

Open Sets of Linear Systems with $n = mp$ That Are Not Stabilizable by Static Output Feedback: New Cases of a Problem of Eremenko

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Abstract

A real linear system $\dot{x} = Ax + Bu$, $y = Cx$ with n states, p inputs and m outputs is stabilizable by static output feedback if, for some real matrix K , all eigenvalues of $A + BKC$ have negative real part. Generic systems are stabilizable when $n < mp$, and when $n = mp$ and the degree $d(m, p)$ of the Grassmannian is odd. Eremenko asks for which (m, p) the generic system with $n = mp$ is stabilizable; the only pair known to fail was $(2, 2)$ (Molander; Byrnes and Anderson). We settle some new cases, all negatively; the classification remains open. For $(m, p) = (2, 4)$, $(2, 5)$ and $(3, 3)$ we give explicit integer polynomial matrices with certificates, checkable in exact arithmetic, which show by the criterion of Byrnes and Anderson that the systems that are not stabilizable contain a non-empty open set; the same follows for $(4, 2)$ and $(5, 2)$. We prove the same for $(2, p)$, $(p, 2)$, $(4, p)$ and $(p, 4)$ with every even p , which answers a question of Eremenko and Gabrielov when $\min(m, p) \in \{2, 4\}$; here the examples are complex polynomial curves regarded as real ones, and the proof is a count of dimensions. As a by-product, real pole placement is not generically possible for $(2, 5)$, $(5, 2)$ and $(3, 3)$, which are cases of a conjecture of Rosenthal and Sottile. We also note that the 2×4 matrix printed in the problem file dated October 3, 2026 differs in a sign from the example of Byrnes and Anderson and is not a counterexample. This is an unrefereed note.

1 Introduction

A real linear system with p inputs, m outputs and n states is

$$\dot{x} = Ax + Bu, \quad y = Cx, \quad \Sigma = (A, B, C) \in \mathfrak{S}_{m,p,n} = \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times p} \times \mathbb{R}^{m \times n}.$$

A static output feedback $u = Ky$, with a real $p \times m$ matrix K , gives the closed loop system $\dot{x} = (A + BKC)x$. The system Σ is *stabilizable* (by real static output feedback) if for some K all eigenvalues of $A + BKC$ have negative real part. The set of stabilizable systems is open, semialgebraic and not empty (take A stable and $K = 0$). As usual, *the generic system is stabilizable* means that this set is dense in $\mathfrak{S}_{m,p,n}$; as its complement is closed and semialgebraic, this is the same as saying that the complement lies in a proper algebraic subset. The negation is: *there is a non-empty open set of systems that are not stabilizable*. All negative statements below have this form; they do not say that most systems fail to be stabilizable. As in [Ere26], p is the number of inputs and m the number of outputs; in [BA84, EG02c, RS98] the letters are interchanged, which is harmless, since (A, B, C) is stabilizable if and only if (A^T, C^T, B^T) is.

For $n < mp$ the generic system is stabilizable, by a theorem of X. Wang on pole placement [Wan92]; see [EG02c, Ere26]. Byrnes and Anderson proved that $mp \geq n$ is necessary [BA84, Theorem 2]. In the

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critical case $n = mp$, for a generic system every monic polynomial of degree n equals $\det(zI - A - BKC)$ for exactly $d(m, p) = 1!2! \cdots (p-1)!(mp)! / (m!(m+1)! \cdots (m+p-1)!)$ complex matrices K , counted with multiplicity; this is the degree of the Grassmannian [BB81]. Hence for odd $d(m, p)$ the generic real system can be assigned every real closed loop polynomial, and is stabilizable. By a result of Bernstein [Ber76], quoted in [RS98], $d(m, p)$ is odd exactly when $\min(m, p) = 1$, or $\min(m, p) = 2$ and $\max(m, p) = 2^k - 1$. The only pair for which the generic system was known not to be stabilizable is $(2, 2)$: this is [BA84, Theorem 3], where it is attributed to unpublished work of P. Molander. Eremenko's list of unsolved problems contains the question [Ere26]: "For which m and p the generic system with $n = mp$ is stabilizable?" It is record AMR-036-0044 of the UnsolvedMath dataset [ula26].

A stronger property is real pole placement: every monic real polynomial of degree n is $\det(zI - A - BKC)$ for some real K . For $n = mp$ it fails on a non-empty open set of systems when $(m, p) = (2, 2)$ [WH78], when $(m, p) = (4, 2)$ [RS98], when $\min(m, p) = 2$ and $\max(m, p)$ is even [EG02a], and whenever m and p are both even [EG02c]; Rosenthal and Sottile conjectured that it fails in this sense whenever $d(m, p)$ is even [RS98]. These results do not decide stabilizability. For example, the open set in [EG02c] is a neighbourhood of the Wronski system, whose closed loop polynomials are, in the notation of [EG02c], Wronskians of p polynomials. That system is stabilizable, because a real gain gives the span of $(z+1)^m, \dots, (z+1)^{m+p-1}$, whose Wronskian is $c(z+1)^{mp}$ with $c \neq 0$; so are all systems close to it. Accordingly, Eremenko and Gabrielov ask at the end of the introduction of [EG02c] whether generic systems with $n = mp$ and even m, p are stabilizable.

Results. For a real $m \times (m+p)$ matrix $V(z)$ of polynomials and a real $p \times (m+p)$ matrix L , let $\begin{bmatrix} V(z) \\ L \end{bmatrix}$ be the square matrix with the rows of $V(z)$ on top of those of L .

Theorem 1.1. *Let (m, p) be $(2, 4)$, $(2, 5)$ or $(3, 3)$, and let $V(z)$ be the integer polynomial matrix of size $m \times (m+p)$ displayed in Section 3. Then no real $p \times (m+p)$ matrix L of rank p satisfies $\det \begin{bmatrix} V(z) \\ L \end{bmatrix} = cz^{mp}$ with a real constant c , the value $c = 0$ included. Consequently, for $(m, p) \in \{(2, 4), (4, 2), (2, 5), (5, 2), (3, 3)\}$ and $n = mp$ there is a non-empty open subset of $\mathfrak{S}_{m,p,n}$ consisting of systems that are not stabilizable by real static output feedback.*

Theorem 1.2. *Let $p \geq 2$ be even and $n = 2p$. Then there are non-empty open subsets of $\mathfrak{S}_{2,p,n}$ and of $\mathfrak{S}_{p,2,n}$ consisting of systems that are not stabilizable by real static output feedback.*

Theorem 1.3. *Let $p \geq 2$ be even and $n = 4p$. Then there are non-empty open subsets of $\mathfrak{S}_{4,p,n}$ and of $\mathfrak{S}_{p,4,n}$ consisting of systems that are not stabilizable by real static output feedback.*

The case $p = 2$ of Theorem 1.2 is the known case $(2, 2)$. Theorems 1.2 and 1.3 answer the question of [EG02c] negatively when $\min(m, p) \in \{2, 4\}$. The pair $(2, 4)$ with $n = 8$ was left open in [BA84, Example 3]. The pairs $(2, 5)$ and $(3, 3)$ show that generic stabilizability can fail when $m+p$ is odd, and when m and p are odd. We do not obtain the classification asked for in [Ere26]; see Table 1.

(m, p) , up to order	$d(m, p)$	generically stabilizable?	source
$\min(m, p) = 1$, or $(2, 2^k - 1)$	odd	yes	[BB81, Ber76]
$(2, 2)$	2	no	Molander, [BA84]
$(2, 4), (2, 5), (3, 3)$	14, 42, 42	no	Theorem 1.1
$(2, p)$ and $(4, p)$, p even	even	no	Theorems 1.2 and 1.3
all others, e.g. $(2, 9), (3, 4), (5, 5)$	even	open	

Table 1: Generic stabilizability by real static output feedback for $n = mp$.

A system that is not stabilizable admits no real K with a stable closed loop polynomial. Hence:

Corollary 1.4. *For $(m, p) \in \{(2, 5), (5, 2), (3, 3)\}$ and $n = mp$ there is a non-empty open set of systems for which the real pole placement map $K \mapsto \det(zI - A - BKC)$ is not surjective.*

These are cases of the conjecture of Rosenthal and Sottile mentioned above (Conjecture 2.5 in the arXiv version of [RS98]): for $n = mp$ and even $d(m, p)$ the pole placement map is not generically surjective over $\mathbb{R}$. We did not find them in the literature (see the end of this note).

