

Euclidean Realizability of Fundamental-Polytope Types

Alper Ferudun*

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Abstract

For a finite metric d , its fundamental polytope is the convex hull of the vectors $(e_i - e_j)/d_{ij}$. We answer the universality clause of Vershik’s question about Euclidean metrics: not every abstract combinatorial type is Euclidean. The four-cycle metric gives a cube, whereas four Euclidean points give a fundamental polytope with 6, 10, or 12 vertices, never 8. Four points is minimal. In contrast, every strict metric has a Euclidean representative of the same labelled type. Adding a common constant c to its off-diagonal distances preserves its face lattice, and $c \geq 2n \operatorname{diam}(d)$ guarantees an affinely independent Euclidean realization. The argument combines the known four-cycle cube, the Gordon–Petrov face criterion, and the classical additive-constant embedding method. It does not classify all degenerate Euclidean types.

1 The realizability question

For a metric d on $X = \{1, \dots, n\}$, with $n \geq 2$, put

$$H = \left\{ x \in \mathbb{R}^n : \sum_i x_i = 0 \right\}, \quad m_{ij} = \frac{e_i - e_j}{d_{ij}}, \quad R(d) = \operatorname{conv}\{m_{ij} : i \neq j\} \subset H.$$

This is the fundamental, or Kantorovich–Rubinstein, polytope. A metric is *strict* if $d_{ij} < d_{ik} + d_{kj}$ whenever i, j, k are distinct. It is *Euclidean* if it is the ordinary distance metric of distinct points in a real Euclidean space. We allow those points to be affinely dependent. The polytope $R(d)$ nevertheless always has dimension $n - 1$: the vectors m_{in} , $i < n$, span H , and central symmetry puts the origin in its interior.

Vershik [5, Problem 1(4)] asks which combinatorial types occur for Euclidean metrics, and in particular whether all types do. The following two statements separate the general question from its strict-metric restriction. A labelled face-lattice isomorphism below identifies the vertices with the same ordered-pair labels.

Theorem 1. *The four-cycle shortest-path metric has a cubical fundamental polytope. No Euclidean metric has a fundamental polytope of that abstract combinatorial type. Four points is the smallest cardinality for which a non-Euclidean type occurs.*

Theorem 2. *Let d be a strict metric on n points, and set*

$$d_{ii}^{(c)} = 0, \quad d_{ij}^{(c)} = d_{ij} + c \quad (i \neq j).$$

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: [AlperTheKing](https://github.com/AlperTheKing).

For every $c \geq 0$, $R(d^{(c)})$ and $R(d)$ have the same labelled face lattice. For every $c \geq 2n \operatorname{diam}(d)$, the metric $d^{(c)}$ has a realization by n affinely independent points in $\mathbb{R}^{n-1}$. Consequently every strict fundamental-polytope type is Euclidean.

The ingredients have substantial prior history. The four-cycle cube is already explicit in Çelik et al. [2, Example 3]. The type-preservation argument uses Gordon and Petrov’s admissible graph criterion [4, Theorem 3]. It is also consistent with the arrangement description in Delucchi, Kühne and Mühlherr [3, Remark 3.4 and Corollary 3.14]. The addition of a constant to ordinary distances to obtain a Euclidean metric is classical [1]. The contributions here are the explicit realizability deductions and a self-contained sufficient bound; no absolute priority claim is made. In particular, the first theorem is not a claim that the cube itself is a new example.

2 Vertices and the four-point obstruction

We include a direct proof of the standard vertex test, so that the obstruction does not rely on a genericity assumption.

Lemma 3. *For any finite metric, m_{ij} is a vertex of $R(d)$ if and only if there is no $k \notin \{i, j\}$ with $d_{ij} = d_{ik} + d_{kj}$.*

Proof. If such a k exists, then

$$m_{ij} = \frac{d_{ik}}{d_{ij}} m_{ik} + \frac{d_{kj}}{d_{ij}} m_{kj}$$

is a proper convex combination of two other molecules. Conversely, suppose all these inequalities are strict and define

$$f(k) = \frac{d_{ij} + d_{kj} - d_{ik}}{2}.$$

This is the average of the two 1-Lipschitz functions d_{kj} and $d_{ij} - d_{ik}$, so it is 1-Lipschitz. The triangle inequalities give $0 \leq f(k) \leq d_{ij}$, while $f(i) = d_{ij}$ and $f(j) = 0$. For $k \notin \{i, j\}$ the two endpoint constraints have the positive slack

$$d_{ik} - (f(i) - f(k)) = d_{jk} - f(k) = \frac{d_{ik} + d_{kj} - d_{ij}}{2} > 0.$$

