\square$

5.2 Time reversal

Let $\vartheta(u) := \alpha + \lambda - u$ on $\mathbb{T}$ and $\tilde{\vartheta}(u) := \alpha + \lambda - u$ on $\mathbb{R}/2\mathbb{Z}$.

Lemma 5.2. (a) *If a regular trajectory leaves the arc A_1 , after a reflection at a point Q , in the state $u \in \mathbb{R}/2\mathbb{Z}$, then the reversed trajectory leaves A_1 at Q in the state $\tilde{\vartheta}(u)$.*

(b) *$\vartheta(\Sigma) = \Sigma'$, and $\vartheta \circ G \circ \vartheta = G^{-1}$ outside a finite set; in particular $G(\vartheta(u)) = \vartheta(G^{-1}(u))$ for $u \notin \Sigma'$. The same holds for $\tilde{\vartheta}$ and $\tilde{G}$.*

- (c) ϑ maps regular points to regular points and their orbits onto orbits, and it permutes the families of G and the closures of the components of G . The same holds for $\tilde{\vartheta}$ and $\tilde{G}$, and $\tilde{\vartheta}$ commutes with σ .

Proof. (a) Let chord k of $\tilde{\gamma}$ start at $c^\varepsilon(Q)$, so that $u = u_k$ and $b_k = 1$; then chord $k-1$ ends on A_1 and $\theta_{k-1} = \theta_k - 2\beta_1$. For the reversed trajectory $\gamma^-(t) = \gamma(-t)$ the function ε is replaced by $1 - \varepsilon$, so its unfolding is $t \mapsto c(\tilde{\gamma}(-t))$, and by Lemma 2.1(a) its chords lie on the lines $\ell_{-\theta_j}$, in the reverse order. Thus γ^- leaves Q along the mirror image of chord $k-1$, and its state is $-\theta_{k-1}/\pi + \lambda/2 = \alpha + \lambda - u_k$.

(b) $\vartheta(0) = \alpha + \lambda = G(\lambda)$, $\vartheta(\delta) = \alpha + \lambda - \delta = G(0)$ and $\vartheta(\lambda) = \alpha = G(\delta)$ by (10). Let $j(u) := \lambda - u$. Then $\vartheta = R_\alpha \circ j$, $j \circ R_\alpha \circ j = R_\alpha^{-1}$, and $j \circ S \circ j = S^{-1}$ outside a finite set by (9). With $G = R_\alpha \circ S$ this gives, outside a finite set, $\vartheta G \vartheta = R_\alpha j R_\alpha S R_\alpha j = j S R_\alpha j = S^{-1} R_\alpha^{-1} = G^{-1}$. Both sides of $G(\vartheta(u)) = \vartheta(G^{-1}(u))$ are continuous at u if $u \notin \Sigma'$, because G is continuous outside Σ , G^{-1} is continuous outside Σ' , and $\vartheta(u) \in \Sigma$ means $u \in \Sigma'$; so the identity holds for all $u \notin \Sigma'$. The proof for $\tilde{\vartheta}$ and $\tilde{G}$ is the same.

(c) A point is regular if and only if its orbit avoids $\Sigma \cup \Sigma'$. By (b), ϑ maps the orbit of a regular point u onto the orbit of $\vartheta(u)$, which therefore avoids $\vartheta(\Sigma \cup \Sigma') = \Sigma \cup \Sigma'$. So $\vartheta(U_G) = U_G$, and the isometry ϑ permutes the components of U_G , hence the families. The closure of a minimal component is the closure of the orbit of any of its regular points (Proposition 3.2(iii)); so ϑ permutes these closures as well. $\square$

5.3 Symmetric points

Consider the following points of $\mathbb{T}$ (the point c_5 only if $\delta \neq 0$):

$$c_1 := \frac{\alpha + \lambda}{2}, \quad c_2 := \frac{\alpha + \lambda + 1}{2}, \quad c_3 := \frac{1 + \lambda}{2}, \quad c_4 := \frac{\lambda + \delta}{2}, \quad c_5 := \frac{\delta}{2}.$$

Geometrically, c_1 and c_2 are the chords which leave A_1 at a right-angled corner, respectively at its apex iR_1 ; c_3 is the horizontal chord of the large half-disc; and c_4, c_5 are the chords whose excursion into the small half-disc is symmetric.

Lemma 5.3. (a) *The fixed points of ϑ are c_1 and c_2 . The points $u \in \mathbb{T} \setminus \Sigma$ with $\vartheta(u) = G(u)$ are c_3, c_4 and c_5 . We call (c_1, I) , (c_2, I) , (c_3, II) , (c_4, II) , (c_5, II) the symmetric elements; there are $|\Sigma| + 2$ of them.*

(b) *A regular orbit O of G satisfies $\vartheta(O) = O$ if and only if it carries a symmetric element, that is, contains its point. In this case it carries exactly two symmetric elements if it is periodic, and exactly one if it is not.*

(c) *If the point of a symmetric element is not regular, it lies on a connection $(e_0, \dots, e_j)$ with $\vartheta(e_i) = e_{j-i}$ for all i . A connection with this property carries exactly one symmetric element.*

(d) *For every lift $\tilde{c}_i \in \mathbb{R}/2\mathbb{Z}$ of c_i : $\tilde{\vartheta}(\tilde{c}_1) = \tilde{c}_1$, $\sigma\tilde{\vartheta}(\tilde{c}_2) = \tilde{c}_2$, and $\sigma\tilde{\vartheta}(\tilde{c}_i) = \tilde{G}(\tilde{c}_i)$ for $i = 3, 4, 5$.*

Connections are defined as in Lemma 3.3, for $T = G$ (for $\delta = 0$ with $\Sigma = \{0, \lambda\}$ and $\Sigma' = G(\Sigma)$); as there, two connections with a common point are equal.

Proof. (a) $\vartheta(u) = u$ means $2u \equiv \alpha + \lambda \pmod{1}$. For $u \notin \Sigma$, $\vartheta(u) = G(u)$ means $2u \equiv (1 - \mu(u))\lambda \pmod{1}$ by (3). On $(\lambda, 1)$ this is $2u \equiv \lambda$, with the only solution c_3 . On (δ, λ) it is $2u \equiv (1 - N)\lambda \equiv \lambda + \delta$, with the only solution c_4 . On $(0, \delta)$ it is $2u \equiv -N\lambda \equiv \delta$, with the only solution c_5 .