Method. Byrnes and Anderson reduced generic stabilizability for $n = mp$ to the assignment of the single polynomial z^{mp} [BA84, Theorem 1]; in the formulation of [Ere26], the generic system is stabilizable if and only if $\det \begin{bmatrix} V(z) \\ L \end{bmatrix} = cz^{mp}$ has a real solution L for every non-degenerate polynomial matrix $V(z)$ of the appropriate type. We prove the direction we need, in a form that also allows degenerate V (Proposition 2.3). It reduces the theorems to finding one V without real solutions; according to [Ere26], no regular procedure for finding such matrices was known. In Theorem 1.1 the absence of real solutions is certified by a combination of quadratic Plücker relations that is positive definite on a linear space. For Theorems 1.2 and 1.3 we regard an $a \times (a + b)$ matrix of complex polynomials of degree $2b$ as a real matrix of size $2a \times (2a + 2b)$. Its real solutions L come in circles $e^{i\theta}L$, unless L is a complex subspace, so the mp equations for L are in effect imposed on a family of dimension $mp - 1$, and a count of dimensions for $a = 1, 2$ shows that the generic complex curve has no real solution.

2 The criterion

Fix $m, p \geq 1$ and put $N = m + p$, $n = mp$. Let $P(m, p)$ be the real vector space of $m \times N$ matrices $V(z) = V_0 + V_1z + \dots + V_pz^p$ with real matrices V_j . For an m -element set $I \subset \{1, \dots, N\}$ let $V_I(z)$ be the minor of $V(z)$ on the columns I ; it is a polynomial of degree at most n . We write $V_I(z) = \sum_{k=0}^n M_k(V)_I z^k$, so that $M_k(V)$ is a vector in $\Lambda = \mathbb{R}^{\binom{N}{m}}$, the space with coordinates y_I . A vector $y \in \Lambda$ is *decomposable* if $y_I = Y_I$ for all I , where Y is a real $m \times N$ matrix and Y_I are its maximal minors. The decomposable vectors form a closed cone $\mathcal{D} \subset \Lambda$, the cone over the real Grassmannian in its Plücker embedding.

Lemma 2.1. *Let L be a real $p \times N$ matrix of rank p , and let Y be a real $m \times N$ matrix whose rows form a basis of $\{y \in \mathbb{R}^N : Ly^T = 0\}$. Then $\det \begin{bmatrix} V(z) \\ L \end{bmatrix} = \kappa \sum_I V_I(z) Y_I$ for every $V \in P(m, p)$, with a constant $\kappa = \kappa(L, Y) \neq 0$. Every non-zero vector of $\mathcal{D}$ is $(Y_I)_I$ for such a pair (L, Y) .*

Proof. The matrix $[Y^T \mid L^T]$ is invertible, and $\begin{bmatrix} V \\ L \end{bmatrix} [Y^T \mid L^T]$ is block triangular with diagonal blocks VY^T and LL^T . So the claim holds with $\kappa = \det(LL^T)/\det[Y^T \mid L^T]$, because $\det(VY^T) = \sum_I V_I Y_I$ by the Cauchy–Binet formula. The last statement is clear. $\square$

Definition 2.2. For $V \in P(m, p)$ let $U(V) = \{y \in \Lambda : \langle M_k(V), y \rangle = 0 \text{ for } 0 \leq k \leq n - 1\}$. We say that V has property (NS) if $U(V) \cap \mathcal{D} = \{0\}$.

By Lemma 2.1, V has (NS) if and only if there is no real L of rank p with $\det \begin{bmatrix} V(z) \\ L \end{bmatrix} = cz^n$, $c \in \mathbb{R}$. The case $c = 0$, in which the determinant vanishes identically, is included.

Proposition 2.3 (Byrnes–Anderson criterion, one direction). *Let $n = mp$. If some $V_0 \in P(m, p)$ has property (NS), then $\mathfrak{S}_{m,p,n}$ and $\mathfrak{S}_{p,m,n}$ contain non-empty open sets of systems that are not stabilizable by real static output feedback.*

Proof. Let $\mathcal{M}$ be the space of real matrices with rows $M_0, \dots, M_n \in \Lambda$. For $M \in \mathcal{M}$ and $y \in \Lambda$ put $P_{M,y}(z) = \sum_{k=0}^n \langle M_k, y \rangle z^k$. If $M(V)$ denotes the matrix with rows $M_k(V)$, then $P_{M(V),y} = \sum_I V_I y_I$. We call M *Hurwitz-free* if there is no $y \in \mathcal{D}$ such that $P_{M,y}$ has degree n and all its roots in $\{\operatorname{Re} z < 0\}$.

For $\varepsilon > 0$ we call M ε -good if there is no $y \in \mathcal{D} \setminus \{0\}$ such that $P_{M,y}$ is identically zero, or has degree n and all its roots in $\{|z| \leq \varepsilon\}$.

Step 1. For some $\varepsilon > 0$, a neighbourhood $\mathcal{U}$ of $M(V_0)$ in $\mathcal{M}$ consists of ε -good matrices. Otherwise there are $M^{(i)} \rightarrow M(V_0)$, $\varepsilon_i \rightarrow 0$ and unit vectors $y_i \in \mathcal{D}$ such that $P_i = P_{M^{(i)}, y_i}$ is zero or has degree n and all roots in $\{|z| \leq \varepsilon_i\}$. We may assume $y_i \rightarrow y$; then $y \in \mathcal{D}$ and $|y| = 1$. Let $c_k^{(i)}$ be the coefficients of P_i . In the second case $|c_k^{(i)}| \leq \binom{n}{k} \varepsilon_i^{n-k} |c_n^{(i)}|$ for $k < n$, and $c_n^{(i)} = \langle M_n^{(i)}, y_i \rangle$ is bounded. In both cases $c_k^{(i)} \rightarrow 0$ for $k < n$, so $\langle M_k(V_0), y \rangle = 0$ for $k < n$. This contradicts (NS).

Step 2. The map $T_\varepsilon: \varphi(z) \mapsto (s-1)^n \varphi(\varepsilon \frac{s+1}{s-1})$ is a linear bijection of the space of polynomials of degree at most n . If $\varphi = c \prod_{j=1}^n (z - r_j)$ with $c \neq 0$, then $T_\varepsilon \varphi = c \prod_j ((\varepsilon - r_j)s + (\varepsilon + r_j))$. If $\varphi \neq 0$ has degree less than n , then $T_\varepsilon \varphi$ vanishes at $s = 1$. Since $s \mapsto \varepsilon \frac{s+1}{s-1}$ maps $\{\operatorname{Re} s < 0\}$ onto $\{|z| < \varepsilon\}$, we conclude: if $T_\varepsilon \varphi$ has degree n and all its roots in $\{\operatorname{Re} s < 0\}$, then φ has degree n and all its roots in $\{|z| < \varepsilon\}$. Let $M \mapsto M^\varepsilon$ be the linear automorphism of $\mathcal{M}$ with $P_{M^\varepsilon, y} = T_\varepsilon P_{M, y}$ for all y . Thus M^ε is Hurwitz-free whenever M is ε -good.

Step 3. For $g \in \operatorname{GL}_N(\mathbb{R})$ let $\hat{g}$ be the matrix of its $m \times m$ minors, $\hat{g}_{J,I} = \det(g_{ji})_{j \in J, i \in I}$. By the Cauchy–Binet formula $(Vg)_I = \sum_J V_J \hat{g}_{J,I}$, that is, $M(Vg) = M(V)\hat{g}$, and $\hat{g}(Y_I)_I = ((Yg^T)_J)_J$. So $\hat{g}$ maps $\mathcal{D}$ onto itself, $P_{M\hat{g}, y} = P_{M, \hat{g}y}$, and $M \mapsto M\hat{g}$ is a linear automorphism of $\mathcal{M}$ that preserves the Hurwitz-free matrices.

Step 4. For a real $p \times m$ matrix K let $y(K) \in \mathcal{D}$ be the vector of maximal minors of $[I_m \mid -K^T]$. Its coordinates are, up to sign, the minors of K of all sizes, the empty minor 1 included, and these are linearly independent functions of K . Let $\Sigma = (A, B, C) \in \mathfrak{S}_{m,p,n}$. We have $\det(zI - A - BKC) = \det \begin{bmatrix} zI - A & B \\ KC & I_p \end{bmatrix}$, and the last p rows of this matrix are $[K \mid I_p] \begin{bmatrix} C & 0 \\ 0 & I_p \end{bmatrix}$. Expanding along these rows, we see that $\det(zI - A - BKC)$ is a linear combination of the $p \times p$ minors of $[K \mid I_p]$, which are again the minors of K , with coefficients that are polynomials in z and in the entries of Σ . Hence there is a unique $M_\Sigma \in \mathcal{M}$, depending continuously on Σ , with $\det(zI - A - BKC) = P_{M_\Sigma, y(K)}(z)$ for all K . If Σ is stabilizable, then M_Σ is not Hurwitz-free. So for every open set $\mathcal{W} \subset \mathcal{M}$ of Hurwitz-free matrices, $\{\Sigma : M_\Sigma \in \mathcal{W}\}$ is an open set of systems that are not stabilizable, and it remains to find $\mathcal{W}$ and one system Σ_0 with $M_{\Sigma_0} \in \mathcal{W}$.