Keep $f(i)$ and $f(j)$ fixed and replace every other value by $(1 - \varepsilon)f(k) + \varepsilon d_{ij}/2$. For sufficiently small $\varepsilon > 0$, all endpoint constraints remain strict, and every difference between two other points shrinks by $1 - \varepsilon$. Thus only the ordered pair (i, j) satisfies $f(u) - f(v) = d_{uv}$. Pairing with this perturbed function exposes exactly m_{ij} in $R(d)$. $\square$

Proof of Theorem 1. Give the four cycle edges length 1 and the two diagonals length 2. The diagonal molecules are nonvertices by Lemma 3. With cyclic indices, put $v_i = e_i - e_{i+1}$. Then

$$R(d) = \operatorname{conv}\{\pm v_1, \pm v_2, \pm v_3, \pm v_4\}, \quad v_1 + v_2 + v_3 + v_4 = 0.$$

The first three vectors form a basis of H . The invertible linear map that sends them, respectively, to

$$(1, 1, 1), \quad (1, -1, -1), \quad (-1, 1, -1)$$

sends v_4 to $(-1, -1, 1)$. The eight signed images are precisely the vertices of $[-1, 1]^3$.

Now take four distinct Euclidean points. Equality in the triangle inequality holds exactly when one of the three points lies between the other two on a line. If there is no collinear triple, Lemma 3

gives all 12 molecules as vertices. If exactly one triple is collinear, exactly one unordered pair has an intervening point, giving 10 vertices. If two distinct triples are collinear, they share two distinct points and hence determine the same line. All four points are then collinear. Exactly the three consecutive pairs have no intervening point, giving 6 vertices. The cases exhaust all configurations, so 8 vertices is impossible.

Vertex count is an invariant of the abstract face lattice, without any labelling condition. Since the dimension is always $n - 1$, any metric representing the three-dimensional cube would have to have four points. Finally, every metric on at most three points is realized by a possibly degenerate Euclidean triangle (or fewer points). This proves both the obstruction and its minimality. $\square$

Remark 4. The same obstruction holds in every real Hilbert space, and even in every strictly convex normed space: equality in its triangle inequality has the same collinearity consequence. The argument excludes all metrics of the cubical type, not merely an isometric embedding of the particular four-cycle metric.

3 Preserving strict types by a common shift

A directed graph G on X is *admissible* for d if the molecules of all its directed edges lie in a common facet of $R(d)$. Equivalently, there is a 1-Lipschitz function with $f(x) - f(y) = d_{xy}$ on those edges. The Gordon–Petrov criterion [4, Theorem 3] says that admissibility is equivalent to the following condition: for every array of edges (x_t, y_t) , $1 \leq t \leq k$, in G , with distinct x_t and distinct y_t , one has

$$\sum_{t=1}^k d_{x_t y_t} \leq \sum_{t=1}^k d_{x_t y_{t+1}}, \quad y_{k+1} = y_1. \quad (1)$$

Proposition 5. *If d is strict and $c \geq 0$, the metrics d and $d^{(c)}$ have exactly the same admissible directed graphs, and hence the same labelled fundamental-polytope face lattice.*

Proof. An admissible graph for a strict metric has disjoint sets of sources and targets. Otherwise it contains edges $x \rightarrow y \rightarrow z$. A supporting function would give

$$d_{xz} \geq f(x) - f(z) = d_{xy} + d_{yz},$$

contradicting strictness if $x \neq z$, and positivity if $x = z$. The shifted metric remains strict. Graphs violating disjointness are therefore inadmissible for both metrics.

For a graph with disjoint source and target sets, every distance on either side of (1) is off-diagonal. Adding c to each off-diagonal distance adds exactly kc to each side. Every inequality, including its equality case, is consequently unchanged. The admissible graphs are identical.