(b) Let $\vartheta(O) = O$, $u \in O$ and $\vartheta(u) = G^a u$. By Lemma 5.2(b), $\vartheta(G^i u) = G^{a-i} u$ for all i . So $G^i u$ is fixed by ϑ if and only if $G^{a-2i} u = u$, and $\vartheta(G^i u) = G(G^i u)$ if and only if $G^{a-2i-1} u = u$. If O is

infinite, this means $2i = a$, respectively $2i = a - 1$: exactly one of the two equations has a solution, and it has one. If $|O| = m$, it means $2i \equiv a$, respectively $2i \equiv a - 1 \pmod{m}$: together there are exactly two solutions $i \in \mathbb{Z}/m$ (one of each kind if m is odd, two of the same kind if m is even). Conversely, an orbit through the point of a symmetric element is mapped to itself by ϑ .

(c) Let c be the point of a symmetric element, and let $G^k c \in \Sigma$ for some $k \in \mathbb{Z}$. Then the forward orbit of c meets Σ and its backward orbit meets Σ' . Indeed, if $G^t c \notin \Sigma$ for all $t \geq 0$, then Lemma 5.2(b) gives $G^{-t} \vartheta(c) = \vartheta(G^t c) \notin \vartheta(\Sigma) = \Sigma'$ for $t \geq 0$; as $\vartheta(c) \in \{c, G(c)\}$, this means $G^s c \notin \Sigma'$ for all $s \leq 0$, hence $G^s c \notin \Sigma$ for all $s < 0$, and c would be regular. The other case is symmetric. Let e_0 be the last point of Σ' on the backward orbit and e_j the first point of Σ on the forward orbit of c . Then $(e_0, \dots, e_j)$ is a connection through c ; it also contains $G(c)$ if the element is of kind II, because then $c \notin \Sigma$. By Lemma 5.2(b), ϑ maps connections onto connections, reversing the order. As $\vartheta(c)$ lies on the connection, ϑ maps it to itself, and $\vartheta(e_i) = e_{j-i}$. The symmetric elements carried by such a connection are (e_i, I) with $2i = j$ and (e_i, II) with $2i + 1 = j$: exactly one.

(d) For a lift $\tilde{u}$ of $u \in \mathbb{T} \setminus \Sigma$ we have $\tilde{\vartheta}(\tilde{u}) - \tilde{u} = \alpha + \lambda - 2\tilde{u}$ and $\tilde{\vartheta}(\tilde{u}) - \tilde{G}(\tilde{u}) = (1 - \mu(u))\lambda - 2\tilde{u}$. The first number is an even integer for $u = c_1$ and an odd integer for $u = c_2$. The second one equals $\lambda - 2\tilde{c}_3, -N\lambda - \delta + (\lambda + \delta) - 2\tilde{c}_4$ and $-N\lambda - \delta + \delta - 2\tilde{c}_5$ for $u = c_3, c_4, c_5$; as $N\lambda + \delta = 1$, it is an odd integer in each case. $\square$

Lemma 5.4. *Every component of G contains a regular point c such that $c = c_2$, or $c \in \{c_3, c_4, c_5\}$.*

Proof. Let a and b be the numbers of families and of minimal components of G , S the number of connections and $\varsigma := |\Sigma| \in \{2, 3\}$. We first note that

$$b \leq 1, \quad a \leq 2, \quad a \leq 1 \text{ if } b = 1, \quad S \leq \varsigma - b. \quad (14)$$

For $\delta \neq 0$ this is Lemma 3.3. For $\delta = 0$, G is the rotation by α and $S \leq |\Sigma'| = 2$. If $\alpha \notin \mathbb{Q}$, there is no family, $\mathbb{T}$ is the only minimal component, and of the two possible connections, from α to λ and from $\alpha + \lambda$ to 0 , at most one exists, because $(i + 1)\alpha \equiv \lambda$ and $(i' + 1)\alpha \equiv -\lambda$ would give $(i + i' + 2)\alpha \in \mathbb{Z}$. If $\alpha = n/p$ in lowest terms, every point is periodic, and the families are the orbits of the arcs between consecutive points of $\frac{1}{p}\mathbb{Z} \cup (\lambda + \frac{1}{p}\mathbb{Z})$; there are one or two of them.

Invariant families. Let P be a family with $\vartheta(P) = P$, with arc $I = (c, c + w)$. Then $\vartheta(I)$ is an arc $G^k I$ of P , and since ϑ reverses the orientation, it maps the orbit of $c + t$ to the orbit of $c + w - t$ ($0 < t < w$). So P contains exactly one orbit which is mapped to itself, the orbit of $c + w/2$. By Lemma 5.3(b) it carries exactly two symmetric elements; at most one of them is (c_1, I) . So P contains a point as required.

Two exchanged families. Suppose $a = 2$ and ϑ exchanges the two families. Then $b = 0$ by (14), every regular orbit lies in a family, and no regular orbit is mapped to itself. By Lemma 5.3(b),(c) the $\varsigma + 2$ symmetric elements lie on connections, at most one on each. This contradicts $S \leq \varsigma$. So this case does not occur, and by Lemma 5.2(c) every family is mapped to itself.

The minimal component. Let $b = 1$, with minimal component M . If $a = 0$, at most $S \leq \varsigma - 1$ symmetric elements lie on connections (Lemma 5.3(c)), so the points of at least three symmetric elements are regular, and they lie in M . At least two of them are different from (c_1, I) . If $a = 1$, then $\delta \neq 0$, and by Lemma 3.3(c) the connections are κ_P and κ_M , the sets of left and of right end points of the arcs of the family P . As ϑ maps arcs of P onto arcs of P and reverses the orientation, $\vartheta(\kappa_P) = \kappa_M$. So no connection is mapped to itself, and by Lemma 5.3(c) the points of all five symmetric elements are regular. Two of the elements are carried by the orbit of P found above; the other three are carried by regular orbits outside P , that is, in M , and at least two of them are different from (c_1, I) . $\square$

5.4 Proof of Theorem 1.3

By Theorem 1.1 the regular trajectories with the caustic $|z| = r$ correspond to the regular orbits of $\tilde{G}$: every such trajectory is reflected by A_1 , and it leaves A_1 in the states of one orbit. For a component $\tilde{K}$ of $\tilde{G}$ let $B(\tilde{K}) \subset \partial D$ be the set of all points at which the trajectories through the regular points of $\tilde{K}$ are reflected, by the arcs or by the segments. These sets are the regions.

Step 1. By (8), $B(\sigma\tilde{K})$ is the mirror image of $B(\tilde{K})$. By Lemma 5.2(a),(c), reversing the trajectories of $\tilde{K}$ gives the trajectories of a component, which we denote by $\vartheta\tilde{K}$; clearly $B(\vartheta\tilde{K}) = B(\tilde{K})$.