Step 5. An ε -good matrix is ε' -good for $\varepsilon' < \varepsilon$, and by (NS) not all minors of V_0 vanish identically. We may therefore assume that $V_0(\varepsilon)$ has rank m . Put $W(s) = (s-1)^p V_0(\varepsilon \frac{s+1}{s-1})$. Then $W \in P(m, p)$, the coefficient of s^p in W is $V_0(\varepsilon)$, and $W_I = T_\varepsilon((V_0)_I)$ for all I , so that $M(W) = M(V_0)^\varepsilon$. Choose $g \in \operatorname{GL}_N(\mathbb{R})$ with $V_0(\varepsilon)g = [I_m \mid 0]$. Then $Wg = [D \mid N]$ with $D(s) = s^p I_m + \sum_{j < p} D_j s^j$ and $N(s) = \sum_{j < p} N_j s^j$. Let $\Sigma_0 = (A_0, B_0, C_0) \in \mathfrak{S}_{m,p,n}$ be

$$A_0 = \begin{pmatrix} -D_{p-1} & I_m & & \\ \vdots & & \ddots & \\ -D_1 & & & I_m \\ -D_0 & 0 & \cdots & 0 \end{pmatrix}, \quad B_0 = \begin{pmatrix} N_{p-1} \\ \vdots \\ N_1 \\ N_0 \end{pmatrix}, \quad C_0 = (I_m \quad 0 \quad \cdots \quad 0).$$

For every K the matrix $A_0 + B_0 K C_0$ is the block companion matrix of the monic matrix polynomial $D(s) - N(s)K$. Hence, by the Cauchy–Binet formula,

$$\det(sI - A_0 - B_0 K C_0) = \det(D(s) - N(s)K) = \det \left([D \mid N] \begin{bmatrix} I_m \\ -K \end{bmatrix} \right) = \sum_I (Wg)_I(s) y(K)_I.$$

So $M_{\Sigma_0} = M(Wg) = M(V_0)^\varepsilon \hat{g}$. By Steps 1–3 this matrix lies in the open set $\mathcal{W} = \{M^\varepsilon \hat{g} : M \in \mathcal{U}\}$ of Hurwitz-free matrices, and Step 4 gives the open set in $\mathfrak{S}_{m,p,n}$. Its image under the homeomorphism $(A, B, C) \mapsto (A^T, C^T, B^T)$ is an open set of non-stabilizable systems in $\mathfrak{S}_{p,m,n}$. $\square$

Remark 2.4. In [Ere26], $Q(m, m+p)$ is the set of the V whose minors V_I are coprime and have maximal degree n ; a matrix V is *degenerate* if $\det \begin{bmatrix} V \\ L \end{bmatrix} \equiv 0$ for some L in the Grassmannian $G(p, m+p)$; and the criterion requires a real solution of $\det \begin{bmatrix} V \\ L \end{bmatrix} = cz^n$ for every non-degenerate $V \in Q(m, m+p)$. The file does not say whether L is real or complex in the definition of degeneracy; in Remark 3.2 we use complex L , which gives the stronger notion of non-degeneracy. In [BA84, Theorem 1] the criterion is stated for non-degenerate systems, and proved by the same two ideas as above, a Möbius transformation and compactness. Proposition 2.3 needs neither $V_0 \in Q(m, m+p)$ nor non-degeneracy, because (NS) excludes the real solutions with $c = 0$; this matters in Section 4. By Step 1, (NS) is an open condition on $V \in P(m, p)$.

3 Certified examples

Let $I = \{i_1 < \dots < i_{m-1}\}$ and $J = \{j_0 < \dots < j_m\}$ be subsets of $\{1, \dots, N\}$. The Plücker relation $R_{I,J}$ is the quadratic form $R_{I,J}(y) = \sum_{t=0}^m (-1)^t y_{I,j_t} y_{J \setminus \{j_t\}}$, where $y_{I,j} = 0$ if $j \in I$, and otherwise $y_{I,j} = \pm y_{I \cup \{j\}}$ with the sign of the permutation that sorts $(i_1, \dots, i_{m-1}, j)$. The forms $R_{I,J}$ vanish on $\mathcal{D}$, and their common complex zeros are exactly the complex decomposable vectors [KL72]. For $m = 2$ they are, up to sign, the Pfaffians $\text{Pf}_{abcd}(y) = y_{ab}y_{cd} - y_{ac}y_{bd} + y_{ad}y_{bc}$, $a < b < c < d$.

Lemma 3.1. *Let $V \in P(m, p)$. If some real linear combination Q of the forms $R_{I,J}$ is positive definite on $U(V)$, then V has property (NS).*

Proof. If $y \in U(V) \cap \mathcal{D}$, then $Q(y) = 0$, hence $y = 0$. □

The three matrices of Theorem 1.1 are $V(z) = \sum_j V_j z^j$ with the following coefficients.

The case $(m, p) = (2, 4)$. Here $n = 8$, $\dim \Lambda = 15$ and

$$\begin{aligned} V_0 &= \begin{pmatrix} 4 & 5 & 1 & 1 & 5 & -2 \\ 2 & -2 & -5 & 3 & 1 & 0 \end{pmatrix}, \quad V_1 = \begin{pmatrix} 3 & -5 & -6 & 0 & 1 & -1 \\ -5 & 0 & 1 & 0 & -5 & -4 \end{pmatrix}, \quad V_2 = \begin{pmatrix} -2 & 3 & 0 & -2 & 0 & -1 \\ 0 & -3 & -1 & -1 & 1 & -6 \end{pmatrix}, \\ V_3 &= \begin{pmatrix} 5 & 1 & 2 & -5 & -1 & -4 \\ 1 & -1 & 3 & 4 & -1 & -6 \end{pmatrix}, \quad V_4 = \begin{pmatrix} 3 & -2 & -1 & 6 & -1 & -2 \\ 0 & -5 & 2 & -2 & 2 & -2 \end{pmatrix}. \end{aligned}$$

The certificate is

$$\begin{aligned} Q &= -\text{Pf}_{1234} + \text{Pf}_{1236} - \text{Pf}_{1245} - 2\text{Pf}_{1246} + \text{Pf}_{1256} - 2\text{Pf}_{1456} \\ &\quad + \text{Pf}_{2346} + \text{Pf}_{2356} - \text{Pf}_{2456} - \text{Pf}_{3456}. \end{aligned}$$

The case $(m, p) = (2, 5)$. Here $n = 10$, $\dim \Lambda = 21$ and

$$\begin{aligned} V_0 &= \begin{pmatrix} -3 & -3 & 3 & 0 & 2 & -1 & 1 \\ 0 & 0 & 1 & -5 & -1 & -2 & -2 \end{pmatrix}, \quad V_1 = \begin{pmatrix} -1 & 1 & 5 & -1 & 0 & -2 & 0 \\ 6 & 3 & 1 & 1 & 1 & -4 & -1 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 1 & -1 & 2 & 4 & 0 & 2 & 2 \\ -3 & -2 & 1 & 0 & -2 & -2 & -1 \end{pmatrix}, \\ V_3 &= \begin{pmatrix} 1 & 0 & 2 & 0 & -2 & -3 & 5 \\ 2 & 5 & -1 & -1 & -3 & -1 & 0 \end{pmatrix}, \quad V_4 = \begin{pmatrix} 0 & 1 & 0 & 4 & 1 & -3 & 2 \\ -2 & 1 & 0 & -3 & 2 & -1 & 0 \end{pmatrix}, \quad V_5 = \begin{pmatrix} 3 & 2 & 2 & -1 & 2 & -3 & -2 \\ 4 & 2 & -4 & -1 & -1 & 0 & -1 \end{pmatrix}. \end{aligned}$$

The certificate is $Q = \sum \mu_{abcd} \text{Pf}_{abcd}$, where the 35 coefficients, for the sets $\{a < b < c < d\} \subset \{1, \dots, 7\}$ in lexicographic order 1234, 1235, ..., 4567, are

$$\begin{aligned} &2, 0, 1, -1, -2, 3, -1, 0, 0, 1, -1, 3, 1, -1, 0, -1, 1, 0, -1, 0, \\ &0, 2, 0, -1, 1, 1, 0, -1, 0, -2, -1, -2, -1, 2, 1. \end{aligned}$$