All molecules are vertices for a strict metric by Lemma 3. Facet vertex sets are precisely the inclusion-maximal admissible edge sets. Identifying the vertices with the same ordered-pair labels therefore identifies all facets, and then all faces by intersections of facets. This also covers strict metrics whose fundamental polytopes are not simplicial. $\square$

Strictness is essential. For the four-cycle metric, every positive shift makes all triangle inequalities strict and increases the number of vertices from 8 to 12. This is why the positive statement does not contradict the cubical obstruction. The proof uses the face criterion directly; it does not require the arrangement fan of [3] to coincide with the fan of combinatorial types.

4 An explicit Euclidean realization

The next bound applies to every finite metric, though preservation of type in Proposition 5 requires strictness.

Proposition 6. *Let d be a metric on $n \geq 2$ points and let $\Delta = \text{diam}(d)$. If $c \geq 2n\Delta$, then $d^{(c)}$ is the ordinary distance metric of n affinely independent points in $\mathbb{R}^{n-1}$.*

Proof. Write $D = (d_{ij})$, $Q = (d_{ij}^2)$, and let $S(c)$ be the matrix with zero diagonal and entries $(d_{ij} + c)^2$ off the diagonal. For $x \in H$,

$$x^\top S(c)x = -c^2\|x\|^2 + 2cx^\top Dx + x^\top Qx. \quad (2)$$

Indeed, the off-diagonal all-ones matrix has quadratic form $-\|x\|^2$ on H . For any $x \in \mathbb{R}^n$,

$$\sum_{i \neq j} |x_i x_j| \leq (n-1)\|x\|^2,$$

so $|x^\top Dx| \leq (n-1)\Delta\|x\|^2$ and $|x^\top Qx| \leq (n-1)\Delta^2\|x\|^2$. Thus

$$x^\top S(c)x \leq [-c^2 + 2c(n-1)\Delta + (n-1)\Delta^2]\|x\|^2.$$

At $c = 2n\Delta$ the bracket equals $-(3n+1)\Delta^2 < 0$, and it is decreasing for $c \geq 2n\Delta$. Consequently $S(c)$ is negative definite on H .

To construct the points, define the $(n-1) \times (n-1)$ matrix

$$G_{ij} = \frac{S(c)_{in} + S(c)_{jn} - S(c)_{ij}}{2} \quad (i, j < n).$$

For $y \in \mathbb{R}^{n-1}$ and $x = (y_1, \dots, y_{n-1}, -\sum_i y_i)$, direct expansion gives $y^\top Gy = -x^\top S(c)x/2$. Hence G is positive definite. Choose linearly independent vectors $p_1, \dots, p_{n-1}$ with Gram matrix G , and put $p_n = 0$. The defining identity for G gives $\|p_i - p_j\|^2 = S(c)_{ij}$, including pairs involving n . Since $d_{ij} + c > 0$, their ordinary distances are exactly $d_{ij} + c$. $\square$

Propositions 5 and 6 prove Theorem 2. No optimality is claimed for the simple bound $2n\Delta$. After rescaling the resulting configurations by $1/c$, all mutual distances tend to 1 as $c \rightarrow \infty$. Thus every strict type has representatives arbitrarily close in their distance data to a regular simplex.

5 Scope and reproducibility

Theorems 1 and 2 answer the universality clause of Vershik's Problem 1(4): the answer is negative in general and positive for strict types. They do not give a classification of all degenerate Euclidean types or resolve the other parts of Problem 1. The classical constructions and criteria cited above are credited as such; their use here is not a claim of new embedding machinery.

The accompanying source archive includes a standard-library Python checker using exact rational arithmetic. It independently enumerates the four-cycle hull, verifies the 6/10/12 Euclidean examples, compares 72 strict four-point facet lattices under shifting, tests 96 Gram and exposing-potential examples on 2 to 9 points, and checks 14,688 balanced-cycle identities. A nonstrict negative control detects the change from 8 to 12 vertices. These finite checks supplement the proofs; they do not establish the arbitrary-size statements by enumeration.

AI tools assisted with literature searches, proof development, regression code and manuscript preparation. This is an unrefereed preprint; neither independent review nor proof-assistant verification is claimed.

References

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