Step 2. Through each point Q of ∂D which is not a reflex corner there is exactly one unoriented trajectory with the caustic $|z| = r$ which is reflected at Q : near Q it consists of the parts in D of the two tangents from Q to the caustic (for $Q = \pm R_1$ only one of the tangents enters D , and the trajectory returns along it). Hence, if $B(\tilde{K})$ and $B(\tilde{K}')$ have a common point, then $\tilde{K}' = \tilde{K}$ or $\tilde{K}' = \vartheta\tilde{K}$. In particular a regular trajectory which is reflected at a point of $B(\tilde{K})$ belongs to $\tilde{K}$ or to $\vartheta\tilde{K}$, and all its reflection points lie in $B(\tilde{K})$.

Step 3: $\vartheta\tilde{K} = \sigma\tilde{K}$ for every component $\tilde{K}$ of $\tilde{G}$. Let K be the component of G with $\pi^{-1}(K) = \tilde{K} \cup \sigma\tilde{K}$ (Lemma 5.1). By Lemma 5.4 it contains a regular point $c \in \{c_2, c_3, c_4, c_5\}$. Let $\tilde{c} \in \tilde{K}$ be a lift of c . By Lemma 5.3(d), $\sigma\vartheta(\tilde{c})$ is $\tilde{c}$ or $\tilde{G}(\tilde{c})$; in both cases it is a regular point of $\tilde{K}$. Hence the component $\sigma\vartheta\tilde{K}$ meets $\tilde{K}$ in a regular point, and $\sigma\vartheta\tilde{K} = \tilde{K}$.

Step 4. By Steps 1–3, two components of $\tilde{G}$ have the same region if they lie in the same orbit of σ , and disjoint regions otherwise; and $B(\tilde{K}) = B(\vartheta\tilde{K}) = B(\sigma\tilde{K})$ is its own mirror image. By Lemma 5.1 the regions correspond bijectively to the components of G . By (14) there are at most two components. The trajectories of a periodic component have the same data (m_1, m_2, v, n) , because μ is constant on the arcs of a family, hence the same period by (2). This proves (a).

(b) Let $\alpha + \lambda \in \mathbb{Q}$. By Theorem 1.2 every regular point is periodic, so $b = 0$ and $a \in \{1, 2\}$. Suppose $a = 1$, with period m_1 and width w . Then the m_1 arcs of the family cover $\mathbb{T}$ up to finitely many points, so $w = 1/m_1$ and their end points form a coset of $\frac{1}{m_1}\mathbb{Z}$. The points of Σ are not regular, so they are end points, and $\lambda \in \frac{1}{m_1}\mathbb{Z}$. Hence $a = 2$ if $\lambda \notin \mathbb{Q}$.

(c) Let $\alpha + \lambda \notin \mathbb{Q}$. By Theorem 1.2 some regular point is not periodic; it lies in a minimal component. So $b = 1$ and $a \leq 1$ by (14). Let $\tilde{M}$ be a minimal component of $\tilde{G}$. The reflection points of a trajectory from one departure from A_1 to the next are the images of its state $\tilde{u}$ under at most $N + 2$ maps, each of which is continuous on the arcs of a finite partition of $\mathbb{R}/2\mathbb{Z}$. The forward orbit of a regular point of $\tilde{M}$ is dense in $\tilde{M}$ (Proposition 3.2(iii)), and every regular point of $\tilde{M}$ is an interior point of $\tilde{M}$; so the reflection points of its trajectory are dense in $B(\tilde{M})$. $\square$

6 The periodic region when $\rho_1 + \rho_2$ is irrational

In this section $\alpha + \lambda \notin \mathbb{Q}$. By Theorem 1.3(c) and its proof, G has at most one family; a periodic region exists if and only if G has a regular periodic point, and its data (m_1, m_2, v, n) in (2) are those of the family. We use the following trivial fact twice.

Lemma 6.1. *A bijection ι of a finite set $J \subset \mathbb{Z}$ with $\iota(i) \leq i$ for all $i \in J$ is the identity.*

Proof. $\sum_{i \in J} (i - \iota(i)) = 0$, and all terms are non-negative. $\square$

Proof of Theorem 1.4(a) and (e). If G has a regular periodic orbit O , Lemma 4.1 gives the generating relation $p\alpha + q\lambda = n$ with $p \geq 1$, $q \geq 0$, $n \geq 1$, and $(m_1, m_2) = (cp, cq)$. Now m_2 is the sum of the $v \leq m_1$ non-zero values of μ on O , and these are at most $N + 1$; so $cq = m_2 \leq (N + 1)m_1 = (N + 1)cp$. This proves (e). If $1, \alpha, \lambda$ are linearly independent over $\mathbb{Q}$, there is no relation at all. If $\lambda \in \mathbb{Q}$, then $\alpha \notin \mathbb{Q}$, and every relation $i\alpha + j\lambda = k$ has $i = 0$. In both cases there is no relation with $p \geq 1$, hence no periodic orbit. This proves (a). $\square$

Proof of Theorem 1.4(b). Let $\alpha = n/p$ in lowest terms; then $\lambda \notin \mathbb{Q}$. A periodic regular orbit O has $m_1 n/p + m_2 \lambda \in \mathbb{Z}$, hence $m_2 = 0$: it avoids $[0, \lambda]$, and G acts on it as the rotation by n/p . So $O = u + \frac{1}{p}\mathbb{Z}$, and this set avoids $[0, \lambda]$ if and only if $\lambda < \frac{1}{p}$ and $u \bmod \frac{1}{p} \in (\lambda, \frac{1}{p})$. Conversely, if $\lambda < \frac{1}{p}$, every such u is a regular periodic point with $m_1 = p$, $m_2 = v = 0$ and $m_1 \alpha + m_2 \lambda = n$. Thus $U_G = \{u : u \bmod \frac{1}{p} \in (\lambda, \frac{1}{p})\}$ for $\lambda < \frac{1}{p}$, one family of width $\frac{1}{p} - \lambda$, and $U_G = \emptyset$ for $\lambda > \frac{1}{p}$. By (2) the trajectories of the family are reflected p times by A_1 and n times by the segments if n is even, and $2p$ and $2n$ times if n is odd. Let $\rho_1 = s/t$ in lowest terms, so that $\alpha = 2s/t$. If t is odd, then $(n, p) = (2s, t)$; if t is even, then s is odd and $(n, p) = (s, t/2)$. In both cases there are t reflections off A_1 and $2s$ off the segments, and $\lambda < \frac{1}{p}$ means $\rho_2 < \frac{1}{2t}$, respectively $\rho_2 < \frac{1}{t}$. $\square$