The case $(m, p) = (3, 3)$. Here $n = 9$, $\dim \Lambda = 20$ and

$$V_0 = \begin{pmatrix} 8 & -7 & -2 & 4 & -11 & 1 \\ -2 & -4 & -10 & -1 & 17 & -2 \\ -7 & 9 & -7 & -4 & 3 & -5 \end{pmatrix}, \quad V_1 = \begin{pmatrix} -5 & 6 & 9 & 2 & -4 & -10 \\ -7 & -4 & -18 & -8 & -4 & 10 \\ 4 & 7 & 0 & -1 & 1 & 2 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 6 & 6 & 6 & 1 & 2 & -11 \\ -16 & -13 & 1 & 5 & -3 & -6 \\ 11 & 3 & -3 & 11 & 3 & 7 \end{pmatrix},$$

$$V_3 = \begin{pmatrix} -6 & 0 & 2 & -13 & 12 & -6 \\ 5 & 7 & 3 & 2 & 9 & 7 \\ 0 & 9 & -3 & -4 & 4 & -8 \end{pmatrix}.$$

The certificate is $Q = 10^{-7} \sum \lambda_{I|J} R_{I,J}$ with 45 integer multipliers $\lambda_{I|J}$, $|\lambda_{I|J}| \leq 10^7$. The sum is over the 45 forms that remain when the pairs (I, J) with $|I| = 2$, $|J| = 4$ are taken in lexicographic order and the forms $R_{I,J}$ that vanish identically, or are equal up to sign to an earlier one, are omitted; 30 of them have three terms and 15 have four. The multipliers are too long to print here. They are in the file `cert_33_nice.json` of the source files, in this order, and with the labels $I|J$ in the file `certificates_readable.txt`.

Proof of Theorem 1.1. For each of the three matrices it was verified in exact rational arithmetic (see the Verification paragraph) that $U(V)$ has dimension $\binom{N}{m} - n$, that is, 7, 11 and 11, and that the restriction of Q to $U(V)$ is positive definite. By Lemma 3.1, V has property (NS), and the second assertion follows from Proposition 2.3. $\square$

Remark 3.2. The same computations show that the minors of each V are coprime with maximal degree n , and that the forms $R_{I,J}$ have no common non-zero complex zero on the kernel of $M(V)$. So $V \in Q(m, m + p)$ is non-degenerate, and the three matrices are counterexamples also in the formulation of [Ere26]. As V_p has rank m , Step 5 applied to V itself gives rational systems with 8, 10 and 9 states for which no real K makes $A + BKC$ nilpotent. The matrices were found numerically: of 1489 random matrices with Gaussian coefficients for $(2, 4)$, six had no real solution; for $(3, 3)$ and $(2, 5)$ all of 203 and 245 random matrices had real solutions, and the examples came from a random walk that never increases the number of real solutions, followed by numerical optimization and rounding.

Remark 3.3 (The case $(2, 2)$). For $m = p = 2$ the space $U(V)$ is two-dimensional for generic V and $\mathcal{D}$ is the cone $\{\text{Pf}_{1234} = 0\}$, so (NS) holds if and only if Pf_{1234} is definite on $U(V)$. The example of [BA84, Example 4] is the system

$$F = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad G = \begin{pmatrix} I_2 \\ 0 \end{pmatrix}, \quad H = (I_2 \quad 0), \quad \text{and we put } V'(s) = \begin{pmatrix} s^2 & 1 & s & 0 \\ s-1 & s^2 & 1 & s \end{pmatrix}.$$

A direct computation with these matrices gives $\det(sI - F - GKH) = \det \begin{bmatrix} V'(s) \\ K \quad I_2 \end{bmatrix}$. For V' the equation $\sum_I V'_I y_I = c s^4$ reads $y_{12} = c$, $y_{23} = y_{14} = -c$, $y_{34} = 0$, $y_{13} + y_{24} = c$, and on these vectors $\text{Pf}_{1234} = c^2 - y_{13}y_{24}$. So y_{13} is a root of $t^2 - ct + c^2$, and there is no real solution $y \neq 0$. (The expansion printed as (4.7) in [BA84] differs from the one obtained from these matrices in some signs; the conclusion is the same.) The matrix printed in [Ere26], in the version dated October 3, 2026, has $z + 1$ in place of $z - 1$. For that matrix the equations are $y_{12} = y_{23} = y_{14} = c$, $y_{34} = 0$, $y_{24} - y_{13} = c$ and $y_{13}y_{24} = c^2$, with two real solutions for $c \neq 0$: the determinant of the 4×4 matrix with rows $(z^2, 1, z, 0)$, $(z + 1, z^2, 1, z)$, $(1, -t, 1, 0)$, $(t + 1, -1, 0, 1)$ is z^4 for $t = (\sqrt{5} - 1)/2$. So the matrix as printed is not a counterexample. The sign is presumably a misprint; the file is kept on the web and may be corrected.

4 Complex curves as real systems; the case $m = 2$

Let $a, b \geq 1$ and $d = 2b$. Let $\mathcal{V}_{a,b}$ be the real vector space of $a \times (a+b)$ matrices $V_{\mathbb{C}}(z)$ whose entries are complex polynomials of degree at most d . We identify $\mathbb{C}^{a+b}$ with $\mathbb{R}^{2(a+b)}$. For $V_{\mathbb{C}} \in \mathcal{V}_{a,b}$ with rows $v_1, \dots, v_a$ let $V_{\mathbb{R}}(z)$ be the real $2a \times 2(a+b)$ polynomial matrix with rows $v_1, iv_1, \dots, v_a, iv_a$. Thus $V_{\mathbb{R}} \in P(m, p)$ with $m = 2a$, $p = 2b$ and $n = 4ab = 2ad$. For a polynomial f with complex coefficients let $f^*(z) = \overline{f(\bar{z})}$, $\operatorname{Re} f = (f + f^*)/2$ and $\operatorname{Im} f = (f - f^*)/2i$; the same notation is used for matrices of polynomials, with $\bar{S}$ for the matrix of the f^* .

Every real linear functional on $\mathbb{C}^{a+b}$, a space of row vectors, is $w \mapsto \operatorname{Re}(w\xi)$ for a unique column $\xi \in \mathbb{C}^{a+b}$. Hence the real subspaces of $\mathbb{C}^{a+b}$ of real dimension $2b$ are the spaces $L_{\Xi} = \{w : \operatorname{Re}(w\Xi) = 0\}$, where Ξ is a complex $(a+b) \times 2a$ matrix whose columns are linearly independent over $\mathbb{R}$.

Lemma 4.1. *Let $V_{\mathbb{C}} \in \mathcal{V}_{a,b}$ and let Ξ be as above. Put $S = V_{\mathbb{C}} \Xi$, an $a \times 2a$ matrix of polynomials, and $\Delta_S = \det \begin{bmatrix} \operatorname{Re} S \\ \operatorname{Im} S \end{bmatrix}$. If the rows of the real matrix L form a basis of L_{Ξ} , then $\det \begin{bmatrix} V_{\mathbb{R}} \\ L \end{bmatrix} = \kappa \Delta_S$ with a constant $\kappa \neq 0$. Consequently $V_{\mathbb{R}}$ has property (NS) if and only if there is no such Ξ with $\Delta_S \in \mathbb{R}z^{2ad}$. Moreover $\Delta_S = (-2i)^{-a} \det \begin{bmatrix} S \\ \bar{S} \end{bmatrix}$, and $\Delta_{e^{i\theta}Sh} = \det(h) \Delta_S$ for $\theta \in \mathbb{R}$ and $h \in \operatorname{GL}_{2a}(\mathbb{R})$.*

Proof. The functionals $w \mapsto \operatorname{Re}(w\xi_c)$, where ξ_c are the columns of Ξ , form a basis of the annihilator of L_{Ξ} . Let Y be the matrix of their coordinate vectors. By Lemma 2.1, $\det \begin{bmatrix} V_{\mathbb{R}} \\ L \end{bmatrix} = \kappa \det(V_{\mathbb{R}} Y^T)$. The entries of $V_{\mathbb{R}} Y^T$ in the rows belonging to v_j and iv_j are $\operatorname{Re}(v_j \xi_c) = \operatorname{Re} S_{jc}$ and $\operatorname{Re}(iv_j \xi_c) = -\operatorname{Im} S_{jc}$. So $\det(V_{\mathbb{R}} Y^T) = \pm \Delta_S$. The other statements follow from $S = \operatorname{Re} S + i \operatorname{Im} S$, $\bar{S} = \operatorname{Re} S - i \operatorname{Im} S$ and $\operatorname{Re}(Sh) = (\operatorname{Re} S)h$, and from the fact that $\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \otimes I_a$ has determinant 1. $\square$

All sets below are semialgebraic, and $\dim$ is the dimension of semialgebraic sets [BCR98, Section 2.8]; a complex vector space of dimension k has dimension $2k$. We use two inequalities.