Proof of Theorem 1.4(c). Let $\alpha + q\lambda = n$ with integers $q \geq 2$ and n ; then $\lambda \notin \mathbb{Q}$ and $0 < \delta < \lambda$. Fix a point $u_0 \in \mathbb{T}$ and put $w_j := u_0 + j\lambda \in \mathbb{T}$ for $j \in \mathbb{Z}$. These points are pairwise distinct. The set $\mathcal{J} := \{j : w_j \in [0, \lambda)\}$ is unbounded in both directions; for $i \in \mathcal{J}$ let i^+ be the next element of $\mathcal{J}$. The computation in Section 2.3, made with the half-open arcs $[0, \delta)$ and $[\delta, \lambda)$, shows that $i^+ - i = \mu(w_i)$ for all $i \in \mathcal{J}$. Since $\alpha \equiv -q\lambda \pmod{1}$, (3) gives

$$G(w_j) = w_{g(j)}, \quad g(j) := j - q \quad (j \notin \mathcal{J}), \quad g(i) := i^+ - q \quad (i \in \mathcal{J}),$$

and g is a bijection of $\mathbb{Z}$, because G and G^{-1} map the set $\{w_j\}$ into itself. Let $\Omega \subset \mathbb{Z}$ be a finite orbit of g and $J := \Omega \cap \mathcal{J}$. Then $J \neq \emptyset$, because g decreases outside $\mathcal{J}$. For $i \in J$ we have $g(i) = i^+ - q$, and g then subtracts q until it reaches $\mathcal{J}$ again. So the first-return map of $g|_\Omega$ to J is $\iota(i) = i^+ - kq$, where $k = k(i) \geq 1$ is minimal with $i^+ - kq \in \mathcal{J}$. As $\iota(i) \in \mathcal{J}$ and $\iota(i) < i^+$, we have $\iota(i) \leq i$, and Lemma 6.1 gives $\iota(i) = i$. Thus q divides $i^+ - i = \mu(w_i)$ for every $i \in J$. Conversely, if $i \in \mathcal{J}$ and $\mu(w_i) = kq$, then g maps $i \mapsto i + (k-1)q \mapsto \dots \mapsto i + q \mapsto i$, because the intermediate indices lie strictly between i and i^+ ; so w_i is periodic, its orbit has k points, of which only w_i lies in $[0, \lambda)$, and $\sum \mu = kq$ on it.

Hence every periodic orbit of G contains a point $u \in [0, \lambda)$ for which q divides $\mu(u)$; if the orbit is regular, then $u \in (0, \delta) \cup (\delta, \lambda)$. Now $\mu = N$ on (δ, λ) and $\mu = N + 1$ on $(0, \delta)$, and at most one of N , $N + 1$ is divisible by q . If q divides N , every $u \in (\delta, \lambda)$ is regular and periodic: by the above its orbit consists of the points $u + iq\lambda$, $0 \leq i < N/q$, and these lie in $(\lambda, 1)$ for $i \neq 0$ (Section 2.3). As $\delta, \lambda \in \Sigma$, the arc (δ, λ) is a component of U_G , and every regular periodic orbit meets it. So there is exactly one family, with $m_1 = N/q$, $m_2 = N$, $v = 1$ and width $\lambda - \delta$. If q divides $N + 1$, the same holds for $(0, \delta)$, with $m_1 = (N + 1)/q$, $m_2 = N + 1$, $v = 1$ and width δ . If q divides neither, $U_G = \emptyset$. In the first two cases, with $s := m_2$, we get $m_1 \alpha + m_2 \lambda = m_1(n - q\lambda) + m_1 q\lambda = sn/q$, and (2) gives the numbers of reflections stated in the theorem. $\square$

Remark 6.2. In Theorem 1.4(c) the answer can be expressed through n . From $\lambda < \alpha < 1$ we get $\frac{n-1}{q} < \lambda < \frac{n}{q+1}$, so $1 \leq n \leq q$. If $n \geq 3$, then $1/\lambda < q/2$, so $N + 1 < q$ and there is no periodic region. If $n = 2$, then $N \leq q - 1$, and a periodic region exists if and only if $N = q - 1$, that is, $\rho_2 < \frac{1}{2(q-1)}$; its trajectories have period $q + 1$: one reflection off A_1 and q off A_2 . If $n = 1$, all values of $N \geq q + 1$ occur, and the criterion is that q divides N or $N + 1$.

Proof of Theorem 1.4(d). Let $p\alpha + \lambda = n$ with integers $p \geq 2$ and n . Then $\alpha, \lambda \notin \mathbb{Q}$ and $0 < \delta < \lambda$. Put $\beta := 1 - \alpha$; then $\beta \in (0, 1)$ and $\lambda \equiv p\beta \pmod{1}$.

Step 1. Fix a point $u_0 \in \mathbb{T}$ and put now $w_j := u_0 + j\alpha$ and $\mathcal{J} := \{j : w_j \in [0, \lambda)\}$. As $\lambda \equiv -p\alpha$, for $i \in \mathcal{J}$ the number $\mu(w_i)$ is the least $l \geq 1$ with $i - lp \in \mathcal{J}$ (Section 2.3, with half-open arcs as in the proof of (c)); put $i^- := i - \mu(w_i)p$. By (3),

$$G(w_j) = w_{g(j)}, \quad g(j) := j + 1 \quad (j \notin \mathcal{J}), \quad g(i) := i^- + 1 \quad (i \in \mathcal{J}),$$

and g is a bijection of $\mathbb{Z}$. Let Ω be a finite orbit of g and $J := \Omega \cap \mathcal{J}$; again $J \neq \emptyset$. For $i \in J$, g maps i to $i^- + 1$ and then adds 1 until it reaches $\mathcal{J}$. So the first-return map of $g|_{\Omega}$ to J maps i to the smallest element $\iota(i)$ of $\mathcal{J}$ which is larger than i^- . As $i \in \mathcal{J}$ and $i > i^-$, $\iota(i) \leq i$, and Lemma 6.1 gives $\iota(i) = i$: no element of $\mathcal{J}$ lies strictly between i^- and i . Conversely, if $i \in \mathcal{J}$ has this property, then g maps $i \mapsto i^- + 1 \mapsto i^- + 2 \mapsto \dots \mapsto i$, and w_i is periodic; its orbit has $\mu(w_i)p$ points, of which only w_i lies in $[0, \lambda)$. Since $w_{i-j} \equiv w_i + j\beta$, this shows: a point $y \in [0, \lambda)$ is periodic if and only if

$$y + j\beta \notin [0, \lambda) \pmod{1} \quad \text{for } 0 < j < \mu(y)p; \quad (15)$$

its orbit is then $\{y + j\beta : 0 \leq j < \mu(y)p\}$, in which only y lies in $[0, \lambda)$, so that $m_1 = \mu(y)p$ and $m_2 = \mu(y)$; and every periodic orbit of G contains such a point.

Step 2. Let $y \in (0, \delta) \cup (\delta, \lambda)$ satisfy (15), let its orbit be regular, and let $l := \mu(y)$. Write $j = kp + i$ with $0 \leq k < l$ and $0 \leq i < p$; then $y + j\beta \equiv (y + i\beta) + k\lambda$. Let $1 \leq i \leq p - 1$ and $z := y + i\beta$. The l points $z + k\lambda$, $0 \leq k < l$, cut $\mathbb{T}$ into $l - 1$ arcs of length λ and one arc, from $z + (l - 1)\lambda$ to z , of length $1 - (l - 1)\lambda$; this is $\lambda + \delta$ if $l = N$ and δ if $l = N + 1$. None of these points lies in $[0, \lambda]$: they are not in $[0, \lambda)$ by (15), and they are different from λ because they belong to the regular orbit of y . So the arc $[0, \lambda]$ lies in one of the open arcs cut out by them, which must be longer than λ ; it is the last one, $l = N$, and $z \in (\lambda, \lambda + \delta) \pmod{1}$. Hence

$$y \in (\delta, \lambda) \quad \text{and} \quad y + i\beta \in (\lambda, \lambda + \delta) \pmod{1} \quad \text{for } 1 \leq i \leq p - 1. \quad (16)$$

Conversely, (16) implies (15) and that the orbit of y is regular: for $i \geq 1$ the points $y + i\beta + k\lambda$, $0 \leq k < N$, lie in $(\lambda, \lambda + \delta + (N - 1)\lambda) = (\lambda, 1)$, and for $i = 0$, $0 < k < N$ they lie in $(\lambda, 1)$ by Section 2.3.

Let (16) hold. For $i = 1$: the real number $y + \beta$ lies in $(\delta, \lambda + 1)$, so $\lambda < y + \beta < \lambda + \delta$ as real numbers, and $\beta < \lambda$ because $y > \delta$. The step β is smaller than the distance $1 - \delta = N\lambda$ between the interval $(\lambda, \lambda + \delta)$ and its translate by 1; so by induction $y + i\beta \in (\lambda, \lambda + \delta)$ as real numbers for $1 \leq i \leq p - 1$. Thus (16) is equivalent to the inequalities between real numbers

$$\delta < y < \lambda, \quad \lambda < y + \beta, \quad y + (p - 1)\beta < \lambda + \delta. \quad (17)$$

They imply $(p - 1)\beta < \lambda + \delta - y < \lambda$. Now $\beta = (p - n + \lambda)/p$, and $n \leq p$ because $n = p\alpha + \lambda < p + 1$. So $(p - 1)\beta < \lambda$ reads $(p - 1)(p - n) < \lambda < 1$, which forces $n = p$ and $\beta = \lambda/p$. Then (17) becomes

$$\max\left(\delta, \frac{(p - 1)\lambda}{p}\right) < y < \min\left(\lambda, \delta + \frac{\lambda}{p}\right), \quad (18)$$

and such y exist if and only if $\frac{(p - 1)\lambda}{p} < \delta + \frac{\lambda}{p}$, that is, $p\delta > (p - 2)\lambda$. As $\delta/\lambda = \{1/\lambda\} = \{1/(2\rho_2)\}$, this is the condition of the theorem.

Step 3. Conversely let $n = p$ and $p\delta > (p - 2)\lambda$, and let y be any point of the open arc I_0 described by (18). Then (17) and (16) hold, and by Steps 1 and 2 y is a regular periodic point whose orbit consists of the Np points $y + i\beta + k\lambda$, $0 \leq i < p$, $0 \leq k < N$; they lie in $(\lambda, 1)$ unless $i = k = 0$. So $m_1 = Np$, $v = 1$, $m_2 = N$ and $m_1\alpha + m_2\lambda = N(p\alpha + \lambda) = Np$. The end points of I_0 are not regular. Indeed, an end point is δ or λ , or it is the left end point $y_0 = \frac{(p - 1)\lambda}{p} > \delta$, or the right end point $y_1 = \delta + \frac{\lambda}{p} < \lambda$. In the last case $G(y_1) = y_1 + \alpha + N\lambda \equiv y_1 - \beta - \delta = 0$. For y_0 the points $y_0 + i\beta + k\lambda$ with $(i, k) \neq (0, 0)$ lie in $[\lambda, 1)$, by the same estimates with $y_0 + \beta = \lambda$; so G acts on them as R_α , and $G^{Np - 1}(y_0) = y_0 + \beta = \lambda$. So I_0 is a component of U_G . By Steps 1 and 2 every regular periodic orbit meets I_0 . Hence there is exactly one family, and (2) gives Np reflections off A_1 , N off A_2 and $Np - 2$ off the segments if Np is even, and twice these numbers if Np is odd. For $p = 2$ the condition $p\delta > (p - 2)\lambda$ always holds. $\square$

7 The case $\rho_1 + \rho_2 = \frac{1}{2}$

Proof of Theorem 1.5. Let $\alpha + \lambda = 1$. Then $\lambda < \frac{1}{2}$, so $N \geq 2$, and $G(u) = S(u) - \lambda$ in $\mathbb{T}$, with S as in (9). Let $u \in (0, \delta) \cup (\delta, \lambda)$ and $s := \mu(u) \in \{N, N+1\}$. By the computation in Section 2.3, $u + s\lambda \equiv S(u)$, and $u + k\lambda$ lies in $(\lambda, 1)$ modulo 1 for $0 < k < s$. Hence $G(u) = u + (s-1)\lambda$, and G subtracts λ at each further step until the orbit returns to u : we get $G^j(u) = u + (s-j)\lambda$ for $1 \leq j \leq s$. So u is a regular periodic point with $m_1 = s$, $v = 1$, $m_2 = s$ and $m_1\alpha + m_2\lambda = s$. Every G -orbit meets $[0, \lambda)$, because G is the rotation by $-\lambda$ on $[\lambda, 1)$. So the regular points are the points on the orbits of $(0, \delta)$ and (δ, λ) , and all of them are periodic. These two arcs are components of U_G . So the families are the orbit of (δ, λ) , with $s = N$ and width $\lambda - \delta$, and, if $\delta \neq 0$, the orbit of $(0, \delta)$, with $s = N+1$ and width δ ; and $\delta = 0$ means that $1/\lambda = 1/(2\rho_2)$ is an integer. By the proof of Theorem 1.3 the regions correspond to the families, and (2) with $n = s$ gives s reflections off each arc and $s-2$ off the segments if s is even, and twice these numbers if s is odd. $\square$

Neither the proof nor the statement uses that ρ_2 is irrational. For example $(\rho_1, \rho_2) = (\frac{1}{3}, \frac{1}{6})$ has $N = 3$, $\delta = 0$ and one region of period $6 \cdot 3 - 4 = 14$, and $(\frac{5-\sqrt{5}}{10}, \frac{\sqrt{5}}{10})$ has $N = 2$ and two regions of periods $3 \cdot 2 - 2 = 4$ and $6 \cdot 3 - 4 = 14$, as in [DR14, §4.3].