Lemma 4.2. *Let $f: X \rightarrow Y$ be a continuous semialgebraic map of semialgebraic sets.*

- (a) $\dim X \leq \dim f(X) + \max_{y \in f(X)} \dim f^{-1}(y)$.
- (b) *If $\dim f^{-1}(y) \geq k$ for all $y \in f(X)$, then $\dim f(X) \leq \dim X - k$.*

Proof. By Hardt's theorem on semialgebraic triviality [BCR98, Section 9.3], $f(X)$ is a finite union of semialgebraic sets T_l such that $f^{-1}(T_l)$ is semialgebraically homeomorphic to $T_l \times F_l$, where F_l is the fibre of f over a point of T_l . The dimension is invariant under semialgebraic homeomorphisms, the dimension of a product is the sum of the dimensions, and the dimension of a finite union is the maximum of the dimensions [BCR98, Section 2.8]. Hence $\dim X = \max_l (\dim T_l + \dim F_l)$ and $\dim f(X) = \max_l \dim T_l$. $\square$

When we apply (b), each fibre contains the image of a k -dimensional semialgebraic set under a semialgebraic map with finite fibres; such an image has dimension k by (a).

Lemma 4.3. *For $\nu \geq 0$ let Z_{ν} be the set of pairs (s_1, s_2) of complex polynomials of degree at most ν with $\operatorname{Im}(s_1^* s_2) \in \mathbb{R}z^{2\nu}$, and let $Z_{\nu}^0 \subset Z_{\nu}$ be given by $\operatorname{Im}(s_1^* s_2) = 0$. Then $\dim Z_{\nu} \leq 2\nu + 4$ and $\dim Z_{\nu}^0 \leq 2\nu + 3$, in a space of dimension $4\nu + 4$.*

Proof. Put $f = s_1 + is_2$ and $g = s_1 - is_2$. Then $4\operatorname{Im}(s_1^* s_2) = gg^* - ff^*$. So $(s_1, s_2) \in Z_{\nu}$ if and only if $ff^* = gg^* - 4cz^{2\nu}$ for some $c \in \mathbb{R}$, and c is determined by (f, g) . For given (g, c) let $R = gg^* - 4cz^{2\nu}$. If $R = 0$, then $f = 0$. If $R \neq 0$ and $ff^* = R$, then the roots of f form one half of the roots of R , and the modulus of the leading coefficient of f is determined; so f is determined up to finitely many choices and a constant factor of modulus 1. Thus the fibres of $(f, g) \mapsto (g, c)$ on Z_{ν} have dimension at most 1, and Lemma 4.2(a) gives $\dim Z_{\nu} \leq (2\nu + 2) + 1 + 1$. On Z_{ν}^0 we have $c = 0$, which gives $2\nu + 3$. $\square$

Proof of Theorem 1.2. Let $a = 1$, $b \geq 1$, $d = 2b$, and write v for the single row of $V_{\mathbb{C}}$. Then $\mathcal{V} = \mathcal{V}_{1,b}$ has dimension $D = 2(b+1)(d+1)$. For $\Xi = (\xi_1, \xi_2)$ we have $S = (s_1, s_2)$ with $s_c = v\xi_c$, and $\Delta_S = \text{Im}(s_1^* s_2)$. By Lemma 4.1 the set $\mathcal{B} \subset \mathcal{V}$ of all v for which $V_{\mathbb{R}}$ fails to have (NS) is the projection to $\mathcal{V}$ of the set $\mathcal{I}$ of all triples (v, ξ_1, ξ_2) such that ξ_1, ξ_2 are linearly independent over $\mathbb{R}$ and $(v\xi_1, v\xi_2) \in Z_d$. We show that $\dim \mathcal{B} \leq D - 1$. There are two types of pairs.

Type (i): ξ_1, ξ_2 are linearly independent over $\mathbb{C}$. For such a pair the map $v \mapsto (v\xi_1, v\xi_2)$ is linear and onto the space of pairs of polynomials of degree at most d . By Lemma 4.3 the set of v with $(v, \xi_1, \xi_2) \in \mathcal{I}$ has dimension at most $D - 2d$. The pairs form a set of dimension $4(b+1)$, so by Lemma 4.2(a) the part $\mathcal{I}_i$ of $\mathcal{I}$ of type (i) has dimension at most $D - 2d + 4b + 4$. If $(v, \Xi) \in \mathcal{I}_i$, then by Lemma 4.1 also $(v, e^{i\theta}\Xi h) \in \mathcal{I}_i$ for $\theta \in \mathbb{R}$ and $h \in \text{GL}_2(\mathbb{R})$, and $e^{i\theta}\Xi h = \Xi$ only if $e^{i\theta}h = I$, that is, $h = \pm I$. So every non-empty fibre of the projection $\mathcal{I}_i \rightarrow \mathcal{V}$ has dimension at least 5, and by Lemma 4.2(b) the image has dimension at most $D - 2d + 4b + 4 - 5 = D - 1$.

Type (ii): $\xi_2 = \mu\xi_1$ with $\mu \in \mathbb{C} \setminus \mathbb{R}$. Then $\text{Im}(s_1^* s_2) = \text{Im}(\mu) s_1 s_1^*$, and this lies in $\mathbb{R}z^{2d}$ only if $s_1 \in \mathbb{C}z^d$. For fixed $\xi_1 \neq 0$ the map $v \mapsto v\xi_1$ is onto the polynomials of degree at most d , so the condition defines a subspace of dimension $D - 2d$. It depends only on the line $\mathbb{C}\xi_1$. The lines form a set of dimension $2b$, so this part of $\mathcal{B}$ has dimension at most $D - 2d + 2b = D - 2b$.

Hence $\mathcal{B} \neq \mathcal{V}$. For $v \notin \mathcal{B}$ the matrix $V_{\mathbb{R}} \in P(2, 2b)$ has property (NS), and Proposition 2.3 with $(m, p) = (2, 2b)$ proves the theorem. $\square$

The set $\mathcal{B}$ is not empty: if the first entry of v is z^d , then $\xi_1 = e_1$, $\xi_2 = ie_1$ is a solution of type (ii). For $a = 1$ there is a convenient reformulation, used in the examples.

Proposition 4.4. *Let v be a row of $b+1$ complex polynomials of degree at most $2b$, and let K_v be the real vector space of Hermitian matrices H of size $b+1$ with $vHv^* \in \mathbb{R}z^{4b}$, where v^* is the column of the v_j^* . Then $V_{\mathbb{R}}$ has property (NS) if and only if K_v contains no non-zero matrix of rank at most 2. Moreover $\dim K_v \geq (b-1)^2$.*

Proof. For $\Xi = (\xi_1, \xi_2)$ let $H_{\Xi} = (\xi_2 \xi_1^* - \xi_1 \xi_2^*)/2i$, where ξ^* is the conjugate transpose. Then H_{Ξ} is Hermitian and $vH_{\Xi}v^* = \text{Im}(s_1^* s_2)$. If ξ_1, ξ_2 are independent over $\mathbb{C}$, then H_{Ξ} has rank 2; if $\xi_2 = \mu\xi_1$, then $H_{\Xi} = \text{Im}(\mu) \xi_1 \xi_1^*$ has rank 1. So if (NS) fails, K_v contains a non-zero matrix of rank at most 2. Conversely, let $H \in K_v$ have rank 1 or 2. If $H = \pm \xi \xi^*$, then $H = H_{\Xi}$ for $\Xi = (\xi, \pm i\xi)$. If $H = \alpha\alpha^* - \beta\beta^*$ with α, β independent, then $H = H_{\Xi}$ for $\Xi = (\alpha + \beta, i(\alpha - \beta))$. If $H = \pm(\alpha\alpha^* + \beta\beta^*)$, then $(v\alpha)(v\alpha)^* + (v\beta)(v\beta)^* = cz^{4b}$; both summands are non-negative on the real line, so both vanish at 0 to order at least $4b$, and $v\alpha \in \mathbb{C}z^{2b}$; then $\alpha\alpha^* \in K_v$ and we are in the first case. Finally, the Hermitian matrices form a space of dimension $(b+1)^2$, and K_v is defined by $4b$ linear equations. $\square$

Example 4.5. (a) $b = 1$. For $v = (1 + iz^2, z)$ we get $V_{\mathbb{R}} = \begin{pmatrix} 1 & z^2 & z & 0 \\ -z^2 & 1 & 0 & z \end{pmatrix}$ and $\sum_I (V_{\mathbb{R}})_{II} y_I = y_{12}(1 + z^4) + (y_{14} - y_{23})z + y_{34}z^2 + (y_{13} + y_{24})z^3$. If this is cz^4 , then $y_{12} = y_{34} = 0$, $y_{14} = y_{23}$, $y_{13} = -y_{24}$, and $\text{Pf}_{1234}(y) = y_{13}^2 + y_{14}^2$. So $V_{\mathbb{R}}$ has property (NS); this is a transparent example for $(2, 2)$.