8 Rational pairs and certificates

Proposition 8.1 (rational pairs). *Let $\alpha = A/M$ and $\lambda = B/M$ with integers $0 < B < A < M$, and let $D_0 := M - NB$, so that $\delta = D_0/M$. Then G maps the arc $I_k := (\frac{k}{M}, \frac{k+1}{M})$, $k \in \mathbb{Z}/M$, onto $I_{\pi(k)}$, where*

$$\pi(k) \equiv \begin{cases} k + A + B - D_0 & (0 \leq k < D_0), \\ k + A - D_0 & (D_0 \leq k < B), \\ k + A & (B \leq k < M) \end{cases} \pmod{M},$$

and μ equals $N+1$, N , 0 on I_k in these three cases. Every point outside $\frac{1}{M}\mathbb{Z}$ is regular and periodic. G has one or two families. The data of a family are those of any cycle C of π whose arcs it contains: $m_1 = |C|$, $m_2 = \sum_{k \in C} \mu(I_k)$, $v = |C \cap \{0, \dots, B-1\}|$ and $n = (m_1A + m_2B)/M$. The common end point of I_k and I_{k+1} is regular, so that the two arcs lie in the same family, if and only if $\mu(I_{\pi^j(k)}) = \mu(I_{\pi^{j(k)+1})}$ for all $j \geq 0$.

Proof. The points of Σ and the translations $\alpha + \mu\lambda$ of G are multiples of $\frac{1}{M}$. This gives π , and it shows that the orbit of a point outside $\frac{1}{M}\mathbb{Z}$ is finite and avoids Σ . A point $\frac{j}{M}$ belongs to Σ if and only if μ takes different values on the two arcs adjacent to it; and if it does not, G translates both arcs by the same amount. This gives the last statement. The number of families is one or two by (14). $\square$

With (2) this gives the regions and periods for every rational pair by a finite computation. Table 2 lists them for denominators up to 7. For denominators up to 12 there are 231 pairs; 30 of them have one region and 201 have two (the list is in the reproducibility package). We have no closed formula (Section 9).

Proposition 8.2 (certificate of non-periodicity). *Let $\alpha + \lambda \notin \mathbb{Q}$, let (p, q, n) generate the relations $i\alpha + j\lambda = k$, with $p \geq 1$, and let $\kappa := q - p\mu$. Let $V \subset \mathbb{T}$ be invariant under G , for instance the complement of a family. If for some $K \geq 1$ the sums $\kappa(u) + \kappa(Gu) + \dots + \kappa(G^{K-1}u)$ have the same strict sign for all regular $u \in V$, then V contains no regular periodic point.*

Proof. Let $O \subset V$ be a regular periodic orbit. Summing the K -sums over $u \in O$ gives $K \sum_O \kappa$, which is 0 by Lemma 4.1; but all terms have the same strict sign. $\square$

ρ_1	ρ_2	periods	ρ_1	ρ_2	periods
1/6	1/7	8 = 6+0+2, 13 = 6+7+0	1/3	2/7	13 = 6+7+0, 31 = 15+14+2
1/5	1/7	31 = 15+14+2, 43 = 20+21+2	2/5	1/7	37 = 15+14+8, 51 = 20+21+10
1/5	1/6	11 = 5+6+0, 18 = 10+6+2	2/5	1/6	13 = 5+6+2, 22 = 10+6+6
1/4	1/7	6 = 4+0+2, 11 = 4+7+0	2/5	1/5	4 = 1+3+0, 8 = 4+2+2
1/4	1/6	5 = 2+3+0, 6 = 4+0+2	2/5	1/4	11 = 5+4+2, 24 = 10+12+2
1/4	1/5	6 = 4+0+2, 9 = 4+5+0	2/5	2/7	26 = 10+14+2, 52 = 25+21+6
2/7	1/7	5 = 2+3+0, 11 = 5+4+2	2/5	1/3	11 = 5+6+0, 21 = 10+9+2
2/7	1/6	15 = 7+6+2, 34 = 14+18+2	3/7	1/7	5 = 1+4+0, 13 = 6+3+4
2/7	1/5	31 = 14+15+2, 45 = 21+20+4	3/7	1/6	17 = 7+6+4, 38 = 14+18+6
2/7	1/4	15 = 7+8+0, 28 = 14+12+2	3/7	1/5	35 = 14+15+6, 51 = 21+20+10
1/3	1/7	5 = 3+0+2, 15 = 6+7+2	3/7	1/4	17 = 7+8+2, 32 = 14+12+6
1/3	1/6	14 = 6+6+2	3/7	2/7	3 = 1+2+0, 7 = 4+1+2
1/3	1/5	13 = 6+5+2, 21 = 9+10+2	3/7	1/3	15 = 7+6+2, 31 = 14+15+2
1/3	1/4	7 = 3+4+0, 12 = 6+4+2	3/7	2/5	29 = 14+15+0, 43 = 21+20+2

Table 2: The regions for all rational pairs $\rho_2 < \rho_1 < \frac{1}{2}$ with denominators at most 7 (Proposition 8.1); a period is written as the sum of the numbers of reflections off A_1 , A_2 and the segments.

If there is no relation with $p \geq 1$, there is no periodic point at all (Theorem 1.4(a)). The K -sum is constant on the arcs of the partition of $\mathbb{T}$ by the points $G^{-i}(d)$, $d \in \Sigma$, $0 \leq i < K$; so for α , λ in a real quadratic field the hypothesis is checked by a finite exact computation. In all cases of Theorem 1.4(b)–(d) which we computed, such a K exists for the complement of the family (Section 10); we do not know whether this is always so.

9 Open questions

1. The periods for rational $\rho_1 + \rho_2 \neq \frac{1}{2}$. Let $\alpha + \lambda = n_0/g$ in lowest terms and $\lambda \notin \mathbb{Q}$. Then there are two regions (Theorem 1.3(b)), each of them has $m_1 = m_2$ (Step 1 in Section 4), g divides m_1 because $n = m_1 n_0/g$ is an integer, and by (2) the period is $m_1(2 + n_0/g) - 2v$, doubled if $m_1 n_0/g$ is odd. For $g = 1$, Theorem 1.5 gives $m_1 \in \{N, N+1\}$ and $v = 1$. For $g \geq 2$ we have no formula for (m_1, v) , and they are not functions of N and n_0/g alone: Table 3 shows four values of ρ_2 with $\rho_1 + \rho_2 = \frac{1}{4}$ and $N = 6$. The periods can be large; for $\rho_1 + \rho_2 = \frac{3}{10}$ and $\rho_2 = \frac{\sqrt{7}}{46}$ the two regions have $m_1 = 465320$ and $m_1 = 466085$, with periods 1102778 and 2209182. For rational pairs the question is to describe the cycles of the permutation of Proposition 8.1.

2. Relations with $p, q \geq 2$. Let $\rho_1 + \rho_2 \notin \mathbb{Q}$ and let the generating relation $p\rho_1 + q\rho_2 = \frac{n}{2}$ have $p \geq 2$ and $q \geq 2$. Does a periodic region exist, and what is its period? The necessary condition $q \leq (N+1)p$ of Theorem 1.4(e) is not sufficient, and on one and the same relation both answers occur: Table 4 shows this for $2\rho_1 + 3\rho_2 = \frac{n}{2}$. In these examples a trajectory may enter the small half-disc several times per period of the quotient ($v \geq 2$), which does not happen in the cases (b)–(d). The proofs of Section 6 use that one of p, q is at most 1: then the orbit can be followed along one arithmetic progression w_j , and the first-return map ι satisfies $\iota(i) \leq i$. We have no substitute for $p, q \geq 2$.

3. Decidability. For a pair with a relation, G is an exchange of rank two. According to its abstract, [Bos88] contains an algorithm which tests such exchanges for aperiodicity and minimality; we have

ρ_2	(m_1, v)	periods
$\sqrt{2}/18$	(32, 5), (38, 6)	70, 166
$\sqrt{5}/28$	(44, 7), (50, 8)	96, 218
$(\sqrt{5} - 1)/16$	(26, 4), (32, 5)	114, 70
$(\sqrt{7} - 2)/8$	(62, 10), (68, 11)	270, 148

Table 3: The two regions for $\rho_1 + \rho_2 = \frac{1}{4}$ and four values of ρ_2 with $N = 6$ (exact computation; the families have total measure exactly 1).

n	ρ_2	N	periodic region: (m_1, m_2, v) and period
1	$(\sqrt{2} - 1)/16$	19	none ($K = 27$)
1	$\sqrt{2}/16$	5	(4, 6, 1), 10 = 4 + 6 + 0
2	$(\sqrt{2} - 1)/16$	19	(26, 39, 2), 87 = 26 + 39 + 22
2	$\sqrt{2}/16$	5	(4, 6, 1), 12 = 4 + 6 + 2
2	$(\sqrt{2} + 1)/16$	3	(2, 3, 1), 5 = 2 + 3 + 0
3	$(\sqrt{2} + 2)/16$	2	none ($K = 7$)
3	$(\sqrt{2} + 3)/16$	1	none ($K = 3$)
4	$(\sqrt{2} + 4)/16$	1	(2, 3, 2), 5 = 2 + 3 + 0

Table 4: The relation $2\rho_1 + 3\rho_2 = \frac{n}{2}$ (exact computation). The absence of periodic points outside the family, or of any periodic point, was certified with Proposition 8.2; K is given where there is no family.

not used it. The certificate of Proposition 8.2 decided most, but not all, sampled cases with $p, q \geq 2$ (Section 10).

4. More arcs. Question 3 of [DR15], on tables bounded by several concentric arcs and radial segments, is not treated here. The unfolding of Section 2.2 has an analogue, but the reduced maps have more discontinuities, while Lemma 3.3, Theorem 4.2 and the results quoted in Remark 4.3 are specific to three discontinuities, to rank two, and to genus two with a single zero.

10 Computations and reproducibility

All theorems, propositions and lemmas of this note are proved without computation. The computations test the reduction against a simulation of the billiard which does not use it, test the explicit statements on many exact instances, and produce the tables. The reproducibility package has five parts.

Programs written with this note (`writing_stage/`). The module `pib.py` implements the objects in the notation of the note: the chord map (1), the map G , the families (found by following the points of Σ' , see Lemma 3.1; the widths are exact), the certificate of Proposition 8.2, and a floating-point billiard in the table D (straight flight, reflection off the arcs and the segments) which knows nothing of the reduction. Numbers are rationals or elements of a real quadratic field, with exact comparisons. The program `check_paper.py` tests the statements as printed: (A) Theorem 1.1(a) and Lemma 2.3, chord by chord, on 200 floating-point trajectories in tables with random radii (466,488 chords; the largest error in u is below 10^{-12}); (B) formula (2) against the period and the three numbers of reflections counted in the table (600 starts in 60 rational pairs), and (3) against (1) (2,400 points); (C) Lemma 2.5 (12,000 points); (D) the examples of Table 1; (E) Theorems 1.2 and 1.3 on 164 irrational pairs with rational sum (total measure of the families exactly 1, two families, $m_1 = m_2$) and on 171 pairs with irrational sum (at most one family); (F) Theorem 1.5 on 436 values of ρ_2 ; (G) Theorem 1.4(b), (c), (d) and Remark 6.2 on 319, 165 and 147 pairs (the predicted family with its data, width and arc, and a certificate for its complement in every case); (H) Proposition 8.1 on the 231 pairs with denominators at most 12, and Table 2 as printed; (I) Theorem 1.3(a) for all 325 rational pairs $(\alpha, \lambda) = (\frac{A}{M}, \frac{B}{M})$ with $M \leq 14$, by an exact computation in the full table; (J) Theorem 1.4(e) on 112 pairs with $p, q \geq 2$; (K) Lemma 5.2(a) and (8) on 2,000 floating-point states, and Lemma 5.2(b), Lemma 5.3 and Lemma 5.4 for families on these 325 pairs and on 76 irrational pairs; (L) Tables 3 and 4 and the long example of Section 9 as printed. All checks passed; the run takes about one minute. The program `check_symmetric.py` tests Lemma 5.3(c), (14) and the case distinctions in the proof

of Lemma 5.4 on the same 325 rational pairs and on 88 irrational pairs (for infinite orbits up to 4000 steps). The program `make_figures.py` produced the coordinates of Figures 1 to 3.

The programs with which the results were first obtained (original/): a floating-point billiard, three exact decomposition engines, and scripts which compare them and test the statements, formulated there with the parameter $x = u - \frac{\lambda}{2}$. Among the recorded results: 1011 irrational pairs with rational sum, all completely periodic with two families; 809 pairs for Theorem 1.4(d), 180 for (c), 250 for (b), 622 for Theorem 1.5; and 186 pairs with $p, q \geq 2$ or a common factor, of which 81 have a family and 105 are certified to have none.

Three independent verification runs, each with its own programs and their recorded outputs, in the folders `verification_run_A/`, `verification_run_B/` and `independent_run_3/`; see the next paragraph.

Verification. The results were first obtained, with proofs and with the programs in `original/`, in the coordinate $x = u - \frac{\lambda}{2}$. Three independent verification runs, all AI-assisted, followed; each wrote its own programs.