(b) $b = 2$. Let $v = (iz, 1 + iz^4, 1 + z^2 + iz^3)$, that is,

$$V_{\mathbb{R}} = \begin{pmatrix} 0 & z & 1 & z^4 & 1 + z^2 & z^3 \\ -z & 0 & -z^4 & 1 & -z^3 & 1 + z^2 \end{pmatrix}, \quad \text{and let} \quad H_0 = \begin{pmatrix} 4 & i & -i \\ -i & 2 & -i \\ i & i & -2 \end{pmatrix}.$$

A direct expansion gives $vH_0v^* = 2z^8$, and the 8 linear equations for K_v have rank 8, so $K_v = \mathbb{R}H_0$. Since $\det H_0 = -20$, the matrix H_0 has rank 3, and $V_{\mathbb{R}}$ has property (NS). This is an explicit example for $(2, 4)$ different from that of Section 3.

(c) $b = 3$ and $b = 4$. For one integer row v in each case, given in the source files, $\dim K_v$ is 4 and 9, and K_v contains no non-zero matrix of rank at most 2: by exact computation, for a basis

$H_1, \dots, H_r$ of K_v the real and imaginary parts of the 3×3 minors of $\sum_i t_i H_i$ generate all forms of degree 5, respectively 7, in $t_1, \dots, t_r$.

(d) The open set of Proposition 2.3 around such a system can be small. For one of three random curves with $b = 2$ and Gaussian integer coefficients in $[-3, 3]$, two real solutions were found numerically after adding Gaussian noise of standard deviation 0.05 or 0.01 to $V_{\mathbb{R}}$. The curve itself has property (NS): by exact computation $K_v = \mathbb{R}H$ with $\det H \neq 0$. But one eigenvalue of H is only about $5 \cdot 10^{-4}$ times the largest one in modulus, which suggests that the curve is close to the set $\mathcal{B}$. As (NS) is open, small enough perturbations have no real solution; our computations do not show how small.

5 The case $m = 4$

Lemma 5.1. *Let F be $\mathbb{R}$ or $\mathbb{C}$, let $e = \dim_{\mathbb{R}} F$, and let $r, d \geq 1$. Let $N_{r,d}$ be the set of $r \times r$ matrices Π of polynomials over F of degree at most d with $\det \Pi \in Fz^{rd}$. Then $\dim N_{r,d} \leq e(r^2(d+1) - rd)$, and the subset where $\det \Pi = 0$ has dimension at most $e(r^2(d+1) - rd - 1)$.*

Proof. We write the proof for $F = \mathbb{R}$; for $F = \mathbb{C}$ all dimensions double.

The nilpotent cone. Let $\mathcal{N}_k$ be the set of nilpotent real $k \times k$ matrices. Every such matrix is conjugate to a strictly upper triangular one. So $\mathcal{N}_k$ is the image of $(\gamma, u) \mapsto \gamma u \gamma^{-1}$, where $\gamma \in \mathrm{GL}_k(\mathbb{R})$ and u is strictly upper triangular. Each fibre contains with (γ, u) all $(\gamma \beta^{-1}, \beta u \beta^{-1})$ with β invertible upper triangular, so it has dimension at least $k(k+1)/2$. By Lemma 4.2(b), $\dim \mathcal{N}_k \leq k^2 + k(k-1)/2 - k(k+1)/2 = k^2 - k$.

Matrices with $\det \Pi = cz^{rd}$, $c \neq 0$. The coefficient Π_d of z^d in Π is invertible, and $\Pi = \Pi_d(z^d I + \sum_{j < d} A_j z^j)$, where the block companion matrix C_A of the second factor has characteristic polynomial z^{rd} , that is, $C_A \in \mathcal{N}_{rd}$. Let X be the set of such $A = (A_0, \dots, A_{d-1})$. Consider $\Phi: \mathrm{GL}_{rd}(\mathbb{R}) \times X \rightarrow \mathcal{N}_{rd}$, $(\gamma, A) \mapsto \gamma C_A \gamma^{-1}$. If $\gamma C_A \gamma^{-1} = \nu$ and $\gamma = [G_1 \mid \dots \mid G_d]$ with blocks of r columns, where C_A has the blocks I_r below the diagonal, then $G_{k+1} = \nu G_k$ for $k < d$, and A is determined by γ . So each fibre of Φ embeds into the space of $rd \times r$ matrices G_1 , and Lemma 4.2(a) gives $(rd)^2 + \dim X \leq (rd)^2 - rd + r^2 d$. Hence this part of $N_{r,d}$ has dimension at most $r^2 + \dim X \leq r^2(d+1) - rd$.

Matrices with $\det \Pi = 0$. A $\rho \times r$ polynomial matrix P whose rows have degrees $\eta_1 \leq \dots \leq \eta_\rho$ is *row-reduced* if the matrix P_h of the coefficients of z^{η_l} in the rows of P has rank ρ . Then $\deg(cP) = \max_l(\deg c_l + \eta_l)$ for polynomial rows c , since the coefficient of this power of z in cP is a non-zero combination of the rows of P_h (the predictable degree property). Let Π have rank $\rho < r$ over $\mathbb{R}(z)$. The polynomial vectors in the row space of Π over $\mathbb{R}(z)$ form a module which has a row-reduced basis P [For75]. Then $\Pi = CP$ with a polynomial $r \times \rho$ matrix C . As the rows of Π have degree at most d , we get $\deg C_{il} \leq d - \eta_l$, where $C_{il} = 0$ if $\eta_l > d$. The latter is impossible, since C has rank ρ ; so $\eta_l \leq d$ for all l . Thus Π lies in the image of the map $(C, P) \mapsto CP$ on the set $\mathcal{P}_\eta$ of pairs with $\deg C_{il} \leq d - \eta_l$ and P row-reduced with row degrees η . We have $\dim \mathcal{P}_\eta \leq \sum_l r(d - \eta_l + 1) + \sum_l r(\eta_l + 1) = r\rho(d+2)$. Let $\mathcal{G}_\eta$ be the set of $\rho \times \rho$ polynomial matrices T with $\deg T_{kl} \leq \eta_k - \eta_l$, in particular $T_{kl} = 0$ if $\eta_k < \eta_l$, such that the matrix T_h of the coefficients of $z^{\eta_k - \eta_l}$ is invertible. Such a T is block triangular with constant diagonal blocks, the same as those of T_h , so $\det T$ is a non-zero constant and T^{-1} is a polynomial matrix. Moreover $\dim \mathcal{G}_\eta = \sum_{\eta_k \geq \eta_l} (\eta_k - \eta_l + 1) \geq \rho^2$, because every pair $k \neq l$ contributes at least 2 to the sum of the terms for (k, l) and (l, k) . For $(C, P) \in \mathcal{P}_\eta$ and $T \in \mathcal{G}_\eta$ the matrix TP has row degrees at most η and $(TP)_h = T_h P_h$, so TP is row-reduced with row degrees η ; and $\deg(CT^{-1})_{il} \leq d - \eta_l$ by the predictable degree property for TP . So $(CT^{-1}, TP) \in \mathcal{P}_\eta$ lies in the same fibre as (C, P) , and $T \mapsto (CT^{-1}, TP)$ is injective. By Lemma 4.2(b) the matrices of rank ρ with indices η form a set of dimension at most $r\rho(d+2) - \rho^2$. This is increasing in ρ for $\rho \leq r-1$, and for $\rho = r-1$ it equals $r^2(d+1) - rd - 1$. $\square$

Proof of Theorem 1.3. Let $a = 2$, $b \geq 1$, $d = 2b$, so that $\mathcal{V} = \mathcal{V}_{2,b}$ has dimension $D = 4(b+2)(d+1)$,

and $2ad = 4d = 8b$. Let $\mathcal{B} \subset \mathcal{V}$ be the set of $V_{\mathbb{C}}$ for which $V_{\mathbb{R}}$ fails to have (NS). By Lemma 4.1, $V_{\mathbb{C}} \in \mathcal{B}$ if and only if $\Delta_{V_{\mathbb{C}}\Xi} \in \mathbb{R}z^{4d}$ for some complex $(b+2) \times 4$ matrix Ξ with $\mathbb{R}$ -independent columns. Let E be the real span of the columns of Ξ and $j = \dim_{\mathbb{C}}(E \cap iE) \in \{0, 1, 2\}$. Multiplying Ξ from the right by a real invertible matrix, which multiplies Δ_S by a non-zero constant, we may assume that Ξ has one of three forms. We show that the corresponding parts $\mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_2$ of $\mathcal{B}$ have dimension less than D .

$j = 0$. The columns of Ξ are independent over $\mathbb{C}$; this requires $b \geq 2$. Then $V_{\mathbb{C}} \mapsto S = V_{\mathbb{C}}\Xi$ is onto the 2×4 matrices of polynomials of degree at most d , and $S \mapsto \begin{bmatrix} \operatorname{Re} S \\ \operatorname{Im} S \end{bmatrix}$ is a bijection onto the real 4×4 matrices of polynomials of degree at most d . By Lemma 5.1 with $r = 4$, $F = \mathbb{R}$, the set of $V_{\mathbb{C}}$ with $\Delta_{V_{\mathbb{C}}\Xi} \in \mathbb{R}z^{4d}$ has dimension at most $D - 4d$ for each Ξ . The matrices Ξ form a set of dimension $8(b+2)$, and the group $\operatorname{GL}_4(\mathbb{R}) \times S^1$, of dimension 17, acts on them by $\Xi \mapsto e^{i\theta}\Xi h$ with stabilizers $\{\pm 1\}$, preserving the condition. As in the proof of Theorem 1.2, $\dim \mathcal{B}_0 \leq D - 4d + 8(b+2) - 17 = D - 1$.

$j = 2$. Here E is a complex plane and $\Xi = (\xi_1, i\xi_1, \xi_2, i\xi_2)$ with ξ_1, ξ_2 independent over $\mathbb{C}$. With $S' = V_{\mathbb{C}}(\xi_1, \xi_2)$, a 2×2 matrix, we get $\Delta_S = \pm(\det S')(\det S')^*$, and this lies in $\mathbb{R}z^{4d}$ only if $\det S' \in \mathbb{C}z^{2d}$. By Lemma 5.1 with $r = 2$, $F = \mathbb{C}$, this holds on a set of dimension at most $D - 4d$ for each pair. The pairs form a set of dimension $4(b+2)$ on which $\operatorname{GL}_2(\mathbb{C})$, of dimension 8, acts freely without changing E . So $\dim \mathcal{B}_2 \leq D - 4d + 4(b+2) - 8 = D - 4b$.

$j = 1$. Here $E \cap iE = \mathbb{C}\xi$ and $\Xi = (\xi, i\xi, \eta_1, \eta_2)$, where ξ, η_1, η_2 are independent over $\mathbb{C}$, since otherwise $E \cap iE$ would be larger. Put $\sigma = V_{\mathbb{C}}\xi$, $q_c = V_{\mathbb{C}}\eta_c$ and $\tau_c = \det(\sigma, q_c)$ for $c = 1, 2$. The map $V_{\mathbb{C}} \mapsto (\sigma, q_1, q_2)$ is onto the triples of columns of two polynomials of degree at most d . Expanding $\det \begin{bmatrix} S \\ S \end{bmatrix}$ along its first two rows, after subtracting i times the first column from the second, we find $\det \begin{bmatrix} S \\ S \end{bmatrix} = 2i(\tau_1\tau_2^* - \tau_2\tau_1^*)$, hence $\Delta_S = -\operatorname{Im}(\tau_1^*\tau_2)$. We claim that the triples with $\operatorname{Im}(\tau_1^*\tau_2) = cz^{4d}$, $c \in \mathbb{R}$, form a set of codimension at least $4d$. The triples with $\sigma = 0$ have codimension $4d + 4$. Let $\sigma \neq 0$ and write $\sigma = g \cdot (\rho_1, \rho_2)^T$ with g monic of degree ϵ and ρ_1, ρ_2 coprime of maximal degree δ , and put $\iota = d - \epsilon - \delta \geq 0$. Then $\tau_c = gT_c$ with $T_c = \rho_1q_{2c} - \rho_2q_{1c}$ of degree at most $\nu = d + \delta$, and $q_c \mapsto T_c$ is onto the polynomials of degree at most ν , because its kernel consists of the multiples $h \cdot (\rho_1, \rho_2)^T$ with $\deg h \leq d - \delta$. The condition reads $gg^* \operatorname{Im}(T_1^*T_2) = cz^{4d}$, and the left side has degree at most $4d - 2\iota$. If $c \neq 0$, then $\iota = 0$, $g = z^\epsilon$ and $\operatorname{Im}(T_1^*T_2) \in \mathbb{R}z^{2\nu}$. By Lemma 4.3 this has codimension at least $2\nu = 4d - 2\epsilon$ in (q_1, q_2) , and the condition $z^\epsilon \mid \sigma$ has codimension 4ϵ ; together at least $4d + 2\epsilon$. If $c = 0$, then $\operatorname{Im}(T_1^*T_2) = 0$, of codimension at least $2\nu + 1$ by Lemma 4.3, and the σ with given ϵ, δ form a set of dimension at most $2\epsilon + 4(\delta + 1)$, that is, of codimension at least $2\epsilon + 4\iota$; together at least $4d + 2\iota + 1$. This proves the claim. The triples (ξ, η_1, η_2) form a set of dimension $6(b+2)$. The space E , up to multiplication by $e^{i\theta}$, is unchanged under $(\xi, \eta) \mapsto (\xi\gamma, e^{i\theta}(\eta h + \xi w))$ with $\gamma \in \mathbb{C}^\times$, $h \in \operatorname{GL}_2(\mathbb{R})$ and $w \in \mathbb{C}^{1 \times 2}$, where $\eta = (\eta_1, \eta_2)$. These parameters form a set of dimension 11, and they are determined by the new triple up to a common sign of $e^{i\theta}$, h and w . So $\dim \mathcal{B}_1 \leq D - 4d + 6(b+2) - 11 = D - 2b + 1 \leq D - 1$.

Hence $\mathcal{B} \neq \mathcal{V}$, and for $V_{\mathbb{C}} \notin \mathcal{B}$ the matrix $V_{\mathbb{R}} \in P(4, 2b)$ has property (NS). Proposition 2.3 with $(m, p) = (4, 2b)$ proves the theorem. $\square$

Remark 5.2 (Machine checks for $m = 4$). (a) $(4, 2)$. For $b = 1$, and for the matrix $V_{\mathbb{R}}$ made from one 2×3 matrix $V_{\mathbb{C}}$ of degree 2 with Gaussian integer coefficients, property (NS) is certified in exact arithmetic by a positive definite combination of Plücker relations as in Lemma 3.1; the data are in the source files. Such certificates were also found, and verified in exact arithmetic, for nine further matrices $V_{\mathbb{C}}$ with random Gaussian integer coefficients.

(b) $(4, 4)$: *numerical evidence only*. For $b \geq 2$ we have no machine-checked instance of Theorem 1.3, and we do not claim an explicit certified example for $(4, 2b)$ with $b \geq 2$. For seven random curves with $b = 2$ a numerical search for a certificate as in Lemma 3.1 found none. For $b = 2$ a floating point computation supports the theorem. If the square matrix Ξ is invertible and $G = \Xi\Xi^{-1}$, then $\det \begin{bmatrix} S \\ S \end{bmatrix} = \det(\Xi) \det \begin{bmatrix} V_{\mathbb{C}} \\ V_{\mathbb{C}}G \end{bmatrix}$, and the coefficients $e_0, \dots, e_{16}$ of the last determinant are linear forms in

the 36 minors of size 2 of G . The closure $X \subset \mathbb{P}^{35}$ of the set of these matrices of minors has dimension 15 and degree 2384, half the degree of $\mathrm{SO}_6(\mathbb{C})$ [BBB⁺17]. By continuity every real solution, of any type, gives a point of X with $e_0 = \dots = e_{15} = 0$; this point is given by a polynomial in Ξ and $\bar{\Xi}$ which does not vanish. For nine random curves $V_{\mathbb{C}}$ all solutions on X of 15 random linear combinations of $e_0, \dots, e_{15}$ were computed by monodromy and parameter homotopy. Each time 2384 distinct regular solutions were found, hence all of them, and at none of them do all of $e_0, \dots, e_{15}$ vanish: their norm, suitably normalized, is at least $9 \cdot 10^{-6}$, while the equations solved hold up to $3 \cdot 10^{-16}$. A planted real solution is found by the same procedure.

6 Remarks and open cases

Corollary 1.4 follows from Theorem 1.1, since for a system that is not stabilizable no polynomial with all roots in the open left half-plane is $\det(zI - A - BKC)$ with real K . A non-empty open set of such systems shows that the pole placement map is not generically surjective, whether generic sets are complements of proper algebraic sets, as in [RS98], or open dense sets, as in [EG02c]. For $(m, p) = (2, 5)$ there is also a non-empty open set of systems, near the Wronski system, for which real pole placement is possible [EG02b]. The settled cases are consistent with a positive answer to the following analogue of the conjecture of Rosenthal and Sottile.