The first run re-derived Theorem 1.1, Lemma 3.3 and the proof of Theorem 1.2 line by line, read the statements of [Bos88] in the paper, and tested the reduction and the criterion: 838,333 chords of floating-point trajectories in 300 tables with random radii; 1,800 floating-point runs and 1,600 exact starts for (2); and 1,286,300 irrational pairs with rational sum, all completely periodic with two families (among them non-quadratic parameters such as $2^{1/3}$, π and a Liouville number, treated with certified rational enclosures). This run also observed that for quadratic irrationals the criterion follows from [McM03].

The second run checked Theorems 1.3 to 1.5, Lemma 3.3 and Propositions 8.1 and 8.2: it re-derived the proofs and tested them on 95 explicit and 4,829 sampled pairs in exact arithmetic, on the period table for denominators up to 12 (two programs of its own and 864 floating-point starts; the table is identical to that of the original program), on all 52,650 exchanges of three arcs of the circles $\mathbb{Z}/M$, $M \leq 26$ (for Lemma 3.3), and on 231 rational and 12 irrational pairs for the regions. It also re-ran part of the original programs and reproduced their recorded numbers. It found the identification (4) with coupled rotations and the derivation of Remark 4.3(b), and it proved the closed form in Theorem 1.4(d), Remark 6.2 and the last statement of Theorem 1.3(b). Among 1,155 sampled pairs with $p, q \geq 2$, $p \neq q$, it found 546 with a family and certified 603 to have none; 13 pairs were not completely decided within its bounds. All corrections required by the two runs were made.

When this note was written, three things were changed after these two runs. The statements were transferred to the coordinate u , in which the reduced map is literally a coupled rotation, and all proofs were written out in this coordinate. The proof of Theorem 1.3 was reorganised around the reflection ϑ of the circle (Lemmas 5.2 to 5.4); the second run had checked the argument in its original form, for the map of two circles. And in the proofs of Theorem 1.4(c) and (d) the index sets were defined with the half-open arc $[0, \lambda)$, so that the auxiliary points need not avoid the discontinuities.

The third run examined the final text, and these three changes first. It compared every formula, interval and discontinuity in the coordinate u with the earlier version; re-derived Section 5 and the proofs of Theorem 1.4(c) and (d) line by line, and then all other proofs; compared the quoted statements of [Bos88, McM15, McM03] with the texts of these papers; read [DR15] with its figures and [DR14] again; and repeated the literature search. It found no wrong theorem, proposition or lemma and no gap that could not be closed. It required ten corrections of the text and of the package, which were made; the mathematical ones are that the last step of the proof of Lemma 5.1(b) and the proof of Lemma 5.1(c) were written out, and that Lemma 5.2(c) is now stated for the orbits of regular points. Its two programs, a floating-point billiard in the table D which does not use the reduction and the maps Φ and G in exact arithmetic, agree in every comparison: (1), Lemma 2.3,

(8) and Lemma 5.2(a) on 410,479 chords of 150 floating-point trajectories; the period and the three numbers of reflections, start by start, on 50 rational pairs (2,000 starts) and on 63 explicit instances of the cases of Theorems 1.4 and 1.5, with a certificate of Proposition 8.2 for the absence of further periodic points in all 45 instances of Theorem 1.4(b)–(d); 80 irrational pairs with rational sum and 58 pairs with irrational sum; the number of regions and their mirror symmetry, determined from the trajectories in the table alone, on 21 pairs; Theorem 1.3(a) and Lemma 5.1(a), exactly in the full table, for $(\alpha, \lambda) = (\frac{A}{M}, \frac{B}{M})$ with all 560 triples of integers $0 < B < A < M \leq 16$; the statements of Section 5 on 92 exact pairs; and Proposition 8.1, Tables 2 to 4 and the long example of Section 9 as printed. It also ran the quick part of the package again from the archive, and the slow programs separately; all recorded outputs were reproduced. The period table of `check_paper.py` for denominators up to 12 is identical to those of the original program and of the second run, and it agrees with the list computed by the third run. No proof depends on a computation.

Scope and priority. The note answers the first part of Question 1 of [DR15] (Theorem 1.2) and the qualitative part of Question 2 (Theorem 1.3). The periods, and the existence of the periodic region when $\rho_1 + \rho_2$ is irrational, are determined only in the cases of Theorem 1.4(a)–(d) and Theorem 1.5, and for rational pairs by the finite procedure of Proposition 8.1; the questions of Section 9 remain open, and Question 3 is not treated. The problem, the conventions and the examples are due to Dragović and Radnović [DR15, DR14]. The periodicity criterion and the bound on the number of components are applications of theorems of Boshernitzan [Bos88] and McMullen [McM15, McM03], as explained in the introduction and in Remark 4.3; Boshernitzan derives his theorem from a criterion of Veech [Vee87] and quotes Keane’s minimality theorem [Kea75]. Only the reduction with the dictionary (4), the period formula (2), Theorem 1.3(a) and the explicit results of Theorem 1.4(b)–(d) and Theorem 1.5 are, to our knowledge, not in print.

What was read: [DR15] in the open version on the journal’s site, with its three figures; [DR14] in the arXiv version (v1), to which our section and theorem numbers refer, the journal version not being compared; of [Bos88], in the publisher’s file, Sections 1 and 2 and the definitions of Section 3, while its Sections 4 and 5 were read by the first verification run and Sections 6–9 by nobody; of [McM15], in the journal version, the definitions and statements of Sections 2, 3, 6, 7 and 9; of [McM03], in the author’s version of 2007 in the repository of Harvard University, whose theorem numbers we use, the definitions and statements of Sections 2, 4, 6 and 7. The proofs of the cited theorems were not checked. We did not read [Kea75, Vee87]; what we say about them is taken from [Bos88]. Of [FSU18] the abstract was read, of [Fr̆a21] the abstract and the introduction.

The searches were made on 8 October 2026 and repeated in part on 9 October 2026 by the third verification run: queries to arXiv (for pseudo-integrable billiards, for the authors of the problem, for billiards in concentric or half-circular tables, for coupled rotations and for interval exchanges of rank two), to Crossref and to zbMATH, the lists of citing works in OpenAlex, and six web searches. No answer to the questions of [DR15] was found. OpenAlex lists 20 works citing [DR15] and 29 citing [DR14]; none of their titles, and none of the abstracts available there, announces such an answer, and their texts were not read. The closest results we found concern almost every parameter: unique ergodicity of the billiard flow on the invariant sets for ellipses with barriers [FSU18] and for tables bounded by arcs of confocal conics [Fr̆a21]. They do not decide the arithmetic question. McMullen’s paper [McM15] does not mention billiards of this kind, and [DR15, DR14] do not mention coupled rotations. This negative search is not a proof of priority.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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