Problem 6.1. *Is it true that for $n = mp$ the generic system is stabilizable by real static output feedback if and only if $d(m, p)$ is odd?*

The open pairs are, up to order: those with $\min(m, p) = 2$ and $\max(m, p)$ odd, at least 9 and not of the form $2^k - 1$; those with $\min(m, p) = 3$ other than $(3, 3)$; those with $\min(m, p) = 4$ and $\max(m, p)$ odd; and those with $\min(m, p) \geq 5$. For $a \geq 3$ the types $j = 0$ and $j = a$ of Section 5 can be treated by Lemma 5.1, but we have not carried out the count for the intermediate types.

Verification. Proposition 2.3, Theorems 1.2 and 1.3 and the lemmas are proved above and do not depend on computation. Theorem 1.1 and the examples depend on finite computations in exact arithmetic. The programs, certificates and outputs are in the source files; all exact programs and the computation of Remark 5.2(b) were run again for this version.

1. A program in pure Python with rational arithmetic computes, from a file with V and the multipliers, the minors V_I , their degrees and greatest common divisor, the rank of $M(V)$ and a basis of $U(V)$, and tests positive definiteness of Q on $U(V)$ by symmetric elimination with positive pivots. It also tests non-degeneracy over $\mathbb{C}$: the restrictions of the $R_{I,j}$ to the kernel of $M(V)$ generate all forms of degree 4 for $(2, 4)$, and of degree 5 for $(2, 5)$ and $(3, 3)$ (a Macaulay matrix has full column rank modulo a prime). The three matrices of Section 3 and five further matrices for the same pairs pass; the matrix of Remark 5.2(a) passes all tests except non-degeneracy, which does not hold for it. For $(3, 3)$ the smallest eigenvalue of Q on the unit sphere of $U(V)$ is only about 10^{-6} , but the test is exact.
2. A second program recomputes the minors, $U(V)$ and the leading principal minors with PARI/GP, and checks that Q vanishes on $\mathcal{D}$, as a polynomial identity in the entries of X for the minors of $[I_m \mid X]$. A third program, written in a first AI-assisted independent verification run without using the other two, computes the minors by the Leibniz formula, has its own elimination and its own generator of the forms $R_{I,j}$, proves the same identity, and tests positive definiteness by fraction-free leading principal minors and by an LDL^T decomposition. The three matrices of Section 3, three of the further ones and the matrix of Remark 5.2(a) pass both.

3. A program by the author reads the matrices of Section 3 and the multipliers for $(2, 4)$ and $(2, 5)$ from the source of this text, and those for $(3, 3)$ from the certificate file, and repeats the test of Lemma 3.1 with its own code. It also checks Remark 3.3, Example 4.5(a),(b), the determinant identities of Sections 4 and 5 and, for the matrix of type $(2, 4)$, the identities of Step 5 exactly.
4. Example 4.5 was checked by two independent exact programs: $K_v = 0$ in (a); $K_v = \mathbb{R}H_0$ and the signature of H_0 in (b); in (c) Macaulay matrices of full rank modulo a prime, 56 in degree 5 for $b = 3$ and 6435 in degree 7 for $b = 4$. For (b) the first verification run also found a certificate as in Lemma 3.1, which passes the first and the third program.
5. The first verification run re-derived the dimension counts in the proofs of Lemmas 4.3 and 5.1 and Theorems 1.2 and 1.3 and reported no gap; the details it asked for in Lemmas 4.2 and 5.1 have been added. Numerically, it verified the identities of Lemma 4.1 and of the three types, and found that the equations defining Z_ν , $N_{r,d}$ and the set treated in type $j = 1$ have independent differentials at random points of these sets for several small parameters. Its least squares searches for real solutions, for random curves of the types $(2, 2)$, $(2, 4)$, $(2, 6)$, $(4, 2)$, $(4, 4)$, $(4, 6)$, found none; planted solutions were found in most control runs, but not in all. These computations, that of Remark 5.2(b), the statistics in Remark 3.2 and the experiment of Example 4.5(d) use floating point arithmetic and are evidence only. For the three matrices of Section 3 all $d(m, p)$ complex solutions were also computed numerically by homotopy continuation; they are distinct and none is real.
6. A second AI-assisted independent verification run was made for this version, with programs of its own. A fourth program for the test of Lemma 3.1 reads the matrices of Section 3 and the multipliers from the source of this text and from the certificate files, generates the forms $R_{I,J}$ from the definition in Section 3, proves that Q vanishes identically on the minors of $[I_m \mid X]$, and tests positive definiteness on $U(V)$ by elimination and by the signs of the coefficients of the characteristic polynomial. The three matrices pass; coprimality of the minors and non-degeneracy over $\mathbb{C}$ were confirmed (Macaulay matrices of ranks 126, 2002, 2002 modulo $2^{31} - 1$), and the numbers printed in the PDF file were compared with the source. The run checked in exact arithmetic Lemma 2.1, Steps 4 and 5 of the proof of Proposition 2.3 (Step 5 for the three matrices), Remark 3.3 and Example 4.5(a),(b),(d); it re-derived the counts of Sections 4 and 5 and reported no gap; and it compared what is said here about [BA84, RS98, EG02c, EG02a, EG02b, RW99, Ere26] with these sources, in the versions named below. For random complex curves with Gaussian integer coefficients it found, in exact arithmetic: property (NS) for 1996 of 2000 curves with $b = 1$ (for the other four $K_v \neq 0$); $K_v = \mathbb{R}H$ with $\det H \neq 0$ for 300 of 300 curves with $b = 2$; Macaulay matrices of rank 56 over $\mathbb{Q}$, as in Example 4.5(c), for 10 of 10 curves with $b = 3$; and the certificates for the type $(4, 2)$ mentioned in Remark 5.2(a). In floating point arithmetic: at planted solutions of type (i) for $b \leq 3$, of type $j = 1$ for $b = 1$ and of type $j = 0$ for $b = 2, 3$, the differential of the n equations with respect to Ξ has rank $n - 1$, the number of parameters of Ξ minus the dimension of the group; and local searches for real solutions of three random curves of type $(4, 4)$, with 2000 starting points each, found none, while planted solutions of three control curves were found from 12, 8 and 5 of 500 starting points. In this run the programs of items 1 to 5 were executed again from the source files, with the same results; the exceptions are the random scans, the searches by which the certificates were found, and the least squares searches of the first run.

Scope and priority. This note settles the cases listed in Theorems 1.1 to 1.3. The question of [Ere26] remains open for all other pairs with $d(m, p)$ even, and Problem 6.1 is a question, not a claim. The criterion is due to Byrnes and Anderson; Proposition 2.3 is one direction of it, in the form we

need. The reduction to linear sections of the Grassmannian is classical [BB81, RS98], and certificates like that of Lemma 3.1 are standard. What we claim as new are the results on stabilizability for the pairs other than $(2, 2)$, the examples, and the construction of Section 4.

We read [Ere26] and the list that links to it on October 3, 2026, the last time at 19:08 GMT; the problem file bears that date, and the list does not mark the problem as solved. We read [BA84] and [RW99] in full, [EG02c] in full in the authors’ preprint version, and the relevant parts of [EG02a] (preprint version) and of [RS98, EG02b] (arXiv versions). We did not see [WH78, Byr89, RRW94, Wan92, Ber76]; the first three were not accessible to us, and what we say about these five papers is taken from [BA84, RS98, EG02c, RW99]. According to [EG02c], the paper [RRW94] solves the problem of generic stabilizability for complex feedback. In [RW99, Section 3.4] it is said, with reference to [BA84, RRW94], that if a system is not generically pole assignable then there is an open set of systems that cannot be stabilized. In [BA84] this is proved for $n > mp$, and we understand the statement in this sense; the same section adds that pole assignability covers the question of stabilizability only in part. Of the works [BCR98, For75, KL72, BBB⁺17], quoted for standard facts, we looked only at the arXiv version of [BBB⁺17].

We searched zbMATH, Crossref, OpenAlex and arXiv for generic stabilizability by static output feedback and for real pole placement, and looked through the works that cite [BA84, EG02c, EG02a, RS98], most recently on October 3, 2026; we also made general web searches. We found no result on generic stabilizability for $n = mp$ other than the case $(2, 2)$, and no earlier use of the construction of Section 4; the arXiv search of that day found only papers on the computational complexity of static output feedback stabilization. As for Corollary 1.4, we found the pairs $(2, 5)$, $(5, 2)$ and $(3, 3)$ neither in the papers just listed nor in the arXiv version of [Sot11], which does not treat pole placement; but we did not search the literature on real Schubert calculus exhaustively, and we did not see the published version of [Sot11]. This negative search is not a proof of priority.

AI-assisted tools were used to search the problem corpus and the literature, to find the examples and the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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